The Minimum Sum Coloring Problem on Circulant Graphs $C_n(1, 2)$

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Abstract. A circulant graph $C_n(a, b)$ is a graph with n vertices $\{v_0, \ldots, v_{n-1}\}$ such that each vertex v_i is adjacent to vertices $v_{(i+a) \bmod n}$ and $v_{(i+b) \bmod n}$. The Minimum Sum Coloring problem consists in finding a vertex coloring $\mathcal{C} : V \to \mathbb{Z}_+$ in such a way that adjacent vertices receive a different color and $\sum_{v_i \in V} c_i$ is minimum. In this paper, we investigate the Minimum Sum Coloring problem on the connected circulant graphs $C_n(a, b)$ with $n \geq 6$, $\gcd(n, a) = 1$, and $b = (\pm 2a) \bmod n$. After a brief discussion about isomorphism, using an array representation of the graphs, we show that an optimal Odd Cycle Transversal can be computed in linear time and in closed form on these graphs, as well as the independence number, the minimum Chromatic Sum, and an optimal Sum Coloring solution. The approach we propose shows that the Sum Coloring problem on these graphs can be solved to optimality in $O(n)$ time and that the strength $s(\cdot)$ of the graph, i.e. the number of colors needed to achieve the minimum Sum Coloring, equals the chromatic number $\chi(\cdot)$ of the graph.

Keywords: sum coloring · circulant graphs · independent set

1 Introduction

Consider $t + 1$ non-negative distinct integers $n, a_1, a_2, ..., a_t$ such that $n > 0$, $a_i \in \{1, \ldots, n-1\}$ for $i = 1, .., t$, and verifying $a_i + a_j \neq n$ for $i, j = 1, .., t$, $i \neq j$. An (undirected) *circulant graph* $C_n(a_1, a_2, ..., a_t) = (V, E)$ has $V = \{v_0, v_1, \ldots, v_{n-1}\}$ as vertex set, and $E = \{(v_i, v_{(i+a_h) \bmod n})$ for $i = 0, ..., n-1$ and $h = 1, ..., t\}$ as edge set [1,2]. Since $\big((i + a_h) \bmod n + (n - a_h)\big) \bmod n = i$, for an arbitrary $i = 0, ..., n-1$ and for an arbitrary entry a_h, the edges $(v_i, v_{(i+a_h) \bmod n})$ and $(v_{(i+a_h) \bmod n}, v_{((i+a_h) \bmod n + (n-a_h)) \bmod n})$ coincide. Thus, without loss of generality, we can limit ourselves to consider $a_i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ for $i = 1, ..., t$. $C_n(a_1, a_2, ..., a_t)$ is $2t$-regular if $a_i \neq \frac{n}{2}$ for $i = 1, .., t$ and $(2t-1)$-regular otherwise.

We will consider (simple undirected) connected circulant graphs with $t = 2$ entries, denoted by a and b, for the sake of simplicity. The graph will be denoted

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as $C_n(a, b)$ (see Fig. 1). The hypotheses on a and b, also called *entries*, imply that $n \geq 4$. Notice that the graph $C_n(a, b)$ consists of $p = \gcd(n, a, b)$ copies of (the connected) $C_{\frac{n}{p}}\left(\frac{a}{p}, \frac{b}{p}\right)$. Hence $C_n(a, b)$ is connected if and only if $\gcd(n, a, b) = 1$ [3].

It is easy to see that, given n, a, and b, $C_n(a', b')$ and $C_n(b', a')$ with $a' \in \{a, n - a\}$ and $b' \in \{b, n - b\}$ are all the same graph.

Circulant graphs form a class of highly symmetric mathematical structures [4], are a special case of the Cayley graph, and are a natural extension of a ring with increased connectivity. These graphs are used to model several practical applications, for example, in the field of small-world networks, discrete cellular neural networks, supercomputing, chemical reactions, and multiprocessor systems [5].

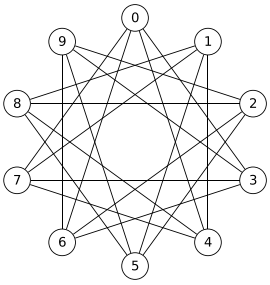

Fig. 1. The 4-regular circulant graph $C_{10}(3, 4)$.

Given a simple (i.e. without loops) graph $G = (V, E)$, a *vertex colouring* $\mathcal{C} : V \to \mathbb{Z}_+$ is a function which assigns a non–negative integer value to each vertex of the graph. A vertex colouring $\mathcal{C}$ is *feasible* if adjacent vertices receive different colors. The *Vertex Colouring Problem* (VCP, for short) consists in finding a feasible vertex colouring which uses the minimum number, $\chi(G)$, of colors. $\chi(G)$ is called the *chromatic number* of G.

The chromatic number of circulant graphs $C_n(a, b)$ is fully characterized by the following theorem.

Theorem 1. [3,6,7] *Let $C_n(a, b)$ be a connected circulant graph. Then*

$$\chi(C_n(a,b)) = \begin{cases} 2 \text{ if } n \text{ even, and } a, b \text{ odd} \\ 4 \text{ if } n = 13 \text{ and } \big(b \equiv \pm 5a \pmod{13} \text{ or } a \equiv \pm 5b \pmod{13}\big) \\ 4 \text{ if } n \neq 5,\ 3 \not| n, \text{ and } \big(b \equiv \pm 2a \pmod{n} \text{ or } a \equiv \pm 2b \pmod{n}\big) \\ 5 \text{ if } n = 5 \\ 3 \text{ otherwise.} \end{cases}$$

A slightly different problem is the *Minimum Sum Colouring Problem* (MSCP, for short). It consists in finding a feasible vertex colouring with minimum cost, where the *cost* of a feasible vertex colouring $\mathcal{C}$ for G is defined as $z(\mathcal{C}) = \sum_{v_i \in V} c_i$. Such minimum cost is the *chromatic sum* of G, and is denoted $\Sigma(G)$ (see Fig. 2).

