Multivariate Generalized Gamma-Type Operators: Approximation Results and Applications to Image Processing

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Abstract. In the present paper, we extend a class of integral operators of probabilistic type, which are constructed by means of the generalized Gamma distribution, to the multidimensional setting. In particular, we discuss their approximation properties in weighted continuous function spaces for a broad range of weights, as well as some shape-preserving properties for them. A graphical example is also shown. The paper concludes with applications to image processing, comparing the performance of the new operators, in terms of image denoising, with other well-established denoising techniques.

Keywords: Generalized Gamma operators · weighted approximation · shape-preserving properties · signal and image processing

1 Introduction

In approximation theory, positive linear operators play a central role due to their ability to approximate functions while preserving qualitative properties such as positivity and monotonicity. Classical examples include the Bernstein and Kantorovich operators, which have been extensively studied and generalized in various directions (see, e.g. [3,5,38] and the references therein).

A significant line of research concerns integral operators defined via kernel functions, which are particularly effective at approximating functions on unbounded domains. In this context, operators generated by probability densities have received considerable attention (see, e.g., [3,11,12,20,24,27]).

Among operators of probabilistic type, those associated with Gamma random processes form an important class. Originally introduced by Lupas and Müller in [26], Gamma operators have been thoroughly investigated in the univariate case with respect to their approximation properties, including rates of convergence and Voronovskaya-type theorems (see, e.g., [13,41]). Various generalizations and modifications have also been proposed in recent years, further highlighting their flexibility and effectiveness in approximation processes (see, e.g. [1,10,28]).

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Despite the extensive literature on univariate Gamma-type operators, their study in the multidimensional setting remains relatively limited (see [18,21]). This observation provides a natural motivation for extending such operators to functions of several variables, combining the probabilistic structure of Gamma kernels with the analytical framework of multivariate approximation theory.

In particular, in this paper we introduce a multidimensional version of the Gamma-type integral operators previously studied in the univariate case in [12]. Their definition (see Sect. 2) is as follows:

$$G_{n,\underline{\nu}}(f)(\underline{x}) = \frac{n^d}{\prod_{i=1}^{d} \Gamma(\nu_i)} \int_{\mathbb{R}_+^d} \underline{u}^{n\underline{\nu}-\underline{1}} e^{-\sum_{i=1}^{d} u_i^n} f(\underline{x} \odot \underline{u}) d\underline{u}, \tag{1}$$

where $\underline{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{R}^d$ with $\nu_i \geq 1$ $(i = 1, \ldots, d)$, $n \geq 1$, and $\underline{x} \in \mathbb{R}_+^d = [0, +\infty)^d$. The symbol $\odot$ denotes the Hadamard product, and f is a real-valued function on $\mathbb{R}_+^d$ such that the integral on the right-hand side of (1) is finite.

In Sect. 3, we investigate the approximation properties of operators $G_{n,\underline{\nu}}$ in weighted continuous function spaces and establish some of their shape-preserving properties. Using a graphical example, we illustrate the behavior of such operators when approximating a test function for different values of n. In addition, in Sect. 4 we explore possible applications in image processing and denoising (for similar studies see, e.g., [2,6,7,25,33]). Specifically, we propose the multivariate generalized Gamma-type operators as a structure-preserving smoothing tool that maintains qualitative features of the original image, such as monotonicity, while reducing noise. This property distinguishes the proposed approach from classical smoothing techniques (Gaussian convolution, bilateral filtering, anisotropic diffusion, Wiener filtering), which provide no explicit guarantee that output images respect structural constraints. Applications include Digital Elevation Models (DEMs) in hydrological analysis, diffusion–weighted MRI, HDR tone mapping, infrared thermography, and aerodynamic pressure mapping–domains, where preserving structural coherence is important.

In this perspective, we compare the performance of operators $G_{n,\underline{\nu}}$, in terms of image denoising, with classical methods, using standard metrics (PSNR and SSIM) and introduce a novel *Monotonicity Violation Index* (MVI) to quantify the preservation of structural properties. Our numerical experiments demonstrate that the operators $G_{n,\underline{\nu}}$ outperforms classical smoothing methods in terms of both denoising quality and structural preservation, establishing it as a theoretically grounded and practically effective tool for applications requiring certified structural fidelity.

