Nanofluid Injection in Poroelastic Media: Possible Applications in Environmental Science

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Abstract. Nanofluids are innovative materials consisting of a suspension of nanoparticles in a fluid, such as water or oil. They are commonly used in engineering and medicine because of their thermal conductivity, which makes them particularly efficient for cooling, heat transport, and tumor treatment. In recent times, the unique properties of nanofluids have led to their use for environmental application. This includes oil extraction processes, the treatment of contaminated water and soil, and the improvement of the seismic response of the soil during strong seismic events. Although these applications are very different from each other, it is necessary in all cases to inject the nanoparticles into the soil. However, predicting the distribution of particles *a priori*, considering the injection parameters, is very difficult due to the intrinsic complexity of the subsoil and uncertainty regarding parameters such as porosity and elastic properties. One possible solution is to use numerical simulations on digital twins of the area of interest. These simulations are computationally expensive, and are affected, in the specific case, by a high level of uncertainty, due to the lack of specific information on soil stratigraphy.

In this paper, we present and discuss a mathematical model that describes the injection of nanofluid into the soil, taking into account its natural poroelasticity, which is often neglected. The subsequent drift and diffusion of the particles, as well as their interaction with the surrounding medium through absorption and desorption processes, can be described using the same model, but with different boundary and initial conditions. After introducing the model, we discuss in detail the effects of poroelasticity and the role of injection parameters on particle distribution. We also present some (*semi*−)analytical representations of the model that could be useful for the preliminar calibration of the numerical simulations and, potentially, of the injection process.

Keywords: Nanofluid · porous media · pollution · fluid injection · poroelasticity · nanoparticles

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1 Introduction

Nanofluids are aqueous (or other fluid) suspensions containing small percentages (max. 1% by volume) of appropriately functionalized nanoparticles (NPs) [5]. Nanofluids are widely used in numerous fields of scientific and technological research thanks to their unique chemical and physical properties and relatively low production cost. Because of anthropogenic pollution, the development of nanotechnologies is regarded as a beneficial environmental tool, especially in the context of in situ remediation, that is, the cleanup of soils and groundwater [3,9]. The enhanced penetration capacity of NPs, attributable to their reduced size, facilitates their injection into the matrix at depth, thereby facilitating the degradation and immobilization of contaminants across a wide range of sites. In addition, the implementation of NPs is often more cost-effective and less destructive than that obtained with ex situ techniques. However, the biological risk associated with the use of these techniques has yet to be fully assessed, given their recent application [13]. Anyway, NPs technology also has many other applications in several fields, some of which are medicine (because their size is similar to that of bio-molecules), cosmetics, restoration and preservation of drinking water supplies, cooling electronics, structural engineering, nuclear engineering, oil industry and injection of nanofluids into the soil to enhance soil properties for agricultural purposes [10] or for improving soil stability [4]. We can also consider emerging trends in green synthesized and bio-based nanoparticles with the aim of safe remediation of the environment [9,14]. In this work, during the injection phase, we reasonably assume that our matrix soil behaves as a poroelastic medium introducing the volumetric dilation in our model. The controlled injection of nanofluids into poroelastic matrices has many important applications, such as in biomedical engineering (drug delivery, hyperthermia cancer treatments [2,7]). Moreover, poroelastic models help predict fluid transport, tissue deformation, and the distribution of nanoparticles, which is critical for targeting therapy effectively and minimizing harm to surrounding healthy tissue. One of the leading application is the nanofluid-enhanced oil recovery: an advanced technique ([11]) used to improve oil production from reservoirs by reducing the interfacial tension between oil and water and increasing rock's wettability (to a more water-wet state). In geothermal systems nanofluids can improve the thermal conductivity of the heat transfer fluid. The process of injecting nanofluids into the soil and their subsequent interaction with the surrounding environment is a highly complex process that depends on numerous factors. Some of these, such as injection parameters, can be calibrated and optimized, while others, such as the geological structure of the sub-soil and the actual porosity of the soil, are very difficult to determine with precision. Hence, it can be very useful to develop accurate mathematical and numerical models capable of supporting the planning phase of NPs injection to maximize the beneficial effect and contribute to cost reduction. On the other hand, thanks to the growing availability of computational resources, it is now possible to simulate complex events on digital twins of real geological domains. However, the accuracy and reliability of these simulations are strongly related to knowledge of the geological structure of the region