It is also important to determine the number of colors necessary to achieve $\Sigma(G)$. This number is called the *strength* $s(G)$ of the graph. Clearly, $s(G) \geq \chi(G)$. MSCP can be generalized to the *Optimum Cost Chromatic Partition* (OCCP) problem [8], where a non decreasing sequence $(k_1, k_2, ..., k_{|V|})$ of color costs is given and the problem consists in finding a feasible vertex colouring such that $\sum_{v_i \in V} k_{c_i}$ is minimized. OCCP reduces to MSCP if $(k_1, k_2, ..., k_{|V|}) = (1, 2, ..., |V|)$.



Fig. 2. The label assigned to each vertex indicates its colour (light grey=1, light yellow=2, and green=3). The vertex colouring on the left (respectively, right) uses two (respectively, three) colours its cost is 12 (respectively, 11). Consequently, the colouring on the left is optimal for VPC as the graph is bipartite, but it is not optimal for MSCP; conversely, the colouring on the left is not optimal for VPC, but it is optimal for MSCP (thus, $\Sigma(G) = 11$ and $s(G) = 3$. (Color figure online)

Minimum Sum Colouring problems arise in many applications, for example, in VLSI design, distributed resource allocation, and scheduling [8,9].

MSCP is an NP-hard problem on general graphs [10], and, indeed, in the literature, many papers focus on computing lower and upper bounds or on heuristic and meta-heuristic algorithms, or exact algorithms for small instances or classes of graphs with special properties (see for example [8,9,11–14]).

MSCP on circulant graphs is studied in [15]. In their paper, the authors state the following proposition, where C_n denotes a cycle with n vertices and $\Sigma(C_n) = \frac{3}{2}n$ if n is even and $\Sigma(C_n) = 3 + \frac{3}{2}(n-1)$ if n is odd.

Proposition 1. *[15] Let $C_n(1,t)$ be a circulant graph, where $n = m(t+1)+r$ and $0 \leq r \leq t$. There exists a proper coloring for $C_n(1,t)$ for which the chromatic sum is*

$$\Sigma(C_n(1,t)) = \begin{cases} \Sigma(C_n) \text{ if } t \text{ is odd and } n \text{ is even} \\ (\leq)\Sigma(C_n) + 2t - 2 \text{ if } t \text{ and } n \text{ are odd} \\ m\Sigma(C_{t+1}) \text{ if } t \text{ is even and } r = 0 \\ (\leq)m\Sigma(C_{t+1}) + \frac{7}{2}r + 2 \text{ if } t \text{ and } r \text{ are even} \\ (\leq)(m-1)\Sigma(C_{t+1}) + \frac{3}{2}(r+1) + \frac{7}{2}t \text{ if } t \text{ is even and } r \text{ is odd.} \end{cases}$$

In the same paper similar results are stated for the circulant graphs $C_n(1, 2, ..., t)$.

An *independent set* (also known as *stable set*) of a graph $G = (V, E)$ is a set of mutually non-adjacent vertices. The cardinality of a largest independent set in G is the *independence number* of G and is denoted by $\alpha(G)$ [16], and the problem of determining $\alpha(G)$ for a graph G is known as the *Maximum Independent Set* problem. Observe that, if $C_n(a,b)$ is not connected $\alpha\big(C_n(a,b)\big) = p{\cdot}\alpha\big(C_{\frac{n}{p}}(\frac{a}{p}, \frac{b}{p})\big)$.

By definition, any vertex colouring $\mathcal{C} : V \to \mathbb{Z}_+$ induces a partition $\mathcal{P} = \{V_1, V_2, ..., V_k\}$ of V into $k \leq n$ independent sets, as vertices assigned the same color are non-adjacent. The viceversa is also true: every partition $\mathcal{P} = \{V_1, V_2, ..., V_k\}$ of V induces a colouring $\mathcal{C}$ where the color c_i assigned to an arbitrary vertex v_i is j if $v_i \in V_j$. For this reason, in the sequel, by vertex colouring $\mathcal{C}$ we mean a partition of V into independent sets. It is easy to see that the cost $z(\mathcal{C}) = \sum_{j=0}^{n-1} c_i$ of a vertex colouring $\mathcal{C}$ can be computed also as $z(\mathcal{C}) = \sum_{j=1}^{k} j|V_j|$. This way of computing $z(\mathcal{C})$ shows that a colouring $\mathcal{C} = \{V_1, V_2, ..., V_k\}$ whose subsets are ordered by non-increasing cardinality is one which minimizes the cost among the $k!$ equivalent vertex colouring obtained by ordering the subsets in a different way. Hence a colouring achieving the minimum chromatic sum $\Sigma(G)$ can be searched among those verifying $|V_1| \geq |V_2| \geq ... \geq |V_k|$. In the sequel we will consider colouring verifying these inequalities, only.