2 Generalized Gamma-Type Operators

From now on, let $d \geq 1$ be a fixed positive integer and denote by $\mathbb{R}^d$ the Euclidean space of dimension d, endowed with the usual norm $\|\cdot\|_2$. If $\underline{x}, \underline{y} \in \mathbb{R}^d$, the inequality $\underline{x} \leq \underline{y}$ means that $x_i \leq y_i$ for every $i = 1, \ldots, d$. Moreover, we use the symbols $\underline{0} = (0, \ldots, 0)$ and $\underline{1} = (1, \ldots, 1)$. Furthermore, for $\underline{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{R}^d$

and $\underline{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with $\underline{x} \geq \underline{0}$, set $\underline{x}^{\underline{\nu}} := \prod_{i=1}^{d} x_i^{\nu_i}$. If $\underline{\nu} \geq \underline{0}$ and $\underline{x} \geq \underline{0}$, then $\underline{x}^{\underline{\nu}}$ is defined as above, assuming that $\underline{0}^{\underline{0}} := 1$.

We are interested in working with real valued functions f defined on $\mathbb{R}_+^d := [0, +\infty)^d$ (in short, $f \in F(\mathbb{R}_+^d)$) belonging to different function spaces. To this end, we fix the following notations used throughout the paper.

First of all, the symbol $C(\mathbb{R}_+^d)$ stands for the space of all continuous real-valued functions on $\mathbb{R}_+^d$. By $C_b(\mathbb{R}_+^d)$ and $C_0(\mathbb{R}_+^d)$ we denote the Banach spaces of all continuous real-valued functions on $\mathbb{R}_+^d$ which are bounded and which vanish at infinity, respectively, endowed with the uniform norm $\|\cdot\|_\infty$ and the pointwise ordering.

Moreover, let w be a weight function on $\mathbb{R}_+^d$, that is $w \in C_b(\mathbb{R}_+^d)$ and $w(x) > 0$ for all $\underline{x} \in \mathbb{R}_+^d$. We consider the space $C_b^w(\mathbb{R}_+^d)$ (resp., $C_0^w(\mathbb{R}_+^d)$) consisting of all those continuous functions f such that $wf \in C_b(\mathbb{R}_+^d)$ (resp., $wf \in C_0(\mathbb{R}_+^d)$). We recall that $C_b^w(\mathbb{R}_+^d)$ (resp., $C_0^w(\mathbb{R}_+^d)$) is a Banach space with respect to the weighted norm $\|\cdot\|_w$ defined as $\|f\|_w = \|wf\|_\infty$ $(f \in C_b^w(\mathbb{R}_+^d))$.

In order to introduce the multivariate operators under study, let us fix $\underline{\nu} \in \mathbb{R}^d$ with $\underline{\nu} \geq \underline{1}$. For every $n \geq 1$, we define the following n-th multivariate operator by setting

$$G_{n,\underline{\nu}}(f)(\underline{x}) = \frac{n^d}{\prod_{i=1}^{d} \Gamma(\nu_i)} \int_{\mathbb{R}_+^d} \underline{u}^{n\underline{\nu}-\underline{1}} e^{-\sum_{i=1}^{d} u_i^n} f(\underline{x} \odot \underline{u}) d\underline{u}, \tag{2}$$

where $\underline{x} \in \mathbb{R}_+^d$, the symbol $\odot$ denotes the Hadamard (componentwise) product, that is $\underline{x} \odot \underline{u} = (x_1 u_1, \ldots, x_d u_d)$, and f is a real-valued function on $\mathbb{R}_+^d$ such that the integral on the right-hand side of (2) is finite.

Note that, for $d = 1$, definition (2) reduces to the unidimensional operators

$$G_{n,\nu}(f)(x) = \frac{n}{\Gamma(\nu)} \int_0^{+\infty} u^{n\nu-1} e^{-u^n} f(xu) du, \quad x \geq 0 \tag{3}$$

which, by a change of variable, turn into the following ones

$$G_{n,\nu}(f)(x) = \frac{n}{x^{n\nu}\Gamma(\nu)} \int_0^{+\infty} t^{n\nu-1} e^{-\left(\frac{t}{x}\right)^n} f(t) dt, \quad x > 0. \tag{4}$$

Operators (4) have been introduced and studied in [12] for suitable functions defined on $(0, +\infty)$. Such operators are called *generalized Gamma-type operators* since they are related to the generalized Gamma distribution (see [35]), whose probability density function is defined by

$$g_{n,x,\nu}(t) = \begin{cases} \frac{n}{x^{n\nu}\Gamma(\nu)} t^{n\nu-1} e^{-\left(\frac{t}{x}\right)^n} & t > 0, \\ 0 & t \leq 0, \end{cases} \tag{5}$$

with $x > 0, n \in \mathbb{N}, n \geq 1$, and $\nu \in \mathbb{R}, \nu \geq 1$. We want to point out that in [12, Proposition 3.2] it was proved that, for every $n \geq 1$, if there exists $\lim_{x \to 0^+} f(x) \in \mathbb{R}$, then $\lim_{x \to 0^+} G_{n,\nu}(f)(x) = \lim_{x \to 0^+} f(x)$.

