A Fractional Epidemic Model for Multi-malware Attacks in Wireless Sensor Networks

Eslam Farsimadan[1(✉)], Leila Moradi[1], Marcello Trovati[2], and Francesco Palmieri[1]

[1] Department of Computer Science, University of Salerno, 84084 Fisciano, Italy
{efarsimadan,lmoradi,fpalmieri}@unisa.it
[2] School of Business, University of Lancashire, Preston, Lancashire, UK
mtrovati@lancashire.ac.uk

Abstract. Nowadays, Wireless Sensor Network (WSN) technology is widely used in many applications requiring distributed monitoring and control capabilities. When large WSN infrastructures are involved, the diffusion of malware among nodes becomes a critical issue, requiring a clear understanding of propagation dynamics to guarantee timely countermeasures. By considering that fractional-order models demonstrated to be particularly effective in forecasting the dynamics of complex real-life processes, this study improves the Susceptible-Exposed1-Exposed2-Infected-Recovered (SE_1E_2IR) model, derived from the traditional SEIR epidemic theory and used for the early detection of multi-malware activities in WSNs. In particular, a fractional SE_1E_2IR epidemic model has been formulated to explain the propagation of multiple malware infections according to a more versatile and realistic view. The model is structured upon a system of fractional differential equations. The basic reproduction number is also determined as a crucial factor for characterizing malware propagation within the network. The model's free and endemic equilibrium points are calculated, and the stability of the equilibrium points is investigated to find requirements for preventing continuous malware propagation in WSNs. To complement the theoretical analyses, MATLAB is used to perform numerical simulations.

Keywords: Wireless sensor networks · Malware propagation · Epidemic models · Basic reproductive number · Stability analysis

1 Introduction

The mechanism of malware propagation in scale-free networked systems is very similar and somewhat comparable to that of biological virus infections in human populations. The propagation of biological viruses in populations has been extensively studied through epidemic theories, resulting in the development of several theoretical models characterizing this process [2]. Consequently, mathematical

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2027
O. Gervasi et al. (Eds.): ICCSA 2026 Workshops, LNCS 16760, pp. 379–397, 2027.
https://doi.org/10.1007/978-3-032-30521-3_24

models that describe the spreading and propagation dynamics of infectious diseases in the population and malware in the network can assume a significant role in both theoretical and practical contexts. Thus, by developing and investigating these models, one can better comprehend the processes, conditions, and attributes of malware as it moves throughout the network. For instance, the Susceptible-Infected-Removed (SIR) model and the Susceptible-Infected-Susceptible (SIS) model are two traditional models derived from epidemic theory, frequently used to characterize the dynamics of worm propagation on the Internet. Nevertheless, limited effort has been devoted to malware propagation models that are specific for WSNs. In several recent studies, researchers have used epidemic models to study how malware spreads in WSNs [18,22].

The majority of models that have been suggested to study the spread of malware in these networks rely on Ordinary Differential Equations (ODEs), which are an ineffective tool for describing the behaviors of a complicated networked system. To address these weaknesses, we consider the use of Fractional Differential Equations (FDEs) as an effective technique for modeling malware propagation in WSNs. Indeed, FDEs possess the capacity to provide a more precise characterization of systems displaying non-smooth or discontinuous behavior due to their capability of capturing the memory effect. Researchers have recently expressed their interest in employing FDEs to model malware propagation in network systems, for instance, in [19,21].

Motivated by the issues mentioned previously, we develop and investigate a new mathematical model, starting from the existing Susceptible-Exposed1-Exposed2-Infected-Recovered (SE_1E_2IR) multi-malware epidemic model, originally based on ODEs [1], that explains malware propagation in WSNs by using FDEs, in order to provide more accurate results and avoid the memory limitations characterizing integer order formulations. More precisely, Caputo fractional derivatives have been used for achieving unrestricted memory and increased flexibility in modeling intricate dynamics because they better capture the memory effect and long-range dependence characteristics of complex systems. Therefore, they are more appropriate for modeling and analyzing real-world phenomena, such as fractal behavior, anomalous diffusion, and non-local interactions. Implementing fractional-order models also shows great potential in precisely depicting the complex behavior of multiple malware infections simultaneously spreading throughout a WSN infrastructure. Therefore, a fractional-order $0 \leq \alpha < 1$ is used to characterize the dimensional oddities characterizing multi-malware propagation and support the detection of malware existence in WSNs. The Fractional Adam-Bashforth-Moulton (FABM) technique is then employed as an appropriate corrective technique to provide hints on the identification or elimination of the malware infection. The stability conditions are also investigated. Simulation results show that the fractional-order model outperforms the integer-order propagation one.