By what above, an immediate lower bound for the chromatic sum $\Sigma(G)$ of an arbitrary graph $G = (V, E)$ can be computed as

$$\sum_{j=1}^{\lfloor \frac{n}{\alpha(G)} \rfloor} j \cdot \alpha(G) + \left\lceil \frac{n}{\alpha(G)} \right\rceil \cdot n \bmod \alpha(G). \tag{1}$$

The proposed lower bound is computed assuming that there exists a colouring $\mathcal{C} = \{V_1, V_2, ..., V_{\lceil \frac{n}{\alpha(G)} \rceil}\}$ where all the subsets are maximal independent set, except, possibly, the last one whose cardinality is $n \bmod \alpha(G)$. Since $0 \leq n \bmod \alpha(G) \leq \alpha(G) - 1\}$, the thesis follows. Observe that if n is a multiple of $\alpha(G)$ in the last term of (1) one has $n \bmod \alpha(G) = 0$. Other lower bounds can be found in [9,11,12].

An important issue about circulant graphs is isomorphism. Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a function $f : V \to V'$ such that $(u, v) \in E$ implies $\big(f(u), f(v)\big) \in E'$ [2,4,17].

A special type of isomorphism can be defined for circulant graphs. Precisely, two circulant graphs $C_n(a_1, a_2..., a_t)$ and $C_n(a'_1, a'_2, ..., a'_t)$ are said to be *Ádám-isomorphic* if there exists a positive integer $\mu \in \{1, ..., n-1\}$, coprime to n, i.e. satisfying $gcd(\mu, n) = 1$, such that $\{a'_1, a'_2, ..., a'_t\} = \{\pm\mu a_1, \pm\mu a_2, ..., \pm\mu a_t\}$, where all the operations are intended to be mod n. The integer μ is called *Ádám's multiplier*. The *Ádám's conjecture* states that two circulant graphs are isomorphic if and only if they are Ádám-isomorphic. The conjecture does not hold in general, and several counterexamples of isomorphic circulant graphs that are not Ádám-isomorphic have been found [2,4]. However, it holds for circulant graphs $C_n(a, b)$ with two entries [18–20]. Bézout Theorem states that, given n and a, there exists an integer h such that $(ha) \bmod n = \gcd(n, a)$. Therefore, it is always possible to transform an arbitrary $C_n(a, b)$ into the Ádám-isomorphic $C_n(\gcd(n, a), (hb) \bmod n)$.

In this paper we focus on MSCP on a connected circulant graph $G = C_n(a, b)$ where $n \geq 6$, $\gcd(n, a) = 1$ and $b = (\pm 2a) \bmod n$, and we characterize its strenght $s(G)$. Actually, the results about the isomorphism of a circulant graph with two entries a and b, and, in particular when $b = (\pm 2a) \bmod n$, state that

every $C_n(a, b = (\pm 2a) \bmod n)$ admits $C_n(1,2)$ as an isomorphic graph. For this reason, we will restrict ourselves to consider the circulant graphs $C_n(1,2)$. The only graphs $C_n(1,2)$ excluded by the assumption $n \geq 6$ are $C_n(1,2)$ for $n = 3,4,5$. Since $C_n(1,2) = K_n$ for $n = 3,4,5$, MSCP is trivially solved in these cases.

The paper is organized as follows. A matrix representation is described in Sect. 2. In Sect. 3 the main results. Concluding remarks are found in Sect. 4.

2 Matrix Representation

Let $C_n(a,b) = (V,E)$ be a connected circulant graph. If $(x \pm a) \bmod n = y$, then $v_x, v_y \in V$ are *a-adjacent* and (v_x, v_y) is an *a-edge*. In particular: if $y = (x+a) \bmod n$ $(y = (x-a) \bmod n$, respectively) and we move from v_x to v_y on the a-edge (v_x, v_y), we say that the a-edge (v_x, v_y) is walked along in the positive (negative, resp.) direction; similarly, for the b-edges. By a-cycle we denote a simple (i.e. loopless) cycle made of a-edges only; a similar definition holds for b-cycles. By definition the graph contains $\gcd(n,a)$ distinct a-cycles with $\frac{n}{\gcd(n,a)}$ vertices each, and $\gcd(n,b)$ distinct b-cycles with $\frac{n}{\gcd(n,b)}$ vertices each.

The circulant graphs we deal with can be represented by an array with an infinite number of rows and columns, which suitably organizes the vertices and the edges of $C_n(a,b)$: the connectedness condition $\gcd(n,a,b) = 1$ ensures that every vertex of the graph is represented in $M_n^\infty(a,b)$. While each element of $M_n^\infty(a,b)$ corresponds to a vertex of the graph, each vertex is represented by infinitely many elements in $M_n^\infty(a,b)$. If $m_{i,j}^\infty$ corresponds to the vertex v_x, we shall write $m_{i,j}^\infty = x$. If $m_{i,j}^\infty = x$ then $m_{i,j-1}^\infty = (x-a) \bmod n$, $m_{i,j+1}^\infty = (x+a) \bmod n$, $m_{i-1,j}^\infty = (x-b) \bmod n$, and $m_{i+1,j}^\infty = (x+b) \bmod n$ (see Fig. 3). Similar structures are defined in [3,21,22]. By construction, the adjacency of two elements in a row (column, resp.) represents an a-edge (a b-edge, respectively) and the two elements are a-adjacent (b-adjacent, respectively). The positive direction for the a-edges is rightwards, and the negative one is leftwards. The positive direction for the b-edges is downwards, and the negative one is upwards. Notice that traversing an arbitrary row (column, resp.) of $M_n^\infty(a,b)$ from left to right (top to bottom, resp.) corresponds to infinitely cycling over the corresponding a-cycle (b-cycle, resp.) in the positive direction, and viceversa. Since an a-cycle and a b-cycle have $\frac{n}{\gcd(n,a)}$ and $\frac{n}{\gcd(n,b)}$ vertices, respectively, $m_{i,j}^\infty = m_{i,j+n/\gcd(n,a)}^\infty = m_{i+n/\gcd(n,b),j}^\infty$, for all $i,j \in \{-\infty, ..., +\infty\}$.