Therefore, if f is right-continuous at $x = 0$, we can define $G_{n,\nu}(f)(0) := f(0)$, thereby extending the definition of operators (4) to the interval $[0, +\infty)$, as in definition (3). This allows us to review the operators defined in (2) as the multidimensional version of the generalized Gamma-type operators studied in [12]. For this reason, we refer to them as *multivariate generalized Gamma-type operators*. On the other hand, such operators can be considered as operators associated with a vector of independent random variables, each of which has a Gamma-type distribution as in (5):

$$G_{n,\underline{\nu}}(f)(\underline{x}) = \int_{\mathbb{R}^d_+} \prod_{i=1}^{d} g_{n,x_i,\nu_i}(t_i) f(\underline{t}) d\underline{t}. \tag{6}$$

The expression (6) also shows that

$$\text{if } f(\underline{x}) = \prod_{i=1}^{d} f_i(x_i), \text{ then } G_{n,\underline{\nu}}(f)(\underline{x}) = \prod_{i=1}^{d} G_{n,\nu_i}(f_i)(x_i), \tag{7}$$

provided that $G_{n,\nu_i}(f_i)$ is well-defined.

Following the same arguments used in [12] for $d = 1$, the operators (2) are well-defined on the space $E_{\underline{\nu}}$ of all functions $f : \mathbb{R}^d_+ \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d_+} h_{\underline{\nu}}(\underline{u}) |f(\underline{a} \odot \underline{u})| \, d\underline{u} < +\infty \tag{8}$$

for any $\underline{a} \in \mathbb{R}^d_+$, where $h_{\underline{\nu}} : \mathbb{R}^d_+ \to \mathbb{R}$ is the integrable function on $\mathbb{R}^d_+$ defined by $h_{\underline{\nu}}(\underline{u}) = \prod_{i=1}^{d} h_{\nu_i}(u_i)$ with

$$h_{\nu_i}(u_i) = \begin{cases} u_i^{\nu_i - 1} e^{-u_i} & 0 \le u_i \le 1 \text{ or } u_i \ge \nu_i, \\ \nu_i^{\nu_i} u_i^{-1} e^{-\nu_i} & 1 < u_i < \nu_i. \end{cases} \tag{9}$$

Hence, we can consider, for every $n \ge 1$, $G_{n,\underline{\nu}} : E_{\underline{\nu}} \to F(\mathbb{R}^d_+)$. Moreover, the operators $G_{n,\underline{\nu}}$ are positive, they are well-defined on $C_b(\mathbb{R}^d_+)$ and, in particular, $\|G_{n,\underline{\nu}}(f)\|_\infty \le \|f\|_\infty$, that is the operators $G_{n,\underline{\nu}}$ are positive linear contractions on $C_b(\mathbb{R}^d_+)$ for every $n \ge 1$.

Example 1. For instance, if $\underline{m} \in \mathbb{N}^d$, the function $f(\underline{x}) = \underline{x}^{\underline{m}}$ $(\underline{x} \in \mathbb{R}^d_+)$ satisfies (8) for every $\underline{\nu} \ge \underline{1}$. This implies that the polynomials on $\mathbb{R}^d_+$ belong to $E_{\underline{\nu}}$ for every $\underline{\nu} \ge \underline{1}$.

Another example is given by the function $f(\underline{x}) = e^{\gamma(\underline{x})}$ $(\underline{x} \in \mathbb{R}^d_+)$, where γ is a positive function such that $\gamma(\underline{x}) = O\left(\sqrt{\prod_{i=1}^{d} x_i}\right)$.

Finally, functions of the type $f(\underline{x}) = \prod_{i=1}^{d} x_i^{\alpha_i} \ln x_i$ $(\alpha_i \ge 0, \underline{x} > \underline{0})$, $f(\underline{0}) = 0$, fulfill condition (8), too.

In what follows, we need the expressions of such operators in the i-th coordinate projections, that is the functions $\mathrm{pr}_i : \mathbb{R}^d_+ \to \mathbb{R}_+$, $i = 1, \ldots, d$, defined by $\mathrm{pr}_i(\underline{x}) = x_i$ $(\underline{x} \in \mathbb{R}^d_+)$.