The rest of this paper is organized as follows: Sect. 2 reports some related work. In Sects. 3, the background, description, and formulation of the proposed model are presented. Section 4 discusses the basic properties of the model. The

system's stability, its relevant and important theorems with their proof are investigated in 5. The simulation outcomes of the modified model and the discussion of the achieved results are presented in Sect. 6. Finally, Sect. 7 presents the conclusion.

2 Related Work

To effectively mitigate and control malware propagation in WSNs, it is necessary to assess and forecast the malware behavior by leveraging reliable models. Recent studies on malware propagation on networked systems, such as WSNs, are based on epidemic theories, which, in turn, rely on mathematical models where differential equations play an essential role.

Verma and Gupta [20] introduced a Vaccination-Susceptible-Carrier-Infectious-Quarantine-Recovered-Susceptible (VSCIQRS) model to mitigate malware propagation in WSNs. The suggested method is employed to prevent malicious activities within the networks. The vaccination technique was employed to mitigate malware propagation in WSNs and could potentially maximize the system's anti-malware abilities. Recently, Dong et al. [7] developed a fractional network-based propagation model featuring a saturated treatment function and developed a fuzzy controller to examine infection management within complex heterogeneous networks. An overview of various malware propagation models in WSNs, highlighting the challenges posed by malware attacks due to the limited defense capabilities of sensor nodes, is discussed in [9,15]. In [10], the authors present a statistical methodology to predict malware propagation in WSNs using HJ-Biplot analysis combined with closed queuing networks. In [22], the authors investigate a new differential equation epidemic model (SUIQR) to describe malware propagation dynamics in WSNs. A mathematical model that describes malware propagation in WSNs, emphasizing the impact of different node types and network topology, is proposed in [18]. The work [21] investigates an improved SEIR model of fractional order to describe malware propagation in WSNs. In [19], the authors present the dynamics of a fractional-order malware propagation model in information networks, examining the stability of equilibrium points and the conditions for Hopf bifurcation. The authors in [14], discussed Securing WSN against multi-malware threats by fractional SE_1E_2VIQR model. However, this study did not consider the effect of varying the fractional order on the system. Furthermore, an analysis of the effect of the main parameters is missing.

3 Background Concepts

This section introduces all the essential elements required to comprehend the suggested fractional-order epidemic model comprehensively while emphasizing the foundational concepts from which this study's ideas emerge. These concepts serve as the basic building blocks upon which the entire proposal is established.

3.1 Fundamentals of Fractional Calculus

Initially, we present fundamental information on FDEs, including Caputo fractional derivatives and the existence and uniqueness of solutions to FDEs.

Definition 1. *[4, 8, 13] Suppose $y \in C^n([0,t])$. The Caputo fractional derivative of the function y, with an order $n-1 < \alpha < n$, where $n \in \mathbb{N}$, is given by:*

$$D_t^{\alpha} y(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(z)}{(t-z)^{\alpha-n+1}} dz, & t > 0,\ 0 < n-1 < \alpha < n, \\ \frac{d^n y(t)}{dt^n}, & \alpha = n \in \mathbb{N}. \end{cases}$$

Lemma 1. *[3, 5] Take the following general FDE, which involves the Caputo derivative:*

$$\begin{aligned} &D_t^{\alpha} y(t) = g(t, y(t)), \quad 0 < \alpha \leq 1, \\ &y(t_0) = y_0, \end{aligned} \tag{1}$$

wherein $g(t, y(t)) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function, called the vector field, related to y and dimension $m \geq 1$. If $g(t, y^*) = 0$, (1) has an equilibrium point at y^*.

Lemma 2. *[13] Suppose $g(t, y(t)) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function related to y and dimension $m \geq 1$. If two positive constants Λ and δ exist such that $\|g(t, y(t))\| \leq \delta + \Lambda \|y(t)\|$ holds for every $(t, y) \in \mathbb{R} \times \mathbb{R}^m$, then a solution to the Caputo FDE (1) exists coming from the point (t_0, y_0).*

Remark 1. [13] If all conditions specified in Lemma 2 are satisfied and $\dfrac{\partial g(t, y(t))}{\partial y(t)}$ is continuous with respect to y, then the solution of the Caputo FDE (1) is certain to be unique.

Lemma 3. *[16] If $0 < \alpha \leq 1$, $y(t) \in C[a,b]$, and $D_t^{\alpha} y(t) \in C(a,b]$, then there exists $z \in [a,t]$ such that, according to the mean value theorem for integrals:*

$$y(t) = y(a) + \frac{1}{\alpha} (D_t^{\alpha} y)(z)(t-a)^{\alpha} \quad \text{for all } t \in (a,b].$$

Remark 2. Assume $0 < \alpha \leq 1$, $y(t) \in C[a,b]$, and $D_t^{\alpha} y(t) \in C(a,b]$. $y(t)$ is non-increasing if $D_t^{\alpha} y(t) \leq 0$ for all $t \in (a,b)$. On the other side, if $D_t^{\alpha} y(t) \geq 0$ for all $t \in (a,b)$, then $y(t) \in C[a,b]$ is non-decreasing for all $t \in [a,b]$.