Without loss of generality we can choose an arbitrary element of $M_n^\infty(a,b)$ corresponding to v_0 and say that it lays in row 0 and column 0, that is to say, $m_{0,0}^\infty = 0$ (see light yellow row and column in Fig. 3). According to the positive directions of a- and b-edges, we have that positively indexed columns are found on the right of column 0, and positively indexed rows are found below row 0. As a consequence, $m_{i,j}^\infty = (ib + ja) \bmod n$.

Let $m_{p,q}^\infty = x$. By construction, the vertices represented in row p are the vertices x verifying $x \bmod \gcd(n,a) = (pb) \bmod \gcd(n,a)$. Since $(x + b\gcd(n,a)) \bmod \gcd(n,a) = x \bmod \gcd(n,a)$, the copies of vertex v_x are found in

row $p+t\gcd(n,a)$, where t denotes an arbitrary integer, but, generally speaking, in a different column.

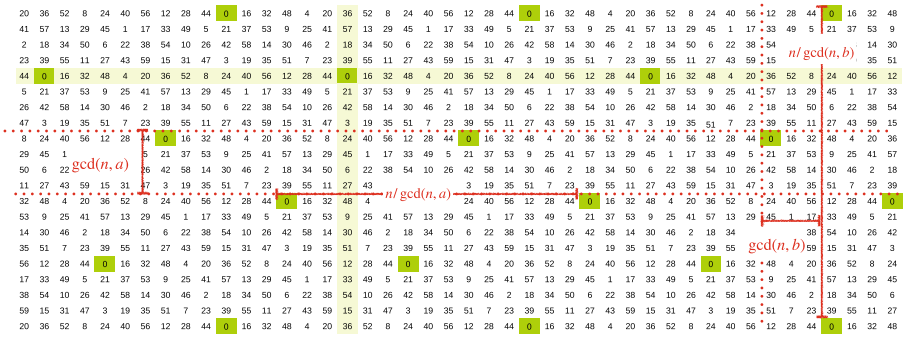

Fig. 3. Part of $M_{60}^{\infty}(16,21)$, in light yellow, row 0 and column 0; all the elements corresponding to v_0 are highlighted in green. In the example, in red, $\gcd(n,a)=4$ rows, $n/\gcd(n,a)=15$ columns, $\gcd(n,b)=3$ columns, and $n/\gcd(n,a)=20$ rows.

For the circulant graphs we deal with, the infinite array $M_n^{\infty}(1,2)$ is shown in Fig. 5 for $n=16,17$. It is easy to see that $\lambda=2$, and that if $m_{i,j}=x$ then $m_{i,j+n}=x$ and $m_{i+\gcd(n,a),j-\lambda}=m_{i+1,j-2}=x$.

3 Optimal Solutions to MSCP on Circulant Graphs $C_n(1,2)$

An Odd Cycle Transversal (OCT, for short) of a graph $G=(V,E)$ is a subset of vertices of the graph that has a nonempty intersection with every odd cycle in G. What graph theorists are interested in is finding an OCT with minimum cardinality for G, say OCT*. By definition, the graph induced by $V\backslash\text{OCT(G)}$ is bipartite, and $\text{OCT}(G)$ is empty if G is bipartite. Recalling that $m_{0,0}^{\infty}=0$, we can give the following definition.

Definition 1. [23] *Consider a circulant graph $C_n(a,b)$ and let $\mathcal{P}$ a subset of pairs of row and column indices (r,s) verifying the following conditions:*

- $m_{r,s}^{\infty}=0$,
- $(r,s)\neq(0,0)$,
- $r+s$ *even*,
- $\nexists z\in\mathbb{Z}$ *such that* $(r,s)=z\cdot(r',s')$ *where* $r'+s'$ *odd.*

Then $(r^*,s^*)\in\mathcal{P}$ is a pair of row and column indices verifying

$$\max\{|r^*|,|s^*|\}=\min\big\{\max\{|r|,|s|\}:(r,s)\in\mathcal{P}\big\}.$$

The elements $m^{\infty}_{r,s} = 0$ with $r+s$ even (odd, respectively) will be called *even (odd,* respectively) *zeroes*. Thus, in $\mathcal{P}$ there are all the even zeroes which are not even multiple of an odd zero.

Theorem 2. [23] *Let $G = C_n(a,b)$ be a circulant graph, then* $|\text{OCT}^*(G)| = \max\{|r^*|, |s^*|\}$.

Theorem 3. [23] *Let $G = C_n(a,b)$ be a circulant graph, and let* $\text{OCT}^*(G)$ *be an* OCT *with minimum cardinality for G. Then* $\alpha(G) = \frac{n-|OCT^*(G)|}{2}$.