Lemma 1. *For every $n \geq 1$, $h, k \geq 0$, $\underline{x} \in \mathbb{R}_+^d$, $\underline{\nu} \geq \underline{1}$, we have*

$$G_{n,\underline{\nu}}(pr_i{}^h)(\underline{x}) = \frac{\Gamma(\nu_i + h/n)}{\Gamma(\nu_i)} x_i^h \tag{10}$$

and

$$G_{n,\underline{\nu}}(pr_i{}^h pr_j{}^k)(\underline{x}) = \frac{\Gamma(\nu_i + h/n)}{\Gamma(\nu_i)} \frac{\Gamma(\nu_j + k/n)}{\Gamma(\nu_j)} x_i^h x_j^k \tag{11}$$

where $i, j = 1, \ldots, d$, $i \neq j$. In particular,

$$G_{n,\underline{\nu}}(pr_i)(\underline{x}) = \frac{\Gamma(\nu_i + 1/n)}{\Gamma(\nu_i)} x_i, \tag{12}$$

$$G_{n,\underline{\nu}}(pr_i^2)(\underline{x}) = \frac{\Gamma(\nu_i + 2/n)}{\Gamma(\nu_i)} x_i^2. \tag{13}$$

Moreover,

$$G_{n,\underline{\nu}}\left(\sum_{i=1}^d pr_i^2\right)(\underline{x}) = \sum_{i=1}^d \frac{\Gamma(\nu_i + 2/n)}{\Gamma(\nu_i)} x_i^2. \tag{14}$$

Proof. Formulas (10) and (11) follow directly from the definition (2) (or, equivalently, from (6)) of operators $G_{n,\underline{\nu}}$, together with [12, Lemma 2.3], since

$$G_{n,\underline{\nu}}(pr_i{}^h)(\underline{x}) = G_{n,\nu_i}(e_h)(x_i)$$

and

$$G_{n,\underline{\nu}}(pr_i{}^h pr_j{}^k)(\underline{x}) = G_{n,\nu_i}(e_h)(x_i) G_{n,\nu_j}(e_k)(x_k),$$

e_h being the univariate function $e_h(t) = t^h$ $(t \geq 0)$. Formulae (12)-(14) can be derived from (10).

3 Approximation and Shape-Preserving Properties

This section is devoted to the approximation and shape-preserving properties of the multivariate operators $G_{n,\underline{\nu}}$.

First we note that property (7) implies that, for every $n, m \geq 1$, the operators (2) map polynomials on $\mathbb{R}_+^d$ of total degree m into polynomials on $\mathbb{R}_+^d$ of total degree m (see [12, Proposition 2.5]). In particular, the operators $G_{n,\underline{\nu}}$ fix the constant functions.

It is easy to show that operators $G_{n,\underline{\nu}}$ also share the following qualitative properties. In what follows, $\mathrm{Lip}(M, 1)$ stands for the usual space of all Lipschitz continuous functions on $\mathbb{R}_+^d$ with Lipschitz constant M. Note that $\mathrm{Lip}(M, 1) \subset E_{\underline{\nu}}$.

Proposition 1. *Fix $\underline{\nu} \geq \underline{1}$; the following properties are satisfied:*

(a) If $f \in E_{\underline{\nu}}$ is an increasing (decreasing) function with respect to each variable, then, for every $n \geq 1$, $G_{n,\underline{\nu}}(f)$ is increasing (decreasing) with respect to each variable.

(b) If $f \in E_{\underline{\nu}}$ is a convex function with respect to each variable, then, for every $n \geq 1$, $G_{n,\underline{\nu}}(f)$ is convex with respect to each variable.

(c) If $f \in \mathrm{Lip}(M,1)$, then, for every $n \geq 1$, $G_{n,\underline{\nu}}(f) \in \mathrm{Lip}\left(\prod_{i=1}^{d} \frac{\Gamma\left(\nu_i+\frac{1}{n}\right)}{\Gamma(\nu_i)} M, 1\right)$.

Proof. Statement (a) and (b) follow directly from (2). For statement (c),

$$\begin{aligned}
&|G_{n,\underline{\nu}}(f)(\underline{x}) - G_{n,\underline{\nu}}(f)(\underline{y})| \\
&\leq \frac{n^d}{\prod_{i=1}^{d} \Gamma(\nu_i)} \int_{\mathbb{R}_+^d} \underline{u}^{n\underline{\nu}-\underline{1}} e^{-\sum_{i=1}^{d} u_i^n} |f(\underline{x} \odot \underline{u}) - f(\underline{y} \odot \underline{u})| \, d\underline{u} \\
&\leq M \|\underline{x} - \underline{y}\|_2 \frac{n^d}{\prod_{i=1}^{d} \Gamma(\nu_i)} \int_{\mathbb{R}_+^d} \underline{u}^{n\underline{\nu}-\underline{1}} e^{-\sum_{i=1}^{d} u_i^n} \|\underline{u}\|_2 \, d\underline{u} \\
&\leq M \|\underline{x} - \underline{y}\|_2 \frac{n^d}{\prod_{i=1}^{d} \Gamma(\nu_i)} \int_{\mathbb{R}_+^d} \underline{u}^{n\underline{\nu}-\underline{1}} e^{-\sum_{i=1}^{d} u_i^n} \sum_{i=1}^{d} u_i \, d\underline{u} \\
&\leq M \|\underline{x} - \underline{y}\|_2 \prod_{i=1}^{d} \frac{\Gamma\left(\nu_i + \frac{1}{n}\right)}{\Gamma(\nu_i)}.
\end{aligned}$$