3.2 Formulation of the Integer-Order Model

Recently, Awasthi et al. [1] designed a deterministic model for multi-malware propagation in WSNs considering five node classes: Susceptible, Exposed1, Exposed2, Infected, and Recovered. Such a model investigates two distinct forms of malware spread in WSNs, considering that the nodes fail or malfunction due to hardware or software issues when subjected to malware attacks. The authors suggested that states in a WSN can be classified into several categories at any time t. They have developed a series of ODEs to describe the dynamic process of malware spread in WSNs. The four distinct states constituting the proposed model are illustrated as follows:

- Susceptible S(t): It refers to the sensor nodes that remain unaffected by malware but are susceptible to infection at any moment.
- Exposed E(t): It denotes malware-infected sensor nodes that are not capable of spreading the malware despite their infection. However, over time, they distribute infection. Given the hypothesis that two distinct types of malware may simultaneously target the network, two categories of Exposed nodes, $E_1(t)$ and $E_2(t)$, are considered.
- infected I(t): It refers to the sensor nodes that are infected and have the potential to propagate the infection to adjacent sensor nodes.
- Recovered R(t): It signifies the category that includes sensor nodes within the network that have been restored following an Infected state.

The dynamical malware propagation formulated in the set of ODEs known as the SE_1E_2IR model is as follows [1]:

$$\begin{cases} \frac{dS}{dt} = b - \varphi\, S\, I - (\sigma + \omega)\, S \\ \frac{dE_1}{dt} = p\varphi\, S\, I - (\lambda_1 + \sigma)\, E_1 \\ \frac{dE_2}{dt} = q\varphi\, S\, I - (\lambda_2 + \sigma)\, E_2 \\ \frac{dI}{dt} = \lambda_1\, E_1 \, + \, \lambda_2\, E_2 \, - \, (\sigma + \gamma)\, I \\ \frac{dR}{dt} = \gamma\, I - \sigma\, R + \omega\, S \end{cases} \tag{2}$$

The system is characterized within the domain $\Gamma = \{(S, E_1, E_2, I, R) \in \mathbb{R}^5\}$ in which $S(0) = S_0$, $E_1(0) = 0$, $E_2(0) = 0$, $I(0) = I_0$, $R(0) = R_0$ for all $t \geq 0$. Because the model observes distinct states for sensor nodes, all state variables are positive for all $t \geq 0$. $N(t)$ uniformly distributed within the designated area L^2, denotes the total number of sensor nodes in the WSN at any given time t ($N(t) = S(t) + E_1(t) + E_2(t) + I(t) + R(t)$). The region covered by a sensor node of the WSN is πr^2, where r represents the communication radius. It should be mentioned that $p+q = 1$, $\lambda_1 > \lambda_2$, and $\varphi = \frac{\beta \pi r^2}{L^2}$. The most important parameters in the SE_1E_2IR model (2) are summarized in Table 1. Figure 1 shows the SE_1E_2IR state transition diagram.

The state of the model (2) at each time is not dependent on the previous history of the system; i.e., it is memoryless. These processes are known as Markovian since they do not have memory. As stated in the introduction, FDEs are invaluable tools for modeling the dynamics of WSNs. Models employing fractional-order derivatives suggest superior agreement between measured and simulated data compared with traditional models reliant on integer-order derivatives.

3.3 The Novel Fractional-Order SE_1E_2IR Model for Malware Propagation in WSNs

We aim to convert the model (2) into a fractional-order scenario that will integrate memory effects into the model. This study generalizes the prior SE_1E_2IR

Table 1. Parameters of the SE_1E_2IR model.

Parameters	Description
b	The addition rate of new nodes in WSN
σ	The failure rate of nodes caused by battery depletion or hardware/software problems
β	The coefficient of malware transmission in WSN
p	The rate of Susceptible nodes that became Exposed and enter into E_1 state
q	The rate of Susceptible nodes that became Exposed and enter into E_2 state
λ_1	The rate of Exposed state E_1 of nodes become Infected
λ_2	The rate of Exposed state E_2 of nodes become Infected
γ	The rate of recovery of Infected nodes
ω	The rate at which Susceptible nodes become immunized and move to the Recovered state R

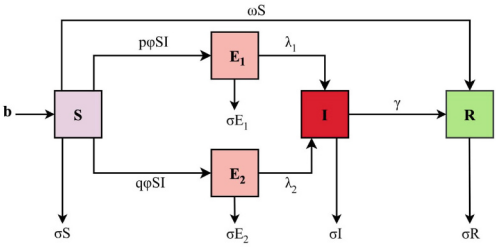


Fig. 1. State transition diagram of the model.