The next theorem shows that r^* and s^*, hence $|\text{OCT}^*|$, can be computed in a closed form for the circulant graphs $G = C_n(1,2)$. In Corollary 1 we express $\alpha(C_n(1,2))$ in closed form and $|\text{OCT}^*|$ as function of it. Finally, in Theorem 5, according to [23], we build OCT^*, an OCT with minimum cardinality. We prove that either OCT^* it is an independent set itself, or it can be partitioned into two independent sets, with cardinality $\alpha(G)$ and cardinality $n \bmod \alpha(G)$, respectively. This allow us to define an optimal solution to MSCP on the $C_n(1,2)$'s, that is a vertex colouring achieving the minimum chromatic sum $\Sigma(G)$. In addition, we prove that $s(G) = \chi(G)$ colors.

Theorem 4. *Consider a circulant graph $C_n(1,2)$, with $n \geq 6$. Then*

$$(r^*, s^*) = \left(\left\lfloor \tfrac{n}{3} \right\rfloor + n \bmod 3, \left\lfloor \tfrac{n}{3} \right\rfloor - n \bmod 3\right) \quad \textit{and} \quad |\text{OCT}^*| = \left\lfloor \tfrac{n}{3} \right\rfloor + n \bmod 3.$$

Proof. The zeroes in $M^{\infty}_n(1,2)$ are found at coordinates $(t, wn-2t)$, for t, w arbitrary integers (see Fig. 4). We now show that some of them do not yield the (r^*, s^*) we are looking for.

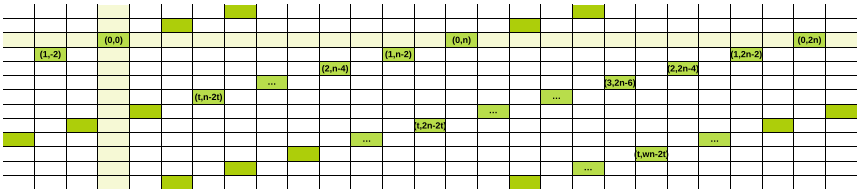

Fig. 4. The coordinates of the zeroes in $M^{\infty}_n(1,2)$

1) Recalling Definition 1, the zeroes with $w = 0$ do not belong to $\mathcal{P}$ as they are either odd zeroes or even zeroes (even) multiple of the odd one $(1,-2)$.
2) By symmetry, we can limit ourselves to consider the zeroes $(t, wn-2t)$ with $w \geq 1$, as $m^{\infty}_{t,wn-2t} = 0$ implies $m^{\infty}_{-t,-wn+2t} = 0$.

Hence, from now on we assume $w \geq 1$.

3) If wn is even, the even zero in $m^{\infty}_{0,wn}$ dominates the zeroes in $m^{\infty}_{t,wn-2t}$ for any integer $t \leq -1$, as $\max\{0, wn\} = wn < |wn-2t| = \max\{|t|, |wn-2t|\}$. If wn is odd, the even zero in $m^{\infty}_{1,wn-2}$ dominates the zeroes in $m^{\infty}_{t,wn-2t}$ for any integer $t \leq -1$, as $\max\{1, wn-2\} = wn-2 < |wn-2t| = \max\{|t|, |wn-2t|\}$.
4) For similar reasons, if wn is even, the even zeroes in $m^{\infty}_{wn/2,0}$ dominates the zero in $m^{\infty}_{t,wn-2t}$ for any integer $t \geq wn/2+1$, as $\max\{wn/2, 0\} = wn/2 < wn/2+1 \leq \max\{t, |wn-2t|\}$. If wn is odd, the even zeroes in $m^{\infty}_{\lfloor wn/2 \rfloor,1}$ dominates the zero in $m^{\infty}_{t,wn-2t}$ for any integer $t \geq \lceil wn/2 \rceil$, as $\max\{\lfloor wn/2 \rfloor, 1\} = \lfloor wn/2 \rfloor < \lceil wn/2 \rceil < \max\{|t|, |wn-2t|\}$.

Hence, from now on we will consider the zeroes in elements $m^{\infty}_{t,wn-2t}$ with $t \geq 0$ and $wn - 2t \geq 0$, only.

5) Consider an even zero Z in $(t, n-2t)$. If $w \geq 2$, Z dominates all the zeroes which are in the region South-East of it, namely all the zeroes in $(s, wn-2s)$ with $s \geq t$, $wn-2s \geq n-2t$, as $\max\{t, n-2t\} \leq \max\{s, wn-2s\}$.

Thanks to the discussion above, we search for (r^*, s^*) among the zeroes in $(t, n-2t)$ for $0 \leq t \leq \lfloor \frac{n}{2} \rfloor$ and integer, and, as observed in [23], t and $n-2t$ must have the same parity of n. By definition, the row index of the zeroes $(t, n-2t)$ increases with t and the column index $|n-2t|$ decreases with t. Hence, the maximum among row and column index is the column index until $t = n-2t$, that is for $0 \leq t \leq \frac{n}{3}$, while it is the row index for $\frac{n}{3} \leq t \leq \frac{n}{2}$. As a consequence, we can limit ourselves to consider the maximum t integer and with the desired parity in $\{0, 1, ..., \lfloor \frac{n}{3} \rfloor - 1, \lfloor \frac{n}{3} \rfloor\}$, and the minimum t integer and with the desired parity in $\{\lceil \frac{n}{3} \rceil, \lceil \frac{n}{3} \rceil + 1, ... \lfloor \frac{n}{2} \rfloor\}$,

We can limit ourselves to study the parity of n with respect to the parity of t, as $n-2t$ has always the same parity of n. Considering all these constraints, we distinguish three cases: *i*) $n \bmod 3 = 0$; *ii*) $n \bmod 3 = 1$; and *iii*) $n \bmod 3 = 2$.