under examination, which is often only partially available. Analytical or semi-analytical solutions, even if related to simplified regimes and models, provide an easy, straightforward guidance, that can be used to prepare numerical simulations, but can also help with the design stages of the injection process, reducing the costs of the injection process and optimizing its effectiveness. In this paper, we present some preliminary results of our work aimed at determining the concentration of nanoparticles following the injection process into the subsoil. In this study we do not consider a particular geological structure. However, to set a realistic context, we can consider as soil model a quartz sand matrix and as NPs model, silica-based nanoparticles, which are among the most promising materials for the ease with which it is possible to modify their characteristics. The paper is organized as follows. In Sect. 2 we introduce the mathematical model able to describe the injection process and the successive phase. In Sect. 3 we discuss the procedure for equation scaling, needed to understand which terms govern the process. In Sect. 4 some simple analytical solutions are derived and discussed to understand the relative contribution of the different parameters during and after the injection phase. In Sect. 5 some conclusions and future developments are presented and discussed.

2 Mathematical Model

In this section, we briefly introduce the mathematical model based on the colloid filtration theory [1,12]. The single-phase nanoparticles flow in three-dimensional domains can be described by the reaction-convection-diffusion equation, as follows:

$$\begin{cases} \dfrac{\partial C}{\partial t} + \nabla \cdot \left(\mathbf{v} \dfrac{C}{\varepsilon} \right) = \nabla \cdot \left(D\, \varepsilon\, \nabla \left(\dfrac{C}{\varepsilon} \right) \right) - k_f C \\ \nabla \cdot \mathbf{v} = 0 \\ \dfrac{\partial C_b}{\partial t} = k_f C \end{cases} \tag{1}$$

where $\mathbf{x} \in R^3$, and $t \geq 0$. C and C_b are the concentration (the number of nanoparticles moles per unit volume of suspension) of suspended unbounded particles and of the bounded ones (i.e. adsorbed) respectively, and $\mathbf{v}$ is the Darcy velocity. D and ε denote the diffusion coefficient and the soil porosity. In addition to the mechanical flow terms, the reaction term, that accounts for the interactions between the NPs and the porous medium, represents the NPs adsorption and it is derived from the classical colloid filtration theory. It is modeled through the filtration rate coefficient as $k_f = \frac{3\eta}{4r_c} \frac{1-\varepsilon}{\varepsilon} \|\mathbf{v}\|$. Roughly speaking the coefficient k_f models how fast the NPs can be captured by the soil grains. It depends on the velocity of the fluid that transports the particles, the porosity, the grain size r_c and the efficiency of the collector η. We observe that since the internalization and desorption phenomena, occur over a longer time-scale, compared to the injection phase, their effect can be neglected at this stage. The equations above must be coupled to the Darcy's law modeling the fluid flow in a porous medium: $\nabla p = -\frac{\mu}{K}\mathbf{v}$. Here p is the pressure, μ the

dynamical viscosity and K the soil permeability. Classical models that deal with transport of NPs in the soil describe the medium as porous. However, especially during the injection phase, it is reasonable to assume that the soil behaves as a poroelastic medium. We can therefore assume that porosity ε varies with respect to the rest value ε_0, according to the following law [6]

$$\varepsilon = \frac{\varepsilon_0 + e}{1 + e}, \tag{2}$$

where $e = \nabla \cdot u$ is the volumetric dilation modeled as the divergence of the soil displacement $\mathbf{u}$. We consider $\mathbf{u}$ governed, in a linear regime, by Navier's equation as follows