In order to discuss the approximation properties of operators in weighted continuous function spaces, let us fix $\underline{\nu} \geq \underline{1}$ and a weight w on $\mathbb{R}_+^d$ such that, for every compact set $K \subset \mathbb{R}_+^d$, there exists $g_{\underline{\nu}} \in L^1(\mathbb{R}_+^d)$ satisfying

$$\frac{h_{\underline{\nu}}(\underline{u})}{w(\underline{a} \odot \underline{u})} \leq g_{\underline{\nu}}(\underline{u}) \quad \text{for every } u \in \mathbb{R}_+^d, a \in K. \tag{15}$$

This implies that $C_b^w(\mathbb{R}_+^d) \subset E_{\underline{\nu}}$ for every $\underline{\nu} \geq \underline{1}$ and, as a consequence, the operators (2) are well-defined also on $C_b^w(\mathbb{R}_+^d)$.

Examples 1. *A particular case of multidimensional weights satisfying condition (15) is given by*

$$w : \mathbb{R}_+^d \to \mathbb{R}, \quad w(\underline{u}) = \prod_{i=1}^{d} w_i(a_i u_i), \tag{16}$$

where, for every $i = 1, \ldots, d$, w_i is a strictly positive bounded function on $[0, +\infty)$ such that, for every compact interval $J \subset [0, +\infty)$, there exists $g_{\nu_i} \in L^1(0, +\infty)$ such that

$$\frac{h_{\nu_i}(u)}{w_i(a_i u_i)} \leq g_{\nu_i}(u) \quad \text{for every } u_i \geq 0, a_i \in J \tag{17}$$

(see (9)). Below we list some examples of univariate weights satisfying (17) which can be used to construct the multivariate weight (16).

1. *Let $w \in C_b([0, +\infty))$ be a weight such that $\inf_{x \geq 0} w(x) > 0$. Hence w satisfies (17). In particular, for every $m \geq 0$, the polynomial weight $w(x) = (1+x^m)^{-1}$ $(x \geq 0)$ satisfies (17).*

2. *Consider $m \geq 1$ and a positive function γ satisfying $\gamma(x) = O(x^{\delta})$, $0 < \delta < 1$. Then $w(x) = e^{-m\gamma(x)}$ $(x \geq 0)$ fulfills* (17).
3. *If $m \in \mathbb{N}$ and $0 < \delta < 1$, then $w(x) = x^{\delta}(1 + x^{m})^{-1}$ $(x \geq 0)$ satisfies* (17). *In all the above examples, condition* (17) *is satisfied as an equality.*

We continue our study with the behavior of the operators on some continuous function spaces.

Proposition 2. *Let w a weight satisfying* (15) *and suppose that, for every $n \geq 1$ and $\underline{\nu} \geq \underline{1}$,*

$$G_{n,\underline{\nu}}(w^{-1}) \in C_b^w(\mathbb{R}_+^d). \tag{18}$$

Then, for every $n \geq 1$ and $\underline{\nu} \geq 1$, the following statements hold:

(a) For every $f \in C_b^w(\mathbb{R}_+^d)$ and $n \geq 1$, $G_{n,\underline{\nu}}(f)$ is continuous on $\mathbb{R}_+^d$. Moreover, $G_{n,\underline{\nu}}(C_b^w(\mathbb{R}_+^d)) \subset C_b^w(\mathbb{R}_+^d)$ and $\|G_{n,\underline{\nu}}\|_{C_b^w(\mathbb{R}_+^d)} = \|G_{n,\underline{\nu}}(w^{-1})\|_w$.
(b) $G_{n,\underline{\nu}}(C_0^w(\mathbb{R}_+^d)) \subset C_0^w(\mathbb{R}_+^d)$ and $\|G_{n,\underline{\nu}}\|_{C_0^w(\mathbb{R}_+^d)} \leq \|G_{n,\underline{\nu}}(w^{-1})\|_w$.