model, represented by ordinary differential equations, by incorporating the dynamics of states. In each case, we replace the conventional ordinary derivative with a fractional derivative to guarantee that both sides maintain equal dimensions. In particular, the following is a representation of the fractional-order malware propagation model that is based on the SE_1E_2IR model (2):

$$\begin{cases} D_t^\alpha S(t) = b^\alpha - \varphi^\alpha S\, I - (\sigma^\alpha + \omega^\alpha)\, S \\ D_t^\alpha E_1(t) = p^\alpha \varphi^\alpha S\, I - (\lambda_1^\alpha + \sigma^\alpha)\, E_1 \\ D_t^\alpha E_2(t) = q^\alpha \varphi^\alpha S\, I - (\lambda_2^\alpha + \sigma^\alpha)\, E_2 \\ D_t^\alpha I(t) = \lambda_1^\alpha E_1 + \lambda_2^\alpha E_2 - (\sigma^\alpha + \gamma^\alpha)\, I \\ D_t^\alpha R(t) = \gamma^\alpha I - \sigma^\alpha R + \omega^\alpha S \end{cases} \tag{3}$$

In (3), D_t^α is the Caputo fractional derivative utilized to integrate hereditary and historical characteristics into the new system. We have elevated all model parameters with dimension $\frac{1}{t^\alpha}$ to the exponent α to maintain dimensional consistency. The adjustment of the dimension mismatch is discussed in [12]. Therefore, both sides of the equations in (3) have identical dimensions. Henceforth, we will focus on the model (3).

4 Basic Results of the Model

To guarantee the validity of the model in monitoring populations of different classes, it is essential to ensure that all state variables remain nonnegative over the entire period of time. In this section, for the sake of convenience and simplicity, we shall refer to:

$$g(t,y(t)) = \begin{bmatrix} g_1(t,y(t)) \\ g_2(t,y(t)) \\ g_3(t,y(t)) \\ g_4(t,y(t)) \\ g_5(t,y(t)) \end{bmatrix} = \begin{bmatrix} b^\alpha - \varphi^\alpha\, S\, I - (\sigma^\alpha + \omega^\alpha)\, S \\ p^\alpha \varphi^\alpha\, S\, I - (\lambda_1^\alpha + \sigma^\alpha)\, E_1 \\ q^\alpha \varphi^\alpha\, S\, I - (\lambda_2^\alpha + \sigma^\alpha)\, E_2 \\ \lambda_1^\alpha\, E_1 + \lambda_2^\alpha\, E_2 - (\sigma^\alpha + \gamma^\alpha)\, I \\ \gamma^\alpha\, I - \sigma^\alpha\, R + \omega^\alpha\, S \end{bmatrix}. \tag{4}$$

4.1 Existence and Uniqueness

Theorem 1. *The fractional SE_1E_2IR model (3) has unique solution for all $t \geq 0$.*

Proof. We apply Lemma 2 to deduce that the fractional SE_1E_2IR model (3) has a solution for all $\mathbb{R}$. In the fractional SE_1E_2IR model (3), suppose $y(t) = (S(t), E_1(t), E_2(t), I(t), R(t))$. It is clear that $g(t, y(t))$ is continuous with respect to $y \in \mathbb{R}^5$. By using (4), we calculate

$\|g(t,y(t))\|_1 = \sum_{j=1}^{5} |g_j(t,y(t))|$ as follows:

$$\|g(t,y(t))\|_1 = |b^\alpha - \varphi^\alpha\, S\, I - (\sigma^\alpha + \omega^\alpha)\, S| + +|p^\alpha \varphi^\alpha\, S\, I - (\lambda_1^\alpha + \sigma^\alpha)\, E_1| + \\ +|q^\alpha \varphi^\alpha\, S\, I - (\lambda_2^\alpha + \sigma^\alpha)\, E_2| + +|\lambda_1^\alpha\, E_1 + \lambda_2^\alpha\, E_2 - (\sigma^\alpha + \gamma^\alpha)\, I| + +|\gamma^\alpha\, I - \sigma^\alpha\, R + \omega^\alpha\, S|. \tag{5}$$

By employing the triangular inequality, we obtain

$$\|g(t,y(t))\|_1 \leq b^\alpha + \varphi^\alpha(1 + p^\alpha + q^\alpha)|S||I| + +(\sigma^\alpha + 2\omega^\alpha)|S| + (2\lambda_1^\alpha + \sigma^\alpha)|E_1| + \\ +(2\lambda_2^\alpha + \sigma^\alpha)|E_2| + (\sigma^\alpha + 2\gamma^\alpha)\,|I| + \sigma^\alpha\,|R|. \tag{6}$$

