On the Travelling Wave Solutions of a Non-linear Reaction-Diffusion-Advection Model for Forest Fire Spread

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Abstract. This preliminary work investigates the dynamics of forest fire propagation using an Advection-Diffusion-Reaction (ADR) model with a non-linear diffusion term. A central focus of this work is the study of specific conditions under which travelling wave solutions occur within the ADR considered model, extending existing literature to a more general, one-dimensional (1D), non-linear diffusion case. The model approximation is validated through numerical simulations, demonstrating its effectiveness in capturing the front dynamics of wildfire propagation. The results confirm the model's reliability in identifying the critical wave speed of propagation, for the studied cases, offering a potential valuable tool for risk mitigation and strategic territorial planning.

Keywords: Forest fire model · advection-reaction-diffusion model · travelling waves

1 Introduction

Forest fires pose a significant threat to vulnerable areas due to the extensive damage they cause to both socio-economic activities and natural ecosystems. Consequently, it is essential to develop robust mathematical models and, subsequently, simulation software useful for estimating the dynamics of forest fires, possibly in real time. These tools are crucial for enhancing damage containment during firefighting operations and for the strategic planning e.g. of industrial sites near forested areas. Fast and reliable computational techniques are indispensable for effective risk mitigation. Therefore, the computational cost required for the simulation itself is crucial. Among the various models available in the literature, this work considers a model based on advection-diffusion-reaction (ADR)

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O. Gervasi et al. (Eds.): ICCSA 2026 Workshops, LNCS 16766, pp. 247–258, 2027.
https://doi.org/10.1007/978-3-032-30539-8_18

equations, originally presented in [1] and recently investigated in [9,10]. In addition to computational techniques, it can be highly useful to study the triggering conditions of travelling wave solutions and to derive analytical estimates for the travelling wave speed. In this paper, we will derive an estimate of the travelling wave speed based on a general solution for ADR equations. The estimate will hold under the assumption of a constant biomass and relates to other estimates of the travelling wave speed under specific conditions, like in [1,5,8,10]. The set of time-evolution parabolic equations of reaction-diffusion type includes advection terms and non-linear diffusion. We compare the analytical predicted minimal travelling wave speed with a numerical solution computed with a spatial centered finite differences method and time integration through an explicit Euler scheme. The use of explicit methods and finite differences is limited by stability conditions on the time step. However, given the preliminary nature of the work and the analytical study carried out in 1D, no particular problems are encountered in this case from a computational point of view. The outline of the work is the following: Sect. 2 illustrates the model, Sect. 3 analyses the existence of travelling wave solutions and estimates the travelling wave speed. Section 4 reports numerical tests to validate the analytical results obtained in Sect. 3, and finally, Sect. 5 reflects the results and conclusions of the work.

2 Mathematical Model

This section provides a brief overview of mathematical models for forest fire dynamics. An extended and detailed taxonomy of the various approaches and models is provided in [2,3]. It is possible to group mathematical forest fire models into three categories such as physics-based models, data-driven models and mechanistic surrogate models. The data-driven models use empirical data to "identify" the fire dynamics in particular cases (sometimes using AI-based tools), whereas mechanistic surrogate models operate independently of the actual physical mechanics of the phenomenon. Finally, the theoretical models are based on the fundamental physical laws that describe the underlying processes. In this work, we utilize a physics-based theoretical model of advection-diffusion-reaction (ADR) type. The model describes the temperature evolution in a fuel-air-mixture during the fire spread in relation to the combustion of a biomass (fuel) fraction. The temperature is defined as a function $T(x,t)$, and the biomass fraction is described by the function $M(x,t)$. Both functions are defined over a one-dimensional domain $L \subset \mathbb{R}$, where the variable $x \in L$ represents a generic point within the domain and $t > 0$ denotes time. The formal derivation of the mathematical model is based on an energy balance, see for example [6,10]. The considered system of equations is