$$G\,\nabla^2 \mathbf{u} + (\lambda + G)(\nabla e) = \nabla p. \tag{3}$$

Here G and λ are the Lamé parameters. In order to calculate the analytical or semi−analytical solutions, we need to simplify the model. Therefore, we consider the soil as an isotropic and homogeneous medium. Obviously, this is a quite strong approximation, under which the system (1) and the relative closure relations can be written in spherical coordinates, assuming radial symmetry. As explained in the introduction, also the analytical solution of this simple model can provide an estimate, albeit approximate, of the distribution of particles during injection and contribute to the careful selection of injection parameters, which are necessary for the validation of the numerical models. First of all, we observe that for spherically symmetric deformation, (3) simply reads:

$$\alpha \nabla\, e = \nabla\, p, \tag{4}$$

where $\alpha = (\lambda + 2G)$. Then (4) can be integrated with respect to r in the interval $[0, +\infty)$, obtaining

$$e = \frac{p - p_0}{\alpha}. \tag{5}$$

Here p_0 is the pressure in the matrix soil and $e = 0$ before the injection. In the assumption of spherical symmetry, Darcy's law reads as $\frac{\partial p}{\partial r} = -\frac{\mu}{K} v$, where gravity has been neglected in this approximation. The soil permeability K, [8], can be expressed as $K = K_0 \exp(\gamma e)$, where γ is the constant that represents the sensitivity of permeability to volumetric strain e and K_0 is the permeability of the soil in the equilibrium condition. Then, in view of the previous model, the variables to consider are: $C(r,t)$, $v(r,t)$, $p(r,t)$, and $C_b(r,t)$. In the new coordinate system, the equations read as

$$\begin{cases} \dfrac{\partial C}{\partial t} + v \dfrac{\partial}{\partial r}\left(\dfrac{C}{\varepsilon}\right) = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 D \varepsilon \dfrac{\partial}{\partial r}\left(\dfrac{C}{\varepsilon}\right)\right) - k_f C, \\ \dfrac{\partial}{\partial r}(r^2 v) = 0, \\ \dfrac{\partial p}{\partial r} = -\dfrac{d_0}{K_0 \exp(\gamma e)} v, \\ \dfrac{\partial C_b}{\partial t} = k_f v C, \end{cases} \tag{6}$$

where $r \in R^+$, is the radius of our spherical domain S, d_0 is the NPs diameter and v now denotes the radial component of the Darcy velocity $\mathbf{v}$, which is positive in our setting.

3 Scaling of the Equation and Spherical Symmetrically Analytical Solution

Before calculating the analytical solution, it is necessary to scale the equations, in order to make any unknown dimensionless. In this section we shortly discuss the procedure for equation scaling, needed to understand which terms govern the process. This is a central step in our analysis, as it allows us to solve the problem analytically, simplifying the possible, subsequent, numerical analysis.

During the injection phase, the nanofluid is pumped into the soil, and, in a first approximation, diffusion term can be neglected (advection-dominated regime), because the movement of particles is mainly due to advection. However, at this stage, because of the high-pressure gradients induced by the injection of the nanofluid, the soil can undergo non-negligible deformations and it behaves as a poro-elastic medium. The variation of porosity is assigned according to (2) and (3). After the injection phase, the NPs diffuse into the soil and interact with the soil grains. In reality, the interaction process is very complex and difficult to model due to the heterogeneity of the substrate in which the particles move and the variability of the conditions caused, for example, by the presence of aquifers. In our paper two different regimes are analyzed and discussed separately: the injection phase and the subsequent phase, when the injection stops. We just point out that, in the present analysis, a homogeneous matrix of quartz sand was chosen as porous medium model, in order to fix realistic parameters. This choice does not represent the composition of a real soil and, for sure, it will influence the dynamics of the problem. In a later work we will discuss a more general model where different materials and a non-homogeneous soil will be considered in a more realistic 3D digital twin of the computational model. To obtain the scaled governing equations we need to introduce the relative reference quantities. In more detail we consider t_s as the characteristic injection duration, v_s as the characteristic fluid speed, C_s is the reference NPs concentrations and p_s is the reference pressure. Then we introduce the following scaled variables, normalizing each quantity with respect to its reference values:

$$\bar{t} = \frac{t}{t_s}, \quad \bar{r} = \frac{r}{t_s v_s}, \quad \bar{v} = \frac{v}{v_s}, \quad \bar{p} = \frac{p - p_0}{p_s}, \quad \bar{C} = \frac{C}{C_s}, \quad \bar{C}_b = \frac{C_b}{C_s}.$$

The value C_s is the reference NPs concentration corresponding to a reference value of the pressure p_s. The bar above the letters indicates the scaled variables.

In the scaled variables system (6) reads

$$\begin{cases} \dfrac{\partial \bar{C}}{\partial \bar{t}} + \bar{v}\dfrac{\partial}{\partial \bar{r}}\left(\dfrac{\bar{C}}{\varepsilon}\right) = \dfrac{1}{\bar{r}^2}\dfrac{\partial}{\partial \bar{r}}\left(\bar{r}^2 D^* \varepsilon \dfrac{\partial}{\partial \bar{r}}\left(\dfrac{\bar{C}}{\varepsilon}\right)\right) - k_f^* \bar{C}, \\ \dfrac{\partial}{\partial \bar{r}}(\bar{r}^2 \bar{v}) = 0, \\ \dfrac{\partial \bar{p}}{\partial \bar{r}} = -\dfrac{d^*}{\exp(\gamma e)}\bar{v}, \\ \dfrac{\partial \bar{C}_b}{\partial \bar{t}} = k_f^* \bar{v} \bar{C}, \end{cases} \tag{7}$$

where $\bar{r} \in R^+$, $D^* = \frac{D}{t_s v_s^2}$, $k_f^* = \frac{(3\,\eta\, t_s\, v_s)(1-\varepsilon)}{4\,\varepsilon\, r_c}$, $d^* = \frac{\eta\, t_s v_s^2}{p_s K_0}$, $\varepsilon = \frac{\varepsilon_0 + \frac{p-p_0}{\alpha}}{1+\frac{p-p_0}{\alpha}}$. Note that from the scaling we can write

$$e = \frac{\bar{p}\, p_s}{\alpha} \quad \text{andthen, from (2), wehave} \quad \varepsilon = \frac{\varepsilon_0 + \frac{\bar{p} p_s}{\alpha}}{1 + \frac{\bar{p} p_s}{\alpha}}. \tag{8}$$

In the previous formulas r_c denotes the typical soil grain radius and the collector efficiency η, which typically refers to the fraction of nanoparticles that are effectively captured or deposited on the sand grain surfaces during the flow, is assumed constant. C_b, the bounded NPs concentration, is assumed to be negligible at the beginning of the injection process. We just point out that we have already exploited the incompressibility of the Darcy velocity field.

4 Injection Phase

In this subsection our aim is to model the injection phase, in which the diffusion phenomenon plays a limited role, therefore we will remove the diffusion term from system (7) and we obtain

$$\begin{cases} \dfrac{\partial \bar{C}}{\partial \bar{t}} + \bar{v}\dfrac{\partial}{\partial \bar{r}}\left(\dfrac{\bar{C}}{\varepsilon}\right) = -k_f^* \bar{C}, \\ \dfrac{\partial}{\partial \bar{r}}(\bar{r}^2 \bar{v}) = 0, \\ \dfrac{\partial \bar{p}}{\partial \bar{r}} = -\dfrac{d^*}{\exp(\gamma e)}\bar{v}, \\ \dfrac{\partial \bar{C}_b}{\partial \bar{t}} = k_f^* \bar{v} \bar{C}. \end{cases} \tag{9}$$