Since, $N = S + E_1 + E_2 + I + R$ we have $S \leq N$. This implies

$$\|g(t,y(t))\|_1 \leq b^\alpha + (\varphi^\alpha(1 + p^\alpha + q^\alpha)N + +(\sigma^\alpha + 2\gamma^\alpha))|I| + (\sigma^\alpha + 2\omega^\alpha)|S| + \\ +(2\lambda_1^\alpha + \sigma^\alpha)|E_1| + (2\lambda_2^\alpha + \sigma^\alpha)|E_2| + \sigma^\alpha\,|R|. \tag{7}$$

Then, we add the subsequent positive terms to the right-hand side of inequality (7)

$$\begin{aligned} &(\varphi^\alpha(1 + p^\alpha + q^\alpha)N + 2(\gamma^\alpha + \lambda_1^\alpha + \lambda_2^\alpha))|S|, \\ &(\varphi^\alpha(1 + p^\alpha + q^\alpha)N + 2(\gamma^\alpha + \omega^\alpha + \lambda_2^\alpha))|E_1|, \\ &(\varphi^\alpha(1 + p^\alpha + q^\alpha)N + 2(\gamma^\alpha + \omega^\alpha + \lambda_1^\alpha))|E_2|, \\ &(2(\omega^\alpha + \lambda_1^\alpha + \lambda_2^\alpha))|I|, \\ &(\varphi^\alpha(1 + p^\alpha + q^\alpha)N + 2(\gamma^\alpha + \omega^\alpha + \lambda_1^\alpha + \lambda_2^\alpha))|R|. \end{aligned} \tag{8}$$

By adding the positive terms (8) to (7), we have:

$$\|g(t,y(t))\|_1 \leq b^\alpha + (\varphi^\alpha(1+p^\alpha+q^\alpha)N + 2(\gamma^\alpha+\omega^\alpha+\lambda_1^\alpha+\lambda_2^\alpha)+\sigma^\alpha)(|S|+|E_1|+|E_2| +|I|+|R|). \tag{9}$$

Since $\|y(t)\| = |S|+|E_1|+|E_2|+|I|+|R|$, we have

$$\|g(t,y(t))\|_1 \leq \delta + \Lambda\|y(t)\|, \quad \forall (t,y) \in \mathbb{R}\times\mathbb{R}^5 \tag{10}$$

where the positive constants are denoted as $b^\alpha = \delta$ and $\Lambda = (\varphi^\alpha(1+p^\alpha+q^\alpha)N+ 2(\gamma^\alpha+\omega^\alpha+\lambda_1^\alpha+\lambda_2^\alpha)+\sigma^\alpha)$. By utilizing Lemma 2, we conclude that the fractional SE_1E_2IR model (3) has a solution for all real numbers $\mathbb{R}$. Since $g(t,y(t)) \in \mathbb{R}^5$, $\dfrac{\partial g(t,y(t))}{\partial y(t)}$ is continuous with respect to y on $\mathbb{R}^5$. Utilizing Remark 1, we ascertain that the fractional SE_1E_2IR model (3) possesses a unique solution for all real numbers.

4.2 Positivity of the Solution

Theorem 2. *The permissible region* $\psi = \{(S,E_1,E_2,I,R) \in \mathbb{R}^5 : S \geq 0,\ E_1 \geq 0,\ E_2 \geq 0,\ I \geq 0,\ R \geq 0\}$ *for the fractional* SE_1E_2IR *model (3) is a positive invariant set.*

Proof. Suppose that $y(t) = (S(t),E_1(t),E_2(t),I(t),R(t))$ is the solution of the fractional SE_1E_2IR model (3). Now we compute $g(t,y(t))$ where $y(t) = (0,0,0,0,0)$. Since all the parameters in the fractional SE_1E_2IR model (3) are positive constants, for all $t \geq 0$, we have

$$g(t,y(t)) = \begin{bmatrix} D_t^\alpha S(t)|_{S=0} \\ D_t^\alpha E_1(t)|_{E_1=0} \\ D_t^\alpha E_2(t)|_{E_2=0} \\ D_t^\alpha I(t)|_{I=0} \\ D_t^\alpha R(t)|_{R=0} \end{bmatrix} = \begin{bmatrix} b^\alpha \\ p^\alpha\varphi^\alpha\, S\, I \\ q^\alpha\varphi^\alpha\, S\, I \\ \lambda_1^\alpha\, E_1 \,+\, \lambda_2^\alpha\, E_2 \\ \gamma^\alpha\, I + \omega^\alpha\, S \end{bmatrix} \geq 0. \tag{11}$$

Given that $g(t,y(t))$ is a continuous function, we deduce that $y(t) \in \psi$ and ψ is a positive invariant set of solution for the fractional SE_1E_2IR model (3) (According to Remark 2).

5 Main Results of the Model and Stability Analysis

This section will identify both equilibrium points. Furthermore, we will calculate the basic reproduction number and conduct a sensitivity analysis based on the free and endemic equilibrium points. This section concludes with an analysis of the stability of the endemic and free equilibrium points.