Consider case $n \bmod 3 = 0$, first. Since $\frac{n}{3}$ and n have the same parity, then

$$|\text{OCT}^*| = \min\left\{ \max\{t, n-2t\} \text{ for } t \geq 0 \text{ and integer} \right\} = \frac{n}{3},$$

and $(r^*, s^*) = (\frac{n}{3}, \frac{n}{3})$, and the theses follow.

Now consider case $n \bmod 3 = 1$. Write n as $n = 3 \cdot \lfloor \frac{n}{3} \rfloor + 1$. Hence, $\lfloor \frac{n}{3} \rfloor$ and n have opposite parity. Since t and n must have the same parity, t can assume either the integer value $\lfloor \frac{n}{3} \rfloor - 1$ or the integer value $\lceil \frac{n}{3} \rceil$. We get

$$\begin{aligned}
|\text{OCT}^*| &= \min\left\{ \max\{t, n-2t\} \text{ for } t = \left\lfloor \tfrac{n}{3} \right\rfloor - 1 \text{ or } t = \left\lceil \tfrac{n}{3} \right\rceil \right\} = \\
&= \min\left\{ \max\left\{ \left\lfloor \tfrac{n}{3} \right\rfloor - 1, n - 2\left(\left\lfloor \tfrac{n}{3} \right\rfloor - 1\right)\right\}, \max\left\{ \left\lceil \tfrac{n}{3} \right\rceil, n - 2\left\lceil \tfrac{n}{3} \right\rceil \right\} = \right. \\
&= \min\left\{ n - 2\left(\left\lfloor \tfrac{n}{3} \right\rfloor - 1\right), \left\lceil \tfrac{n}{3} \right\rceil \right\} = \\
&= \min\left\{ \left\lfloor \tfrac{n}{3} \right\rfloor + n \bmod 3 + 2, \left\lceil \tfrac{n}{3} \right\rceil \right\} = \\
&= \left\lceil \tfrac{n}{3} \right\rceil.
\end{aligned}$$

The zero which gives such minimum is $(t, n-2t)$ for $t = \lceil \frac{n}{3} \rceil$. Hence, $(r^*, s^*) = \left(\lceil \frac{n}{3} \rceil, n - 2\lceil \frac{n}{3} \rceil \right) = \left(\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor - 1 \right)$. Recalling that $n \bmod 3 = 1$, the theses follow.

The third case is when $n \bmod 3 = 2$. Write n as $n = 3 \cdot \lfloor \frac{n}{3} \rfloor + 2$. Hence, $\lfloor \frac{n}{3} \rfloor$ and n have the same parity. Since t and n must have the same parity, t can assume either the integer value $\lfloor \frac{n}{3} \rfloor$ or the integer value $\lfloor \frac{n}{3} \rfloor + 2$. We get

$$\begin{aligned} |\text{OCT}^*| &= \min\left\{ \max\{t, n-2t\} \text{ for } t = \lfloor \tfrac{n}{3} \rfloor \text{ or } t = \lfloor \tfrac{n}{3} \rfloor + 2 \right\} = \\ &= \min\left\{ \max\{\lfloor \tfrac{n}{3} \rfloor, n - 2\lfloor \tfrac{n}{3} \rfloor\}, \max\{\lfloor \tfrac{n}{3} \rfloor + 2, n - 2(\lfloor \tfrac{n}{3} \rfloor + 2)\} = \right. \\ &= \min\left\{ \lfloor \tfrac{n}{3} \rfloor + n \bmod 3, \lfloor \tfrac{n}{3} \rfloor + 2 \right\} = \\ &= \lfloor \tfrac{n}{3} \rfloor + 2. \end{aligned}$$

Since $n \bmod 3 = 2$, both zeroes considered give us the minimum OCT. That is $(r^*, s^*) \in \left\{(\lfloor \frac{n}{3} \rfloor, n - 2\lfloor \frac{n}{3} \rfloor\}), (\lfloor \frac{n}{3} \rfloor + 2, n - 2(\lfloor \frac{n}{3} \rfloor + 2))\right\} = \left\{(\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor + n \bmod 3), (\lfloor \frac{n}{3} \rfloor + n \bmod 3, \lfloor \frac{n}{3} \rfloor + n \bmod 3 - 4)\right\}$. For the sake of simplicity, we will consider as (r^*, s^*) the second one, which can be written as $(\lfloor \frac{n}{3} \rfloor + n \bmod 3, \lfloor \frac{n}{3} \rfloor - n \bmod 3)$, as $n \bmod 3 = 2$, and the theses follow.

We get the following corollary to Theorems 2, 3, 4.

Corollary 1. *Consider a circulant graph $G = C_n(1,2)$. Then*

$$\alpha(C_n(1,2)) = \lfloor \tfrac{n}{3} \rfloor \qquad \textit{and} \quad |\text{OCT}^*| = \alpha(C_n(1,2)) + n \bmod 3.$$

We can now prove the main theorem.