Proof. Let $f \in C_b^w(\mathbb{R}_+^d)$, $n \geq 1$, and $\underline{x} \in \mathbb{R}+^d$. Consider a sequence $(\underline{x}_p)_{p\geq 1}$ in $\mathbb{R}_+^d$ such that $\lim_{p\to+\infty} \underline{x}_p = \underline{x}$. Then,

$$\lim_{p\to+\infty} \underline{u}^{n\underline{\nu}-\underline{1}}\, e^{-\sum_{i=1}^d u_i^n}\, f(\underline{x}_p \odot \underline{u}) = \underline{u}^{n\underline{\nu}-\underline{1}}\, e^{-\sum_{i=1}^d u_i^n}\, f(\underline{x} \odot \underline{u}),$$

for all $\underline{u} \geq \underline{0}$. Now, fix a compact set K of $\mathbb{R}_+^d$ such that $(\underline{x}_p)_{p\geq 1} \subset K$; then there exists $g_{\underline{\nu}} \in L^1(\mathbb{R}_+^d)$ such that

$$\left|\underline{u}^{n\underline{\nu}-\underline{1}}\, e^{-\sum_{i=1}^d u_i^n}\, f(\underline{x}_p \odot \underline{u})\right| \leq \|f\|_w h_{\underline{\nu}}(\underline{u}) w^{-1}(\underline{x}_p \odot \underline{u}) \leq \|f\|_w g_{\underline{\nu}}(\underline{u}).$$

By dominated convergence, $\lim_{p\to+\infty} G_{n,\underline{\nu}}(f)(\underline{x}_p) = G_{n,\underline{\nu}}(f)(\underline{x})$, that is, $G_{n,\underline{\nu}}(f) \in C(\mathbb{R}_+^d)$. Now, taking into account (18) if $f \in C_b^w(\mathbb{R}_+^d)$, for every $\underline{x} \in \mathbb{R}_+^d$,

$$w(\underline{x}) G_{n,\underline{\nu}}(f)(\underline{x}) \leq \|f\|_w w(\underline{x}) G_{n,\underline{\nu}}(w^{-1})(\underline{x}) \leq \|f\|_w \|G_{n,\underline{\nu}}(w^{-1})\|_w.$$

This yields $G_{n,\underline{\nu}}(f) \in C_b^w(\mathbb{R}_+^d)$ and $\|G_{n,\underline{\nu}}\|_{C_b^w(\mathbb{R}_+^d)} \leq \|G_{n,\underline{\nu}}(w^{-1})\|_w$. On the other hand, $\|G_{n,\underline{\nu}}(w^{-1})\|_w \leq \|G_{n,\underline{\nu}}\|_{C_b^w(\mathbb{R}_+^d)} \|w^{-1}\|_{w_m} = \|G_{n,\underline{\nu}}\|_{C_b^w(\mathbb{R}_+^d)}$, that is statement (a).

To prove statement (b), fix $f \in C_0^w(\mathbb{R}_+^d)$; hence, for every $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{R}_+^d$ such that $|f| \leq \max_K |f| + \varepsilon w^{-1}$. Therefore,

$$|G_{n,\underline{\nu}}(f)| \leq \max_K |f| + \varepsilon G_{n,\underline{\nu}}(w^{-1}),$$

from which it follows that $G_{n,\underline{\nu}}(f) \in C_0^w(\mathbb{R}_+^d)$, too.

We now provide approximation results for the operators $G_{n,\underline{\nu}}$ within the framework of weighted continuous function spaces.

Theorem 2. *Let $w \in C_0(\mathbb{R}^d_+)$ be a weight satisfying (15) and (18). Moreover, assume that*

$$\|\cdot\|^2 w \in C_0(\mathbb{R}^d_+). \tag{19}$$

Then, for every $\underline{\nu} \geq \underline{1}$ and $f \in C_0^w(\mathbb{R}^d_+)$,

$$\lim_{n\to+\infty} G_{n,\underline{\nu}}(f) = f \quad \text{with respect to } \|\cdot\|_w,$$

the convergence being uniform on any compact subset of $\mathbb{R}^d_+$.

In particular, for every $\underline{\nu} \geq \underline{1}$ and $f \in C_b(\mathbb{R}^d_+)$,

$$\lim_{n\to+\infty} G_{n,\underline{\nu}}(f) = f$$

uniformly on any compact subset of $\mathbb{R}^d_+$.

Proof. It is a consequence of formulas (12) and (14), hypothesis (19), and Korovkin's result in [4, Corollary 6.13].

Remark 1. For every $\underline{m} > \underline{1}$, consider the polynomial weight defined by setting

$$w_{\underline{m}}(\underline{x}) = \frac{1}{1 + x_1^{2m_1} + \ldots + x_d^{2m_d}} \quad \text{for any } \underline{x} \in \mathbb{R}^d_+.$$

Then, $w_{\underline{m}} \in C_0(\mathbb{R}^d_+)$ and it satisfies (15), (18), and (19).