5.1 Free Equilibrium Points

The free equilibrium points of the system of the fractional SE_1E_2IR model (3) can be identified by solving the equations $D_t^\alpha S(t) = 0, D_t^\alpha E_1(t) = 0, D_t^\alpha E_2(t) = 0, D_t^\alpha I(t) = 0, D_t^\alpha R(t) = 0$. Consequently, we set

$$\begin{cases} b^\alpha - \varphi^\alpha S\, I - (\sigma^\alpha + \omega^\alpha)\, S = 0, \\ p^\alpha \varphi^\alpha S\, I - (\lambda_1^\alpha + \sigma^\alpha)\, E_1 = 0, \\ q^\alpha \varphi^\alpha S\, I - (\lambda_2^\alpha + \sigma^\alpha)\, E_2 = 0, \\ \lambda_1^\alpha E_1 + \lambda_2^\alpha E_2 - (\sigma^\alpha + \gamma^\alpha)\, I = 0, \\ \gamma^\alpha I - \sigma^\alpha R + \omega^\alpha S = 0. \end{cases} \tag{12}$$

A straightforward computation will determine the free equilibrium points (Ξ_0) as follows:

$$\Xi_0 = \left(\frac{b^\alpha}{(\sigma^\alpha + \omega^\alpha)}, 0, 0, 0, \frac{\omega^\alpha b^\alpha}{\sigma^\alpha(\sigma^\alpha + \omega^\alpha)}\right). \tag{13}$$

5.2 Basic Reproduction Number

The basic reproduction number, R_0, is the primary parameter for categorizing the spread of malware in WSN. R_0 represents the total number of secondary Infected nodes caused by a single infected node over a certain period. Hence, R_0 is a dimensionless metric identifying the probability of malware propagation. Various methodologies exist for determining R_0 in the context of epidemic spread [6]. This study employs the next-generation matrix technique. Just E_1, E_2, and I are presently directly engaged in the spread of malware. Upon examination of the issue, we have delineated three classes, denoted as (E_1, E_2, I), to determine the parameter R_0. Consequently, we recall the fractional SE_1E_2IR model (3) here:

$$\begin{cases} D_t^\alpha E_1(t) = p^\alpha \varphi^\alpha S\, I - (\lambda_1^\alpha + \sigma^\alpha)\, E_1 \\ D_t^\alpha E_2(t) = q^\alpha \varphi^\alpha S\, I - (\lambda_2^\alpha + \sigma^\alpha)\, E_2 \\ D_t^\alpha I(t) = \lambda_1^\alpha E_1 + \lambda_2^\alpha E_2 - (\sigma^\alpha + \gamma^\alpha)\, I \end{cases} \tag{14}$$

Employing the approach (the next generation matrix [6]), the system (14) can be expressed as $D_t^\alpha X = P(X) - Q(X)$, where

$$X = \begin{bmatrix} E_1 \\ E_2 \\ I \end{bmatrix}, \quad P(X) = \begin{bmatrix} p^\alpha \varphi^\alpha S\, I \\ q^\alpha \varphi^\alpha S\, I \\ \lambda_1^\alpha E_1 + \lambda_2^\alpha E_2 \end{bmatrix}, Q(X) = \begin{bmatrix} (\lambda_1^\alpha + \sigma^\alpha)\, E_1 \\ (\lambda_2^\alpha + \sigma^\alpha)\, E_2 \\ (\sigma^\alpha + \gamma^\alpha)\, I \end{bmatrix}.$$

The Jacobian matrices of $P(X)$ and $Q(X)$ are calculated as follows:

$$\mathcal{J}(P(X)) = \begin{bmatrix} 0 & 0 & p^\alpha \varphi^\alpha S \\ 0 & 0 & q^\alpha \varphi^\alpha S \\ \lambda_1^\alpha & \lambda_2^\alpha & 0 \end{bmatrix},$$

and

$$\mathcal{J}(Q(X)) = \begin{bmatrix} (\lambda_1^\alpha + \sigma^\alpha) & 0 & 0 \\ 0 & (\lambda_2^\alpha + \sigma^\alpha) & 0 \\ 0 & 0 & (\sigma^\alpha + \gamma^\alpha) \end{bmatrix}.$$

The Jacobian matrices of $P(X)$ and $Q(X)$ at the free equilibrium points (Ξ_0) are calculated according to (13), as follows:

$$K := \mathcal{J}(P|_{\Xi_0}) = \begin{bmatrix} 0 & 0 & \frac{p^\alpha \varphi^\alpha b^\alpha}{(\sigma^\alpha + \omega^\alpha)} \\ 0 & 0 & \frac{q^\alpha \varphi^\alpha b^\alpha}{(\sigma^\alpha + \omega^\alpha)} \\ \lambda_1^\alpha & \lambda_2^\alpha & 0 \end{bmatrix}, H := \mathcal{J}(Q|_{\Xi_0}) = \begin{bmatrix} (\lambda_1^\alpha + \sigma^\alpha) & 0 & 0 \\ 0 & (\lambda_2^\alpha + \sigma^\alpha) & 0 \\ 0 & 0 & (\sigma^\alpha + \gamma^\alpha) \end{bmatrix}. \tag{15}$$