$$\begin{cases} \rho_0 C\left(\frac{\partial T}{\partial t} + \mathbf{c} \cdot \nabla T\right) & = \nabla \cdot (k_t(T) \cdot \nabla T) - h(T - T_\infty) + \psi(T)\rho_0 H M \\ \frac{\partial M}{\partial t} & = -\psi(T) M \end{cases} \tag{1}$$

where

$$\psi(T) = s(T)A\exp\left(-\frac{T_{\text{ac}}}{T}\right) \text{ with } s(T) = \begin{cases} 0, & \text{if } T < T_{\text{pc}} \\ 1, & \text{if } T \geq T_{\text{pc}} \end{cases}$$

is the combustion function, and $k_t(T) = 4\sigma\epsilon\delta T^3 + k$ gives a nonlinear diffusion term. The first equation of the system (1) has an advection term, a non linear diffusion term $k_t(T)$ where k is the thermal diffusivity, ϵ is an emissivity factor, δ is the optical path length for radiation, considered smaller than the characteristic length of the considered computational domain, and σ is Stefan-Boltzmann constant. The term $h(T - T_\infty)$ models Newton's cooling law related to convection and the environment temperature T_∞, and $\psi(T)\rho_0 HM$ models the combustion process in interaction with the biomass M, described by the second equation in (1). C is the specific heat and ρ_0 is the density of the air-fuel mixture described by the density of the fuel mass fraction. We define $T_{\text{ac}} = \frac{E_A}{R}$ with the activation energy E_A and the universal gas constant R, and T_{pc} is the ignition temperature. We examine two formulations for the advection term: the first incorporates a spatially constant velocity $v(t)$ to represent wind effects, while the second employs $c(x,t) = v(t) + \nabla z(x)$. From a physical standpoint, this latter configuration accounts for the influence of terrain topography on fire spread by integrating the elevation gradient. The topography gradient approach has been studied as well e.g. in [7].

3 Travelling Wave Solutions

In this section, we briefly recall the basic concepts related to travelling waves solution (TWS) for ADR equations. We do not claim to exhaustively explain all the theory relating to this type of solution, so please refer to [4,5,8,10] for further details. The basic idea for TWS for an ADR equation is to find a solution $u(x,t) = u(x + c_s t)$ where c_s is the velocity of the travelling wave and is, usually, unknown. Travelling wave solutions for the advection-reaction diffusion system (1) have been investigated with a linear diffusion term $k_t(T) = k$ and without Newton's cooling term in [8]. A linear approximations of the travelling wave speed, under the assumption of constant biomass and linear diffusion, was done in [10]. In [5], the nonlinear system in a persisting fire case has been transformed to an ODE system, which is a promising direction for proving the existence of TWS in this case. One target of this work is to define an estimate of the minimal velocity c^* for which the forest fire model (1) admits TWS. We will study the existence of TWS under the assumptions that:

1. we are in a setting with one spatial dimension;
2. the biomass can be considered constant in the time window relevant for the propagation of the wave front;
3. the fire is already ignited, by focusing the analysis on the characterization of the early-stage fire front dynamics.

Considering a one-dimensional model, we can rewrite equation (1) as

$$\begin{cases} \rho_0 C\Big(\frac{\partial T}{\partial t} + (v(t) + \frac{\partial Z}{\partial x})\frac{\partial T}{\partial x}\Big) & = \frac{\partial}{\partial x}(k_t(T)\frac{\partial T}{\partial x}) - h(T - T_\infty) + \psi(T)\rho_0 HM, \\ \frac{\partial M}{\partial t} & = -\psi(T)M. \end{cases} \tag{2}$$

Given the preliminary nature of this work, we will treat M as constant in order to study the behaviour of the ADR solutions over a short time scales, with the aim of understanding whether and how a travelling wave might be generated. As we assume M to be constant in time, we consider only the first equation in (2), which is a particular case of the more general equation

$$\left(\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x}\right) = \frac{\partial}{\partial x}\left(K(u)\frac{\partial u}{\partial x}\right) + R(u)\,, \tag{3}$$

where the function $u(x,t)$ will be a general function and in the same way we can define the functions: $F(u)$ as a general non-linear advection term, $K(u)$ as a general non-linear diffusion term and finally $R(u)$ as a general reaction term depending on the function $u(x,t)$. For Eq. (3) the existence of TWS has been investigated in [4]. For this class of equations, the following theorem holds:

Theorem 1 (Theorem 1.4 in [4]). *Assume that the functions $f = \frac{dF(u)}{du}$, $K(u)$ are continuous in the interval $[0,1]$, with $\min K(u) > 0$. If the Dini derivative $D^+(KR)(0) < +\infty$, then there exists a constant velocity c^* with*

$$2\sqrt{D_+(KR)(0)} - f(0) \le c^* \le 2\sqrt{\sup_{s\in]0,1]} \frac{K(s)R(s)}{s}} - \min_{s\in[0,1]} f(s) \tag{4}$$

such that Eq. (3) admits non-decreasing solutions if and only if $c > c^*$. Moreover, the solution is unique (up to translation of the origin). Whereas, if the Dini derivative $D_+(KR)(0) = +\infty$, Eq. (3) does not admit monotone solutions for any wave speed c.

The definition of Dini's derivatives is given for ease of reading and discussion:

Definition 1. *For the general continuous function $u(\cdot,t) : \mathbb{R} \to \mathbb{R}$ Dini's derivatives are*

$$D_+u(x,t) = \liminf_{h\to 0^+} \frac{u(x+h,t) - u(x,t)}{h},$$
$$D^+u(x,t) = \limsup_{h\to 0^+} \frac{u(x+h,t) - u(x,t)}{h}.$$

Non-dimensional Equation. According to Theorem 1, we need to analyze and rewrite our model in a non-dimensional way. As reported in [1], we use the transformations $\tilde{x} = \frac{x}{\rho_0}$, $\eta = \frac{M}{\rho_0}$, $\tau = \frac{t}{t_0}$, and $u = \frac{T - T_\infty}{\epsilon T_\infty}$, $w = \frac{t_0}{l_0}v$, where we also

have $\epsilon = \frac{RT_\infty}{E_A}$, $q = \frac{HM_0}{CT_\infty}$, $t_0 = \frac{e^{\frac{1}{\epsilon}}\epsilon}{qA}$, and finally $l_0 = \sqrt{\frac{t_0 k}{\rho_0 C}}$. We will drop the tilde for $\tilde{x}$ and write x instead. The non-dimensional equation then is

$$\frac{\partial u}{\partial t} + \left(w + \frac{\partial Z}{\partial x}\right)\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(K(u)\frac{\partial u}{\partial x}\right) + g(u,\beta)\,, \tag{5}$$

with $K(u) = \tilde{k}(1+\epsilon u)^3 + 1$, and $g(u,\beta) = s(u)^+ \beta e^{\frac{u}{1+\epsilon u}} - \alpha u$ where $\tilde{k} = \frac{4\sigma\delta T_\infty^3}{k}$ and $\alpha = \frac{t_0 h}{\rho_0 C}$. The function $s(u)^+$ is the part of the step function, where $s(u) = 1$, so for $u > 0$.

Case 1: Advection by Wind, Negligible Topographic Effects. In this section, we study the case where the advection term $F(u)$ is defined as $F(u) = wu$ with $w = w(t)$. Therefore, $\frac{\partial F}{\partial x} = \frac{\partial w}{\partial x}u + w\frac{\partial u}{\partial x} = w\frac{\partial u}{\partial x}$. We investigate the conditions required in Theorem 1 to have a non-decreasing solution of (5). The first condition is the continuity of the function $K(u)$ that is satisfied according to the definition. The second condition to verify is $\min_{u\in[0,1]} K(u) > 0$. According to the definition of the function $K(u)$ in Eq. (3), we have $\min_{u\in[0,1]} K(u) = \tilde{k} + 1 > 0$, because $u > 0$ in physical meaningful cases relevant for fire events where $T > T_\infty$. So, $\min K(u) > 0$ in the interval $[0,1]$. The second condition that we have to verify is related to the Dini's derivative $D^+(K(u)g(u,\beta))$, where we have

$$K(u)g(u,\beta) = \left(\tilde{k}(1+\epsilon u)^3 + 1\right)\left(s(u)^+ \beta e^{\frac{u}{1+\epsilon u}} - \alpha u\right).$$