To this system we need to add the following suitable initial boundary conditions

$$\begin{cases} \bar{C}(\bar{r}, 0) = C_0(\bar{r}), \\ \bar{C}(r_i, t) = C_i \\ \bar{p}(r_i, t) = p_i \\ 4\pi\, r_i^2\, v(r_i, t) = Q_i \\ \bar{C}_b(\bar{r}, 0) = 0, \end{cases} \tag{10}$$

where $\bar{r} \in R^+$, Q_i is the infusion flow rate, C_i is the initial nanofluid concentration in the injection point, $r_i = r(0)$, is the radius of the injection area, p_i is the pressure in the injection point and p_0 is the value of the soil pressure before the injection ($p_i \geq p_0$). Here p_i, C_i, and r_i can be considered as design parameters. Within the constraints of current technological limits, these variables can be tuned to optimize the injection process and enhance the overall delivery efficiency. We note also that we assume $C_0(\bar{r})$ small but different from zero since we mean to apply the characteristics method to try to find an analytical solution to $(9)_1 - (10)_{1,2}$. Indeed, as a first order hyperbolic PDE, the transport equation $(9)_1$ can be solved with this method which, instead of looking at the entire surface $C(r,t)$, reconstruct the solution along the trajectories of the single particle of the nanofluid, called characteristic curves. Taking the total derivative of C along the characteristics the partial differential equation becomes an ordinary differential equation. In the sequel, to simplify the notation, we will drop the bar above the scaled variables. Note that from (2) and (5) we have that $\varepsilon = \varepsilon(e(p))$ and $K = K(e(p))$. We start from $(9)_2$and $(10)_4$ and we get v is a function of r only as $v(r) = \frac{Q_i}{4\pi\, r^2}$. While, recalling (8), we can write $(9)_3$, as $\frac{\partial p}{\partial r} = -\frac{d^* Q_i}{4\pi\, r^2} \exp\left(-\frac{\gamma p_s}{\alpha p}\right)$. By separation of variables, taking into account the boundary condition $p(r_i, t) = p_i$, we obtain the explicit solution

$$p(r) = \frac{\alpha}{\gamma p_s} \ln\left[\exp\left(\frac{\gamma p_s p_i}{\alpha}\right) + \frac{\gamma p_s d^* Q_i}{4\pi\alpha}\left(\frac{1}{r} - \frac{1}{r_i}\right)\right]. \tag{11}$$

We point out that the pressure is also independent of time. Concerning the transport equation for the nanofluid concentration, $(9)_1$ with $(10)_1$, we will distinguish three different situations depending on the generality with which the porosity ε is chosen, as discussed in the following subsections.

4.1 General Porosity: $\varepsilon(r)$

According to the expression of the quantities (2), (5) and (11), we have that the general formulation for the porosity is just a function of r. Therefore, equation $(9)_1$ can be written as $\frac{\partial C(r,t)}{\partial t} + \frac{Q_i}{4\pi\, r^2}\frac{\partial}{\partial r}\left(\frac{C(r,t)}{\varepsilon(r)}\right) = -k_f^*(\varepsilon(r))C(r,t)$ and then $\frac{\partial C(r,t)}{\partial t} + \frac{Q_i}{4\pi\, r^2\, \varepsilon(r)}\frac{\partial C(r,t)}{\partial r} = \left[-k_f^*(\varepsilon(r) + \frac{Q_i}{4\pi\, r^2}\frac{\varepsilon'(r)}{\varepsilon^2(r)}\right] C(r,t)$. Applying the characteristic method we have that characteristic curves equations are

$$(\dot{r}(s) = \frac{Q_i}{4\pi r^2 \varepsilon(r)}, \dot{t}(s) = 1) \quad \text{from which we have } \int_{r(0)}^{r(t)} 4\pi\, \rho^2\, \varepsilon(\rho) d\rho = Q_i\, t.$$