Theorem 5. *Consider a circulant $C_n(1,2)$. Let V_1, V_2, V_3, and V_4 be defined as follows.*

$$\begin{cases} V_1 = \{v_{1+3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 1\} \\ V_2 = \{v_{2+3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 1\} \\ V_3 = \{v_{3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 1\} \\ V_4 = \emptyset \end{cases} \quad \textit{if } n \bmod 3 = 0$$

$$\begin{cases} V_1 = \{v_1, v_{5+3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 2\} \\ V_2 = \{v_{3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 1\} \\ V_3 = \{v_0, v_{4+3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 2\} \\ V_4 = \{v_2\} \end{cases} \quad \textit{if } n \bmod 3 = 1$$

$$\begin{cases} V_1 = \{v_1, v_5, v_{9+3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 3\} \\ V_2 = \{v_3, v_{7+3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 2\} \\ V_3 = \begin{cases} \{v_0, v_4\} & \textit{if } n = 8 \\ \{v_0, v_4, v_{8+3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 3\} & \textit{if } n > 8 \end{cases} \\ V_4 = \{v_2, v_6\} \end{cases} \quad \textit{if } n \bmod 3 = 2.$$

Then $\mathcal{C}^* = \{V_1, V_2, V_3, V_4\}$ is an optimal solution to MSCP*,*

$$\sum(C_n(1,2)) = 2(n + n \bmod 3), \qquad \textit{and} \qquad s(C_n(1,2)) = \chi(C_n(1,2)).$$

Proof. Let $G = (V, E) = C_n(1,2)$. The proof of Theorem 5.1 in [23] states that the graph $G(V \backslash \text{OCT}^*)$ is bipartite and the two subsets of vertices have the same

cardinality $\alpha(G)$. These two subsets of vertices will be V_1 and V_2 in the vertex colouring $\mathcal{C}^*$.

According to [23], given (r^*, s^*) an optimum OCT* is obtained as we now describe.

When $n \bmod 3 = 0$, $(r^*, s^*) = (\frac{n}{3}, \frac{n}{3})$. We define OCT* as follows: $\{v_{3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 1\}$. We now show that this set is an independent set. In OCT* there are the vertices v_h with $h \bmod 3 = 0$, for $h = 0, ..., n-1$. The four vertices adjacent to the vertices v_h are the vertices v_k with $k = (h \pm 1) \bmod n$ or $k = (h \pm 2) \bmod n$. Since $k \bmod 3 \neq 0$, for all k, none of them belongs to OCT*. Let $V_3 =$ OCT*. Since $V_1 \cup V_2 \cup V_3 = V$ then $V_4 = \emptyset$. Therefore $\{V_1, V_2, V_3, V_4\}$ is a feasible vertex colouring. Since $|V_j| = \alpha(G)$ for $j = 1, 2, 3$, and its cost $z(\{V_1, V_2, V_3, V_4\}) = (1 + 2 + 3)\frac{n}{3} = 2n$ equals the lower bound (1), $\mathcal{C}^* = \{V_1, V_2, V_3, V_4\}$ is an optimal colouring for MSCP, and, by Theorem 1, $s(G) = \chi(G) = 3$.

When $n \bmod 3 = 1$, we define OCT* as follows: OCT$^* = \{v_0, v_2, v_{4+3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 2\}$ (see Fig. 5 (*a*)). This set is not an independent set as v_2 is *b*-adjacent to v_0 and v_4 (on the other hand, it could not have been an independent set as its cardinality $\alpha(G) + 1$ exceeds the cardinality $\alpha(G)$ of a maximum independent set in G). However, we can partition OCT* into $V_3 =$ OCT$^* \setminus \{v_2\}$ and $V_4 = \{v_2\}$. V_4 is trivially an independent set. As for V_3, we now prove that it is an independent set. Consider the vertices in $V_3 \setminus \{v_0\}$. These are the vertices v_h with $h \bmod 3 = 1$, for $h = 4, ..., n-1$. The four vertices adjacent to v_h are the vertices v_k with $k = (h \pm 1) \bmod n$ or $k = (h \pm 2) \bmod n$. Since $k \bmod 3 \neq 1$, for all k, none of them belongs to $V_3 \setminus \{v_0\}$. Now consider v_0. Its four adjacent vertices are v_1, v_2, v_{n-2}, and v_{n-1}. None of them belongs to $V_3 \setminus \{v_0\}$ as the indices 1 and 2 are not ≥ 4, $(n-1) \bmod 3 = 0$, and $(n-2) \bmod 3 = 2$. Hence, V_3 is an independent set, as claimed. Therefore $\{V_1, V_2, V_3, V_4\}$ is a feasible vertex colouring. Since $|V_j| = \alpha(G)$ for $j = 1, 2, 3$, $|V_4| = n \bmod \alpha(G)$, and the cost $z(\{V_1, V_2, V_3, V_4\}) = (1 + 2 + 3)\frac{n-1}{3} + 4 = 2n + 2$ equals the lower bound (1), $\mathcal{C}^* = \{V_1, V_2, V_3, V_4\}$ is an optimal colouring for MSCP, and, by Theorem 1, $s(G) = \chi(G) = 4$.