According to the hypothesis to study an already ignited fire event, we can consider the term $s(u)^+ = 1$ and rewrite the previous equation, considering also the continuity of the reaction term:

$$K(u)g(u,\beta) = (\tilde{k}(1+\epsilon u)^3 + 1)(\beta e^{\frac{u}{1+\epsilon u}} - \alpha u).$$

We assume positivity of all parameters $\tilde{k}, \epsilon, \beta, \alpha$. According to Theorem 1 we have to check the condition $D^+ Kg(0) < +\infty$. For the sake of simplicity, we define the function $m(u) = K(u)g(u,\beta)$, so

$$m(u) = \left(\tilde{k}(1+\epsilon u)^3 + 1\right)\left(\beta e^{\frac{u}{1+\epsilon u}} - \alpha u\right).$$

We have to compute the Dini derivative $D^+ m(0)$ for $u = 0$ which is $D^+ m(u_0) = \lim_{h\to 0^+} \sup_{u\in[u_0,u_0+h]} \frac{(m(u)-m(u_0))}{h}$, we can compute $m(0) = (\tilde{k}+1)\cdot\beta$. The second step is to compute the behavior of the function $m(u)$ when $u \to 0^+$. We analyze the term $\tilde{k}(1+\epsilon\cdot u)^3 + 1$ for $u \to 0$, where for $(1+\epsilon\cdot u)^3$ using the Taylor formula we get $(1+\epsilon u)^3 = 1 + 3\epsilon u + \mathcal{O}(u^2)$. So we can rewrite the previous term as

$$\tilde{k}(1+\epsilon u)^3 + 1 = \tilde{k}(1 + 3\epsilon u + \mathcal{O}(u^2)) + 1. \tag{6}$$

Following, we can analyze the reaction term $\beta e^{\frac{u}{1+\epsilon u}} - \alpha u$. Using again the Taylor formula for the exponential term $e^{\frac{u}{1+\epsilon u}}$ per $u \to 0$, we get $\frac{u}{1+\epsilon u} =$

$u\left(1-\epsilon u+\mathcal{O}(u^2)\right)$. So, we can write

$$\beta e^{\frac{u}{1+\epsilon u}}-\alpha u=\beta+(\beta-\alpha)u+\beta(\frac{1}{2}-\epsilon)u^2+\mathcal{O}(u^3). \tag{7}$$

Combining the results in (6) and (7), we get

$$m(u)\approx(\tilde{k}+1+3\tilde{k}\epsilon u+\mathcal{O}(u^2))\cdot(\beta+(\beta-\alpha)u+\beta(\frac{1}{2}-\epsilon)u^2+\mathcal{O}(u^3))$$

and therefore $m(u)\approx(\tilde{k}+1)\beta+\left[(\tilde{k}+1)(\beta-\alpha)+3\tilde{k}\epsilon\beta\right]u+\mathcal{O}(u^2)$.

We can now compute the required Dini's derivative:

$$D^+m(0)=\left[(\tilde{k}+1)(\beta-\alpha)+3\tilde{k}\epsilon\beta\right],$$

and due to the positivity of the parameters $\tilde{k}$, ϵ, β and α, we have $D^+m(0)<\infty$. We now consider equation (4) to compute the minimal velocity c^*. First, we have to evaluate the term $\sqrt{\sup_{s\in(0,1]}\frac{K(s)g(s)}{s}}$. Therefore, we can define the function $l(s)=\frac{m(s)}{s}$ with $s\in(0,1]$, and $\tilde{k},\epsilon,\beta,\alpha>0$.