The inverse of H is given by

$$H^{-1} = \begin{bmatrix} \frac{1}{(\lambda_1^\alpha + \sigma^\alpha)} & 0 & 0 \\ 0 & \frac{1}{(\lambda_2^\alpha + \sigma^\alpha)} & 0 \\ 0 & 0 & \frac{1}{(\sigma^\alpha + \gamma^\alpha)} \end{bmatrix}. \tag{16}$$

The next generation matrix $M = KH^{-1}$ is:

$$M = \begin{bmatrix} 0 & 0 & \frac{p^\alpha \varphi^\alpha b^\alpha}{(\sigma^\alpha + \omega^\alpha)(\sigma^\alpha + \gamma^\alpha)} \\ 0 & 0 & \frac{q^\alpha \varphi^\alpha b^\alpha}{(\sigma^\alpha + \omega^\alpha)(\sigma^\alpha + \gamma^\alpha)} \\ \frac{\lambda_1^\alpha}{(\lambda_1^\alpha + \sigma^\alpha)} & \frac{\lambda_2^\alpha}{(\lambda_2^\alpha + \sigma^\alpha)} & 0 \end{bmatrix}. \tag{17}$$

Consequently, the basic reproduction number R_0 (which indicates the largest eigenvalue) is calculated as follows:

$$R_0 = \frac{b^\alpha \varphi^\alpha}{(\sigma^\alpha + \omega^\alpha)(\sigma^\alpha + \gamma^\alpha)} \Big(\frac{p^\alpha \lambda_1^\alpha}{(\lambda_1^\alpha + \sigma^\alpha)} + \frac{q^\alpha \lambda_2^\alpha}{(\lambda_2^\alpha + \sigma^\alpha)} \Big). \tag{18}$$

The scalar R_0 is a dimensionless threshold value commonly referred to as the basic reproduction number when $\alpha = 1$.

5.3 Endemic Equilibrium Points

If $R_0 > 1$, there exists another kind of equilibrium, termed endemic equilibrium $\Xi^* = (S^*, E_1^*, E_2^*, I^*, R^*)$, that occurs when the infection remains continuously present throughout the system. The endemic equilibrium points of the system of the fractional SE_1E_2IR model (3) can be identified by solving equations $D_t^\alpha S^*(t) = 0, D_t^\alpha E_1^*(t) = 0, D_t^\alpha E_2^*(t) = 0, D_t^\alpha I^*(t) = 0, D_t^\alpha R^*(t) = 0$. A straightforward computation will determine the endemic equilibrium points Ξ^* as follows:

$$\begin{aligned} \Xi^* = \Big(& \frac{b^\alpha}{(\sigma^\alpha + \omega^\alpha) R_0}, \frac{(R_0 - 1) p^\alpha b^\alpha}{R_0 (\sigma^\alpha + \lambda_1^\alpha)}, \frac{(R_0 - 1) q^\alpha b^\alpha}{R_0 (\sigma^\alpha + \lambda_2^\alpha)}, \Big(\frac{(R_0 - 1)(\sigma^\alpha + \omega^\alpha)}{\varphi^\alpha} \Big), \\ & \frac{1}{\sigma^\alpha} \Big(\frac{\omega^\alpha b^\alpha}{R_0 (\sigma^\alpha + \omega^\alpha)} + \frac{\gamma^\alpha (\sigma^\alpha + \omega^\alpha)(R_0 - 1)}{\varphi^\alpha} \Big) \Big) \end{aligned} \tag{19}$$

5.4 Stability Analysis of the Fractional SE_1E_2IR Model in Free Equilibrium Points

Regarding the introduced fractional SE_1E_2IR model (3), we should investigate the stability of equilibrium points. This section seeks to identify the local and global stability of free equilibrium points. Indeed, this subsection evaluates local and global stability concerning the free equilibrium points Ξ_0.