When $n \bmod 3 = 2$, we define OCT* as follows: OCT$^* = \{v_0, v_2, v_4, v_6, v_{8+3i}, i = 0, ..., \lfloor \frac{n}{3} \rfloor - 3\}$ for $n > 8$ (see Fig. 5 (*b*)) and OCT$^* = \{v_0, v_2, \ v_4, v_6\}$ for $n = 8$. This set is not an independent set as v_2 is *b*-adjacent to v_0 and v_4, and v_4 is *b*-adjacent to v_2 and v_6 (on the other hand it could not have been an independent set as its cardinality $\alpha(G) + 1$ exceeds the cardinality $\alpha(G)$ of a maximum independent set in G). However, we can partition OCT* into $V_3 =$ OCT$^* \setminus \{v_2, v_6\}$ and $V_4 = \{v_2, v_6\}$. V_4 is trivially an independent set. As for V_3, we now prove that it is an independent set. Consider the vertices in $V_3 \setminus \{v_0, v_4\}$. These are the vertices v_h with $h \bmod 3 = 2$, for $h = 8, ..., n-1$. The four vertices adjacent to v_h are the vertices v_k with $k = (h \pm 1) \bmod n$ or $k = (h \pm 2) \bmod n$. Since $k \bmod 3 \neq 2$, for all k, none of them belongs to $V_3 \setminus \{v_0, v_4\}$. Now consider v_0 and v_4. Their four+four adjacent vertices are $v_1, v_2, v_{n-2}, v_{n-1}$ and v_2, v_3, v_5, v_6. None of them belongs to $V_3 \setminus \{v_0, v_4\}$ as the indices $1, 2, 3, 5$, and 6 are not ≥ 8, $(n-1) \bmod 3 = 1$, and $(n-2) \bmod 3 = 0$.

Hence, V_3 is an independent set, as claimed. Therefore $\{V_1, V_2, V_3, V_4\}$ is a feasible vertex colouring. Since $|V_j| = \alpha(G)$ for $j = 1, 2, 3$, $|V_4| = n \bmod \alpha(G)$, and the cost $z(\{V_1, V_2, V_3, V_4\}) = (1 + 2 + 3)\frac{n-2}{3} + 4 \cdot 2 = 2n + 4$ equals the lower bound (1), $\mathcal{C}^* = \{V_1, V_2, V_3, V_4\}$ is an optimal colouring for MSCP, and, by Theorem 1, $s(G) = \chi(G) = 4$.

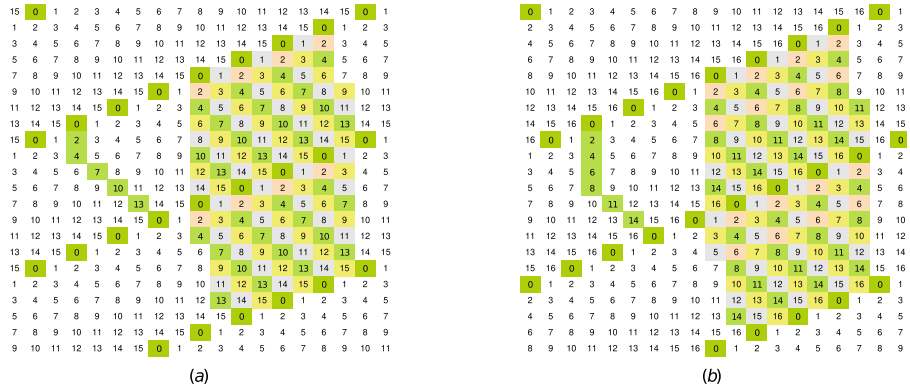

Fig. 5. (*a*) Part of $M_{16}^{\infty}(1,2)$: on the left, in green, OCT*, with 6 elements, $(r^*, s^*) = (6,4)$; (*b*) Part of $M_{17}^{\infty}(1,2)$: on the left, in green, OCT*, with 7 elements, $(r^*, s^*) = (7,3)$. (*a,b*) on the right, the optimum solution $\{V_1, V_2, V_3, V_4\}$ to MSCP (V_1 in light grey, V_2 in light yellow, V_3 in green, V_4 in light orange.

4 Conclusions

In this paper we focussed on the MSCP on circulant graphs $C_n(1,2)$ with $n \geq 6$. The approach is purely combinatorial and the main result is Theorem 5, where we define a vertex colouring attaining the chromatic sum $\Sigma(\cdot)$ on the circulant graphs $C_n(1,2)$ for all values of n. In addition, we show that such a coloring uses a $s(\cdot) = \chi(\cdot)$ colors.

Referring to Proposition 1, one has $m = \lfloor \frac{n}{t+1} \rfloor$, $r = n \bmod (t+1)$. When $t = 2$, in particular, one has $m = \lfloor \frac{n}{3} \rfloor$, $r = n \bmod 3 \in \{0, 1, 2\}$. Recalling that $\lfloor \frac{n}{3} \rfloor = \frac{n - n \bmod 3}{3}$, and since $\Sigma(C_3) = 6$, the results become

$$\Sigma(C_n(1,2)) = \begin{cases} 2n \text{ if } n \bmod 3 = 0 \\ (\leq) 6 \left\lfloor \frac{n}{3} \right\rfloor + \frac{7}{2} 2 + 2 = 2n + 5 \text{ if } n \bmod 3 = 2 \\ (\leq) \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) 6 + \frac{3}{2} \cdot 2 + \frac{7}{2} \cdot 2 = 2n + 2 \text{ if } n \bmod 3 = 1. \end{cases}$$

When $n \bmod 3 = 0$ or 1 this result coincides with our result. When $n \bmod 3 = 2$ our result states that $\Sigma(C_n(1,2)) = 2n + 4$, thus proving that the result by [15] is not tight, as the "$(\leq)$" suggests.

As a side result, we were able to express in a closed form the coordinates of an even zero corresponding to the minimum Odd Cycle Transversal of a circulant graph $C_n(1,2)$, as well as the independence number $\alpha(C_n(1,2))$.

Recalling the isomorphism issue, the results obtained in this paper apply to every $C_n(a, b = (\pm 2a) \bmod n)$ and show that MSCP on circulant graphs $C_n(a, (\pm 2a) \bmod n)$ is solvable in linear time, improving upon the general NP-hard case.

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