We can now compute $u\to 0^+$ using the Taylor formula near $u=0$ for $m(u)$ in Eq. (3), giving $m(u)\approx(\tilde{k}+1)\beta+\left[(\tilde{k}+1)(\beta-\alpha)+3\tilde{k}\epsilon\beta\right]u+\mathcal{O}(u^2)$ and then

$$l(s)=\frac{m(s)}{s}\approx\frac{(\tilde{k}+1)\beta}{s}+\left[(\tilde{k}+1)(\beta-\alpha)+3\tilde{k}\epsilon\beta\right]+\mathcal{O}(s).$$

This gives $\lim_{s\to 0^+}l(s)=+\infty$. We now compute $l(s)$ in $s=1$ and so we can write:

$$l(1)=m(1)=\left(\tilde{k}(1+\epsilon)^3+1\right)\left(\beta e^{\frac{1}{1+\epsilon}}-\alpha\right).$$

This value is a finite value depending on the parameters $\tilde{k},\epsilon,\beta,\alpha$. We can write now that $l(s)\to+\infty$ for $s\to 0^+$ and $l(1)$ is a finite value, so the supremum is $\sup_{u\in(0,1]}l(u)=+\infty$. To compute the c^* values, we have to consider again the function $m(u)$ and compute for the function the Dini derivative

$$D_+m(0)=\liminf_{h\to 0^+}\frac{m(h)-m(0)}{h},$$

where $m(0)=(\tilde{k}+1)\beta$. Considering now $u\to 0^+$, we get: $D_+m(0)=(\tilde{k}+1)(\beta-\alpha)+3\tilde{k}\epsilon\beta$. Recalling that $f=\frac{dF}{du}=w$ considering now w constant and so $f(0)=w$ and $\min_{s\in[0,1]}f(s)=w$. So finally we can write with Eq. (4)

$$c^*\geq 2\sqrt{(\tilde{k}+1)(\beta-\alpha)+3\tilde{k}\epsilon\beta}-w. \tag{8}$$

Obviously, to have the physical and mathematical meaning, it is required to have $(\beta-\alpha)\geq 0$ and that $2\sqrt{(\tilde{k}+1)(\beta-\alpha)+3\tilde{k}\epsilon\beta}\geq w$.

Case 2: Advection by Wind and Topography. In this section, we consider the case where $F(u) = \left(w + \frac{\partial Z}{\partial x}\right) \frac{\partial u}{\partial x}$ with a linear function $Z(x) = ax$ defined by $\frac{dZ(x)}{dx} = a$ with $a \in \mathbb{R}$ constant. Depending on the sign of the parameter $a \geq 0$ (or $a \leq 0$), the function Z represents an up-hill ramp or a down-hill ramp. We study these two cases in a similar way that we followed for the no topography case, indeed the advection term will still be linear with a modified function f. For applying Theorem 1, we have to check that $\min K(u) > 0$ in the interval $[0, 1]$. This condition will be satisfied according to the definition of the function $K(u)$ and the hypothesis of the problem. Indeed, we suppose to work with fire events already ignited and this hypothesis leads to having a temperature $T(x) > T_\infty$ and thus a function $u(x) > 0$ in the considered interval. Again, all requirements of Theorem 1 are fulfilled, and therefore we get estimated for the minimal spreading speed depending on the sign of a. For the up-hill case $a > 0$ we have $c^* \geq 2\sqrt{(\tilde{k}+1)(\beta-\alpha) + 3\tilde{k}\epsilon\beta} - (w + a)$, and for the down-hill case, we find $c^* \geq 2\sqrt{(\tilde{k}+1)(\beta-\alpha) + 3\tilde{k}\epsilon\beta} - (w - a)$.

Table 1. Simulation parameters with applicable range values and measure units for the considered ADR model as reported in [10]. In the last column have been reported the values used for the numerical simulations.