Let $V(x)$ be a differentiable, strictly increasing function such that $V'(x) = 4\pi\, x^2\, \varepsilon(x)$, then from $V(r(t)) - V(r(0)) = Q_i\, t$ it follows $r(t) = V^{-1}\left[V(r_i) + Q_i t\right]$ where $r_i = r(0)$. Along the characteristic curves $r(t)$, the material derivative of the concentration reveals an additional term proportional to the porosity gradient $\varepsilon'(r)$. This term accounts for the local compression or expansion of

the fluid volume as it moves through a medium of varying porosity, effectively modulating the filtration rate k_f^* as $\frac{dC(r(t),t)}{dt} = \left[-k_f^*(\varepsilon(r) + \frac{Q_i}{4\pi\, r^2}\frac{\varepsilon'(r)}{\varepsilon^2(r)} \right] C(r,t)$,

from which

$$C(r,t) = C_0(r(0)) \exp\Big[\int_0^t \left(-k_f^*(\varepsilon(r)) + \frac{Q_i}{4\pi\, r^2}\frac{\varepsilon'(r)}{\varepsilon^2(r)} \right) ds \Big], \tag{12}$$

that is

$$C(r,t) = C_0(V^{-1}\big[V(r(t)) - Q_i t\big]) \exp\Big[\int_0^t \left(-k_f^*(\varepsilon(r) + \frac{Q_i}{4\pi\, r^2}\frac{\varepsilon'(r)}{\varepsilon^2(r)} \right) ds \Big].$$

From $(9)_4$ and $(10)_5$ we get $C_b(r,t) = \frac{Q_i}{4\pi\, r^2} \int_0^t k_f^*(\varepsilon(r,s))\, C(r,s) ds$.

4.2 Linearized Porosity

In this subsection, with the aim of finding a closed form for the concentration of the nanofluid, let's assume that the value of the volumetric dilation e is small. This hypothesis allows to approximate (2) as

$$\varepsilon \approx \varepsilon_0 + (1-\varepsilon_0)e = \varepsilon_0 + \frac{1-\varepsilon_0}{\alpha}(\bar{p} - p_0) = \varepsilon_0 + \frac{1-\varepsilon_0}{\alpha}(p\, p_s).$$

Moreover, we simplify also the pressure equation in $(9)_3$ assuming $\gamma \approx 0$ into the permeability expression. Then, instead of $(9)_3$, we have $\dfrac{\partial p}{\partial r} = -\dfrac{d^* Q_i}{4\pi r^2}$ with $p(r_i,t) = p_i$ and the corresponding solution is $p(r) = p_i + \frac{d^* Q_i}{4\pi}\left(\frac{1}{r} - \frac{1}{r_i}\right)$.

To solve the equation for concentration

$$\frac{\partial C(r,t)}{\partial t} + \frac{Q_i}{4\pi\, r^2}\frac{\partial}{\partial r}\Big(\frac{C(r,t)}{\varepsilon(r)}\Big) = -k_f^*(\varepsilon(r)) C(r,t), \tag{13}$$

we set $U(r,t) = \frac{C(r,t)}{\varepsilon(r)}$, obtaining $\varepsilon(r)\frac{\partial U}{\partial t} + \frac{Q_i}{4\pi r^2}\frac{\partial U}{\partial r} = -k_f^* \varepsilon(r) U$. Then, we divide by $\varepsilon(r)$ and apply the characteristic method to

$$\frac{\partial U}{\partial t} + \frac{Q_i}{4\pi r^2 \varepsilon(r)}\frac{\partial U}{\partial r} = -k_f^* U.$$