3.1 Graphical Representations

In this section, we apply the *multivariate generalized Gamma-type operators* $G_{n,\underline{\nu}}$ defined in (2) in order to approximate the test function

$$f(x_1, x_2) = x_1 x_2 e^{-(x_1+x_2)}, \qquad (x_1, x_2) \in \mathbb{R}^2_+.$$

For the sake of simplicity, in the following graphical experiments, we consider the case

$$\nu_0 = (\nu_1, \nu_2) = (1.4, 1.4).$$

Figure 1 represents an example of the approximation behavior of G_{n,ν_0} for $n = 5, 40$, over the domain $[0, 3] \times [0, 3]$. In the figure, the original function f is displayed by a black dashed mesh, while the approximation $G_{n,\nu_0}(f)$ is represented by a colored surface.

This graphical example confirms that, as n increases, the operators $G_{n,\nu_0}(f)$ approximate the function f, in agreement with the theoretical convergence results. The accuracy of the approximation is further quantified by the Root Mean Square Error (RMSE), computed over an equally spaced grid of 40×40 points over $[0, 3] \times [0, 3]$, for n varying in $\{5, 10, 20, 40\}$, and shown in Table 1.

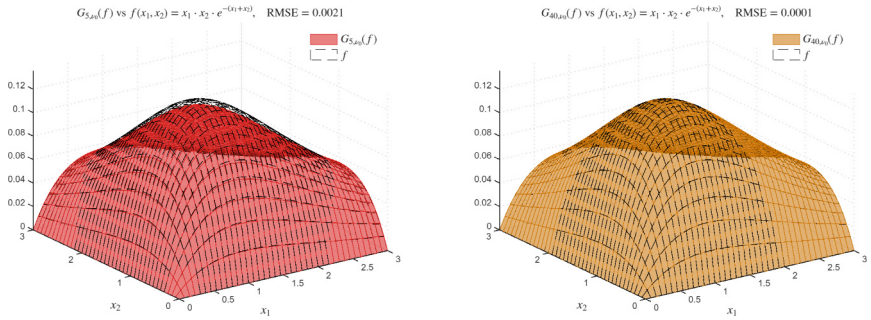

Fig. 1. Approximation of $f(x_1, x_2) = x_1 x_2 e^{-(x_1+x_2)}$ (black dashed mesh) by G_{5,ν_0} (left) and G_{40,ν_0} (right), over $[0,3] \times [0,3]$.

Table 1. RMSE values for the multivariate generalized Gamma-type operators G_{n,ν_0} with $\nu_0 = (1.4, 1.4)$, for the test function f, computed on a 40×40 grid over $[0,3] \times [0,3]$.

RMSE			
$n = 5$	$n = 10$	$n = 20$	$n = 40$
0.0021	0.0006	0.0002	0.0001

4 Structure-Preserving Image Smoothing

Image smoothing is a fundamental pre-processing technique widely employed to suppress noise, which usually occurs in a wide range of applications, such as medical imaging, remote sensing, and engineering fields, causing a degradation of the original signal.

Given a noisy image $f : [0,1]^2 \to [0,1]$, the objective is to produce an approximation $\tilde{f}$ that reduces high-frequency noise while maintaining perceptually relevant structures. Classical methods, such as Gaussian convolution, Bilateral filtering, Anisotropic diffusion or Wiener filter (see, e.g., [29,34,36]) perform well in terms of peak signal-to-noise ratio (PSNR) and structural similarity (SSIM) indices (see [39]). However, they share a common limitation: they provide no explicit guarantee that the output respects structural properties of the input. For instance, if f is known to be monotone (e.g., a digital elevation model, a pressure map, or a radial-density field in MRI), standard smoothing procedures may alter the local ordering of values when applied to noisy and discretized data, potentially introducing artifacts that are not consistent with the underlying physical model. Therefore, in this paper we propose the use of the *multivariate generalized Gamma-type operators* $G_{n,\underline{\nu}}$ as a structure-preserving smoother for such images. In particular we exploit the qualitative properties established in Proposition 1, which, in general, are not shared by any of the classical methods in our comparison. This makes $G_{n,\underline{\nu}}$ the natural choice whenever an application requires certified structural fidelity, not merely statistical denoising quality.

Definition 1. *Let $f : [0,1]^2 \to \mathbb{R}$ be an image satisfying a structural constraint $\mathcal{C}$ (e.g., monotonicity or convexity in a given direction), and let $f_\varepsilon = f + \varepsilon$ be the observed image, where ε denotes noise. A smoother $\mathcal{S}$ is said to be* structure-preserving *with respect to $\mathcal{C}$ if*

$$f \in \mathcal{C} \implies \mathcal{S}(f_\varepsilon) \in \mathcal{C}.$$

Operator $G_{n,\underline{\nu}}$ satisfies the requirement of Definition 1 exactly for monotonicity and convexity, as established in Proposition 1, unlike classical convolutional smoother. Preserving structural properties such as monotonicity and positivity is important in several image processing applications.