Parameter	Symbol	Values [10]	Measure Units	Simulation Values
Density	ρ_0	40–640	$kg \cdot m^{-1}$	40.0
Specific heat	C	1.0–3.0	$kJ \cdot kg^{-1} \cdot K^{-1}$	1.0
Pre-exponential factor	A	0.05	s^{-1}	0.05
Heating value	H	10–10000	$kJ \cdot kg^{-1}$	10000
Convection coefficient	h	0.1–10	$kW \cdot m^{-3} \cdot K^{-1}$	5
Ambient air temperature	T_∞	280–320	K	300
Ingnition temperature	T_{pc}	400–500	K	400
Activation temperature	T_{ac}	400–500	K	400
Thermal diffusivity	k	0–4	$kW \cdot m^{-1} \cdot K^{-1}$	4

4 Numerical Method, Simulations and Results

In this section, we introduce a discretization of the studied ADR model, for a dimensional case in 1D, using a finite difference method (FDM) scheme. The problem is solved using a FDM on a uniform spatial grid with step width Δx and an explicit time integration (Forward Euler) with step size Δt. In the numerical simulations we considered a general constant transport term named as v with a first-order upwind scheme: $v\frac{\partial T}{\partial x} \approx v\frac{T_j^n - T_{j-1}^n}{\Delta x}$. The diffusive term with variable conductivity $k(T)$ has been discretized by evaluating the heat fluxes at the cell interfaces $(j \pm 1/2)$: $\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \approx \frac{1}{\Delta x}\left[k_{j+1/2}\frac{T_{j+1}^n - T_j^n}{\Delta x} - k_{j-1/2}\frac{T_j^n - T_{j-1}^n}{\Delta x}\right]$,

where the interface conductivities are calculated using the arithmetic mean: $k_{j+1/2} = \frac{1}{2}(k_j + k_{j+1})$ and $k_{j-1/2} = \frac{1}{2}(k_j + k_{j-1})$. The final discretized equation for updating the temperature at computational step time $n+1$ for each internal node named with j as counting variable is:

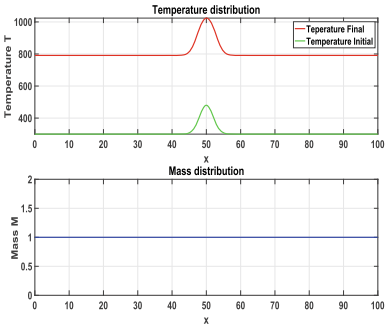
$$T_j^{n+1} = T_j^n + \frac{\Delta t}{\rho_0 C \Delta x^2}\left[k_{j+1/2}(T_{j+1}^n - T_j^n) - k_{j-1/2}(T_j^n - T_{j-1}^n)\right] - \frac{v\Delta t}{\Delta x}(T_j^n - T_{j-1}^n) - \frac{h\Delta t}{\rho_0 C}(T_j^n - T_\infty) + \frac{\Delta t}{C} A e^{-T_{\text{ac}}/T_j^n} H M_j. \quad (9)$$

The accuracy and stability of the numerical solution depends on the choice of spatial resolution, time step, and the physical properties. Since an explicit Forward Euler scheme is used, the time step Δt must satisfy the stability requirement for the diffusion term. Given the non-linear thermal diffusivity $\alpha(T) = k(T)/(\rho_0 C)$, the stability condition is defined as: $\Delta t \leq \min\left\{\frac{\Delta x}{\max(|v|)}, \frac{\rho_0 C \Delta x^2}{2\max(k(T))}\right\}$. The simulation is initialized with a Gaussian temperature profile with the hypothesis to study the front propagation when the ignition process already happened:

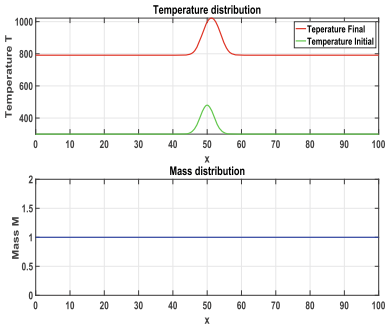
$$T(x,0) = T_{\text{base}} + (T_{\text{peak}} - T_{\text{base}}) \exp\left(-\frac{(x-x_0)^2}{2\sigma_g^2}\right), \quad (10)$$