Working similarly as in the previous subsection we have as solution

$$C(r,t) = \frac{\varepsilon(r)}{\varepsilon(r_i)} C_0 \left(t - \int_{r_i}^r \frac{4\pi\rho^2 \varepsilon(\rho)}{Q_i} d\rho \right) \exp\left(-\frac{3\eta t_s v_s}{4\, r_c} \int_{r_i}^r \frac{(1-\varepsilon(\rho))}{\varepsilon(\rho))} d\rho \right)$$

but this time we can compute explicitly the integrals that appear. Indeed, substituting the simplified value of the pressure $p(r) = p_i + \frac{d^* Q_i}{4\pi}\left(\frac{1}{r} - \frac{1}{r_i}\right)$ in

$\varepsilon(r) = \varepsilon_0 + \frac{1-\varepsilon_0}{\alpha}(p(r)\,p_s)$, we have $\varepsilon(r) = \varepsilon_0 + \frac{(1-\varepsilon_0)p_s}{\alpha}\left(p_i + \frac{d^*Q_i}{4\pi}\left(\frac{1}{r} - \frac{1}{r_i}\right)\right)$. We first consider

$$\int_{r_i}^{r} \frac{4\pi\rho^2\varepsilon(\rho)}{Q_i}d\rho = \frac{4\pi}{Q_i}\int_{r_i}^{r} \rho^2\left[\varepsilon_0 + \frac{(1-\varepsilon_0)p_s}{\alpha}\left(p_i + \frac{d^*Q_i}{4\pi}\left(\frac{1}{\rho} - \frac{1}{r_i}\right)\right)\right]d\rho.$$

The result is a combination of powers of order three and two of r and r_i. Concerning the second integral, we have

$$\int_{r_i}^{r} \frac{1-\varepsilon(\rho)}{\varepsilon(\rho)}d\rho = \left(\frac{1-A}{A}\right)(r - r_i) - \frac{B}{A^2}\ln\left|\frac{Ar+B}{Ar_i+B}\right|$$

where $A = \varepsilon_0 + \frac{(1-\varepsilon_0)p_s}{\alpha}\left(p_i - \frac{d^*Q_i}{4\pi r_i}\right)$ and $B = \frac{(1-\varepsilon_0)p_s d^*Q_i}{4\pi\alpha}$. In this framework, the variable porosity $\varepsilon(r)$ acts as a structural constraint that dictates both the local front velocity and the particle capture efficiency, thereby coupling nanofluid transport directly to the mechanical deformation of the medium.

4.3 Constant Porosity

In this subsection we consider the behavior of the concentration of the nanofluid when the volumetric dilation can be assumed negligible, then we can set a constant porosity that equals the original one before the injection, that is ε_0. Then equation $(9)_1$ becomes $\frac{\partial C}{\partial t} + \frac{v}{\varepsilon_0}\frac{\partial C}{\partial r} = -k_f^* C$ and recalling $v(r) = \frac{Q_i}{4\pi r^2}$ and $k_f^* = \dfrac{(3\,\eta\,t_s\,v_s)(1-\varepsilon_0)}{4\,\varepsilon_0\,r_c}$, we have

$$\frac{\partial C}{\partial t} + \frac{Q_i}{4\pi\,r^2\,\varepsilon_0}\frac{\partial C}{\partial r} = -\frac{(3\,\eta\,t_s\,v_s)(1-\varepsilon_0)}{4\,\varepsilon_0\,r_c}\,C. \tag{14}$$