For instance, in hydrological analysis of Digital Elevation Models (DEMs), smoothing procedures should not significantly alter the local ordering of elevation values, since this may affect drainage patterns and flow accumulation algorithms (see, e.g., [17,31,40,42]). In practice, standard smoothing techniques applied to noisy and discretized data may introduce small-scale artifacts such as spurious sinks.

Another example is given by diffusion-weighted MRI, in which the apparent diffusion coefficient (ADC) is expected to vary consistently along white matter tracts, reflecting anisotropic diffusion processes (see, e.g., [9,19,22,30]). Smoothing procedures that distort this structural coherence may affect downstream analysis.

Furthermore, in HDR tone mapping, monotonicity of the mapping operator is required to preserve the relative ordering of luminance values, ensuring that brighter scene points correspond to brighter display intensities (see, e.g., [14,15,23,32]).

Similarly, in applications such as infrared thermography and aerodynamic pressure mapping, preserving qualitative structural properties of the data is important for physical interpretability (see, e.g., [8,16,37]).

In this context, the operator $G_{n,\underline{\nu}}$ preserves monotonicity, positivity, and convexity by construction, due to its representation as a convex combination of sampled values. This provides a theoretical guarantee that such structural properties are not violated during the smoothing process.

To quantify the structural fidelity of a smoother, we introduce the following metric.

Definition 2 (Monotonicity Violation Index). *Let $f : \{0,\dots,H-1\} \times \{0,\dots,W-1\} \to [0,1]$ be a discrete image that is non-decreasing in the horizontal direction, i.e.,*

$$f(i,j) \le f(i,j+1) \quad \text{for every } i \in \{0,\dots,H-1\},\ j \in \{0,\dots,W-2\}.$$

For a processed image $\tilde{f}$ defined on the same domain, the ***Monotonicity Violation Index*** *is defined as*

$$\mathrm{MVI}(\tilde{f}) = \frac{1}{H(W-1)} \sum_{i=0}^{H-1} \sum_{j=0}^{W-2} \max\Big(0, \big(f(i,j+1) - f(i,j)\big)\big(\tilde{f}(i,j) - \tilde{f}(i,j+1)\big)\Big).$$

By Proposition 1, $G_{n,\underline{\nu}}$ achieves MVI = 0 on any monotone input, regardless of n and $\underline{\nu}$.

We now compare $G_{n,\underline{\nu}}$ with the classical smoothing methods, namely Gaussian blur, Bilateral filter, Anisotropic Diffusion and Wiener filter, in terms of PSNR, SSIM, and MVI (Definition 2). We report numerical experiments on a synthetic monotone ramp image $f(i,j) = j$, defined on a 256×256 grid with pixel coordinates normalized to $[0,1]^2$, which is strictly increasing in the horizontal direction. Additive Gaussian noise with zero mean and variance $\sigma^2 = 0.05$ is added. For $G_{n,\underline{\nu}}$ we fix $\nu_1 = \nu_2 = 2$ and select $n = 20$.

Table 2. PSNR (dB), SSIM and MVI resulting by the comparison between the original monotone ramp image and the noisy one ($\sigma^2 = 0.05$). Best value in bold.

Method	PSNR (dB)	SSIM	MVI
Noisy input	14.19	0.0286	0.0004
Gaussian blur	23.61	0.3315	0.0001
Bilateral filter	22.88	0.2368	0.0001
Anisotropic diffusion	14.66	0.0338	0.0003
Wiener filter	23.52	0.3983	0.0001
$G_{n,\underline{\nu}}$, $n = 20$	**27.36**	**0.7897**	**0.0000**

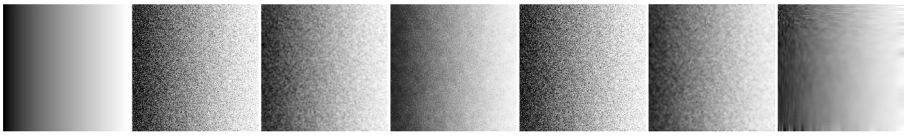

Fig. 2. Monotone ramp image. From left to right: original, noisy ($\sigma^2 = 0.05$), and filtered images.

The results in Table 2 confirm that $G_{n,\underline{\nu}}$ is not only the only method achieving MVI= 0, but also the best performing in terms of PSNR and SSIM, with a significant gap with respect to other classical denoising techniques.

Figure 2 illustrates the visual comparison, showing the original ramp, the noisy version, and the outputs of each method.

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