where $T_{\text{peak}} = 480\,\text{K}$, $T_{\text{base}} = 300\,\text{K}$, $x_0 = L/2$, and $\sigma_g = 0.02L$. We considered a linear computational domain with length $L = 100$. Finally adiabatic boundary conditions (zero-gradient) have been applied in $x = 0$ and $x = L$ with $\frac{\partial T}{\partial x}\big|_{x=0} = 0$, $\frac{\partial T}{\partial x}\big|_{x=L} = 0$. In the discrete domain, these conditions have been implemented by equating the temperature of the boundary nodes to their immediate internal neighbors at each time step: $T_1^{n+1} = T_2^{n+1}$ and $T_{N_x}^{n+1} = T_{N_x-1}^{n+1}$. These conditions simulate an isolated system at the boundaries, ensuring that no heat flux enters or leaves the domain through conduction as also reported in [10]. As mentioned above, the computational domain has a physical length of $L = 100.0$ length units, and has been discretized into $N_x = 1000$ nodes. This choice leads to a spatial resolution of $\Delta x = 0.10$. The temporal evolution is integrated with a fixed time step of $\Delta t = 10^{-4}\,\text{s}$ over a total simulation time duration of 2.5 s. Finally we considered the Stefan-Boltzmann constant σ and scaled parameters $\epsilon = 1.0$ and $\delta = 0.09$. The other parameters used for the numerical simulations are summarized in Table 1. Specifically, four numerical simulations were conducted to evaluate TWS sensitivity with respect to the minimal velocity. The minimal velocity values have been computed for each test case, considering the dimensional effects, and denoted as c^{*d}. As reported in Fig. 1 we executed the four tests varying the advection velocities v with the aim to check the cases $v < c^{*d}$ and $v > c^{*d}$. Particularly, we considered the effects of the biomass using a constant or variable values for M, and v velocity equal to 0.030 or 0.5000 m/s with c^{*d} equal to 0.5388 and 0.0688 m/s respectively. About M in the constant cases, we considered M equal to 1.0 with $\frac{\partial M}{\partial t} = 0$, and in the variable case we started from M equal to 1.0 following the dynamics reported in equation (1). From the test results it is possible to see that for the cases 1(a) and 1(c), with

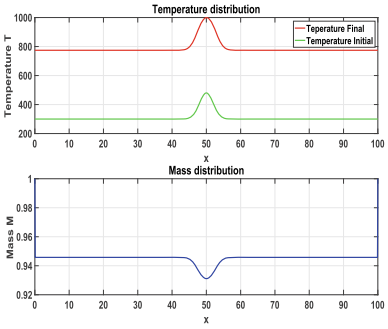
an analytical velocity limit equal to 0.5388 m/s, we have a solution almost fixed in the starting point without TWS as expected. In the other two cases 1(b) and 1(d), according to the analytical condition equal to 0.0688 m/s, we have a TWS that moves in the right direction with a numerical velocity comparable to the estimated limit analytical velocity.



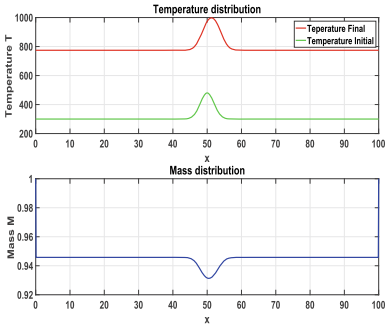
(a) M constant, $w = 0.03$ (m/s)



(b) M constant, $w = 0.50$ (m/s)



(c) M variable, $w = 0.03$ (m/s)



(d) M variable, $w = 0.50$ (m/s)

Fig. 1. Test cases with low (0.03 m/s) and high (0.50 m/s) advection velocity v and with M constant or variable for a simulation time equal to 2.5 s. Tests (a) and (c) show that the main front is almost stationary in the initial position, whereas in the cases (b) and (d) the main front moves according to the direction of advection term.