Applying the characteristic method we have that the characteristic curves equations are

$$\begin{cases} \dot{r}(s) = \frac{Q_i}{4\pi\,r^2\,\varepsilon_0} \\ \dot{t}(s) = 1, \end{cases}$$

from which we have $r(t) = \sqrt[3]{r_i^3 + \frac{3Q_i\,t}{4\pi\,\varepsilon_0}}$ and so the trajectory that passes through $(r(t), t)$ is related to the initial position by the relation $r(0) = r_i = \sqrt[3]{r(t)^3 - \frac{3Q_i\,t}{4\pi\,\varepsilon_0}}$. From (14) one has $\frac{dC}{dt} = -\dfrac{(3\,\eta\,t_s\,v_s)(1-\varepsilon_0)}{4\,\varepsilon_0\,r_c}\,C$ and then

$$C(r,t) = C_0(r_i)\exp\left(-\int_0^t \frac{(3\,\eta\,t_s\,v_s)(1-\varepsilon_0)}{4\,\varepsilon_0\,r_c}\,ds\right),$$

that is $C(r,t) = C_0\left(\sqrt[3]{r(t)^3 - \frac{3Q_i\,t}{4\pi\,\varepsilon_0}}\right)\exp\left(-\dfrac{(3\,\eta\,t_s\,v_s)(1-\varepsilon_0)}{4\,\varepsilon_0\,r_c}\,t\right)$. Clearly, once $C(r,t)$ is determined, the concentration $C_b(r,t)$ of the bounded nanoparticles can be easily obtained from $C_b(r,t) = \frac{Q_i}{4\pi\,r^2}\int_0^t k_f^*(\varepsilon(r,s))\,C(r,s)ds$. In this

simplified scenario, the expression for $C(r,t)$ represents an attenuated traveling wave propagating through the porous medium. The functional form $C_0(r_i(r,t))$ defines the wave shape, which is determined by the initial injection profile and translated radially according to the cubic trajectory $r(t)$. As the wave front advances, its amplitude undergoes a homogeneous exponential decay due to the filtration term. Unlike the variable porosity case, where the wave would also experience structural distortion (due to the $\varepsilon'(r)$ gradient term), here the wave maintains its relative profile, being scaled only by the constant filtration coefficient. This allows for a clear decoupling between the kinematics of the front (the cubic root dependence) and the dynamics of particle deposition (the exponential attenuation).

5 Conclusion

In this study, we developed a hierarchical analytical framework to investigate nanofluid transport in deformable porous media, providing a fundamental benchmark for the validation of future 2D and 3D numerical simulations within high-performance computing (HPC) environments. By comparing three distinct configurations, we have highlighted how the structural representation of the medium dictates the predicted efficiency of the injection process. The transition between the three considered configurations (fixed, linearized, and general porosity) reveals an increasing degree of physical coupling, which nevertheless corresponds to a rise in the analytical complexity of the proposed solutions, as summarized below:

- **Fixed Porosity Model (ε_0)**: Acting as a "zero-order" approximation, this case describes the transport as a standard attenuated traveling wave. While ideal for a rapid, order-of-magnitude estimation of the design parameters, it fails to capture the spatial redistribution of particles caused by the medium's heterogeneity.
- **Linearized Porosity Model**: This intermediate stage illustrates the fundamental interaction between the injection pressure p_i and the response of the porous medium. In this case, the porosity acts as a structural constraint that deforms the advancing concentration front. The width of the front no longer depends solely on the initial concentration but is modulated by the ratio between the porosity value at r and that at the injection point. This demonstrates that the 'mechanical control' of the injection can be strategically utilized to promote deeper particle penetration by locally modifying the pore space.
- **General Variable-Porosity Model ($\varepsilon(r)$)**: This formulation provides the most rigorous analytical reference. By accounting for the porosity gradient, it captures the local acceleration and deceleration of the fluid, offering a precise map of the deposition patterns. This model serves as the ultimate "gold standard" for calibrating complex numerical solvers.

Although preliminary, this work provides a strategic road map for future research. The exact solutions derived here allow for the calibration of numerical solvers before they are deployed to simulate large-scale, non-linear deformation regimes and asymmetric geometries. Moving toward 2D and 3D models will enable the exploration of medium anisotropy, using these analytical benchmarks to ensure physical consistency and mathematical convergence.

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