Numerical Results Analysis

In the considered non-linear models, the wave profile of $T(x,t)$ is subject to simultaneous phenomena: advection, diffusion and reaction. Figure 1 shows how the Gaussian profile used as initial data changes over time so tracking the wave-front velocity using a single point, such as the peak position $T_{\max}(t)$, could provides only a local description. Indeed the wave front may be subject to asymmetric deformations and the peak velocity could deviate from the rate at which

energy is transferred. For this reason, to describe the global propagation velocity, we also treated the numerical solutions as a spatial energy distributions, employing the center of mass ($X_{cm}(t)$) to compute the velocity of the 'mean position' of the heat across the studied cases

$$X_{cm}(t) = \frac{\int_0^L x \cdot [T(x,t) - T_\infty]\,\mathrm{d}x}{\int_0^L [T(x,t) - T_\infty]\,\mathrm{d}x}. \tag{11}$$

Finally the numerical velocity has been derived from the center of mass, $V_{cm} = \Delta X_{cm}/\Delta t$. In the light of application, the wave speed of the wave front has an high relevance. Therefore, we reported in Table 2 both the centre-of-mass velocity (following V_{cm}), which provides an estimate of the average speed at which the heat moves within the system, and the velocity of the maximum value $T_{\max}(t)$ named V_{fmax}. In Table 2 we reported also the used test velocities v and the limit velocities c^{*d} computed for the test cases described above. The numerical results show in the tested cases a good agreement with theory: in Table 2 we show the results of the four tests named (a), (b), (c) and (d) reporting for each test the biomass M as constant or variable, and the c^{*d}, v, V_{cm} values. Particularly, in the cases (a) and (c) we got a general good agreement between the theoretical wave behavior and the numerical tests with some almost stable solutions around the position of the initial values. Indeed, for these cases we used a v value that is smaller than the c^{*d} limit to generate TWS. In these two cases, for the sake of completeness, we computed the V_{cm} finding values close to zero and $V_{fmax} < v$ as expected (see Table 2). Conversely, in the cases (b) and (d) we got as expected a TWS with $V_{cm} > v$ and $V_{fmax} \approx v$ (see Table 2). Further, the comparison between the test cases with constant and variable biomass for the same wind velocity shows that the approximation of a constant biomass, used in the analytical derivation using Theorem 1, is reasonable in the studied time framework.

Table 2. Summary of the results related to the four tests (a), (b), (c) and (d) ref. Figure 1. Velocities V_{cm}, V_{fmax}, c^{*d} and v in m/s .

Test case	M	V_{cm}	V_{fmax}	c^{*d}	v
(a)	constant	0.0001	0.0015	0.5388	0.030
(b)	constant	0.5313	0.5009	0.0688	0.500
(c)	variable	0.0001	0.0015	0.5388	0.030
(d)	variable	0.5313	0.5009	0.0688	0.500

5 Conclusions and Future Works

This preliminary study explores the existence of travelling wave solutions for a non-linear advection-reaction-diffusion (ADR) system used in forest fire modeling. Specifically, following Theorem 1, we reduced the ADR system to a broader

class of equations and we verified that its properties satisfy the necessary theoretical requirements for the existence of travelling wave. We established the conditions for the existence of travelling wave solutions. Furthermore, numerical tests were performed to verify and support the analytical existence conditions. From the numerical results, we got a good agreement between the analytical and the numerical tests in the studied cases. Future research will focus on analyzing the problem under more complex topographical features, incorporating higher-order derivatives and extending the model to a two-dimensional case. Particularly, we will investigate a more general case study with a polynomial function for the effects, $Z(x) = ax^n$, with a first derivative with respect to x equal to $\frac{dZ(x)}{dx} = anx^{n-1}$. According to Eq. (5), it will be required to recompute the formula for the minimal wave speed c^* considering the new general advection term. Finally, further studies will conduct a sensitivity analysis with respect to all physical parameters that characterize the model and perform some comparisons with experimental cases insofar as useful data are reported in the literature.

Funding and Acknowledgments. This work was supported by European Union - NextGenerationEU under the Italian Ministry of University and Research (MUR) National Centre for HPC, Big Data and Quantum Computing CN_00000013-CUP:E13C22001000006.

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