Definition 2. *A steady state solution y^* of an FDE system is called locally asymptotically stable if each solution $y(t)$ that initiates close to y^* converges to y^* as $t \longrightarrow \infty$.*

Definition 3. *A steady state solution y^* of an FDE system is globally asymptotically stable if every solution of FDE converges to y^* as $t \longrightarrow \infty$.*

Theorem 3. *The fractional SE_1E_2IR model (3) demonstrates local asymptotic stability at Ξ_0 when $R_0 < 1$ and unstable when $R_0 > 1$ (.i.e., if all eigenvalues of the matrix (5.4) are negative).*

Proof. The Jacobian matrix of the fractional SE_1E_2IR model (3) at the free equilibrium Ξ_0, can be expressed as follows:

$$\mathcal{J}|_{\Xi_0} = \begin{bmatrix} -(\sigma^\alpha + \omega^\alpha) & 0 & 0 & -\varphi^\alpha S_0 & 0 \\ 0 & -(\lambda_1^\alpha + \sigma^\alpha) & 0 & p^\alpha \varphi^\alpha S_0 & 0 \\ 0 & 0 & -(\lambda_2^\alpha + \sigma^\alpha) & q^\alpha \varphi^\alpha S_0 & 0 \\ 0 & \lambda_1^\alpha & \lambda_2^\alpha & -(\sigma^\alpha + \gamma^\alpha) & 0 \\ \omega^\alpha & 0 & 0 & \gamma^\alpha & -\sigma^\alpha \end{bmatrix}$$

The characteristic equation of the matrix $\mathcal{J}$ is:

$$P_j = (\eta + (\sigma^\alpha + \omega^\alpha))\,(\eta + \sigma^\alpha)\,(\eta^3 + \xi_1 \eta^2 + \xi_2 \eta + \xi_3)$$

where

$$\begin{cases} \xi_1 = Tr(A) = (\lambda_1^\alpha + \lambda_2^\alpha) + 3\sigma^\alpha + \gamma^\alpha, \\ \xi_2 = (A_{ji}A_{ij} - A_{ii}A_{jj}) = (\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \gamma^\alpha) + (\lambda_1^\alpha + \sigma^\alpha)(\sigma^\alpha + \gamma^\alpha) + (\lambda_1^\alpha + \sigma^\alpha)(\lambda_2^\alpha + \sigma^\alpha) - \\ (\lambda_1^\alpha p^\alpha \varphi^\alpha S_0 - \lambda_2^\alpha q^\alpha \varphi^\alpha S_0), \\ \xi_3 = det(A) = p^\alpha \varphi^\alpha S_0 \lambda_1^\alpha (\lambda_2^\alpha + \sigma^\alpha) + q^\alpha \varphi^\alpha S_0 \lambda_2^\alpha (\lambda_1^\alpha + \sigma^\alpha) - (\lambda_1^\alpha + \sigma^\alpha)(\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \gamma^\alpha). \end{cases}$$

where A is the matrix as follows:

$$A = \begin{bmatrix} -(\lambda_1^\alpha + \sigma^\alpha) & 0 & p^\alpha \varphi^\alpha S_0 \\ 0 & -(\lambda_2^\alpha + \sigma^\alpha) & q^\alpha \varphi^\alpha S_0 \\ \lambda_1^\alpha & \lambda_2^\alpha & -(\sigma^\alpha + \gamma^\alpha) \end{bmatrix}$$

After precise verification, it is obvious that for $R_0 < 1$, all coefficients ξ_1, ξ_2, and ξ_3 of the cubic equation $\eta^3 + \xi_1 \eta^2 + \xi_2 \eta + \xi_3$ are strictly positive. Consequently, the characteristic equation has five roots, $\eta_1, \eta_2, \eta_3, \eta_4$, and η_5, all of which have negative real parts. This fact, in turn, indicates that all five eigenvalues of the matrix $\mathcal{J}|_{\Xi_0}$ possess negative real parts; hence, the fractional SE_1E_2IR model (3) demonstrates local stability at the malware-free equilibrium points Ξ_0. Given these findings, the proof is considered complete.

Theorem 4. *If $R_0 \leq 1$, then the free equilibrium points Ξ_0 are globally asymptotically stable.*

Proof. According to the Lyapunov stability theorem [11], if the system possesses a definite positive function, referred to as $\mathcal{L}(t)$, and its full derivative $D_t^\alpha \mathcal{L}(t) \leq 0$ along Ξ_0 with respect to t, consequently, the system is globally asymptotically stable at Ξ_0. We can select some parameters from the model and its state equations to construct a Lyapunov function. Suppose

$$\mathcal{L} = \Delta\Big(\lambda_1^\alpha(\lambda_2^\alpha + \sigma^\alpha)E_1 + \lambda_2^\alpha(\lambda_1^\alpha + \sigma^\alpha)E_2 + (\lambda_1^\alpha + \sigma^\alpha)(\lambda_2^\alpha + \sigma^\alpha)I\Big)$$

be a positive Lyapunov function, where

$$\Delta = \frac{b}{(\sigma^\alpha + \lambda_1^\alpha)(\sigma^\alpha + \lambda_2^\alpha)(\sigma^\alpha + \gamma^\alpha)}.$$

The fractional derivative of $\mathcal{L}$ with respect to t is

$$D_t^\alpha \mathcal{L}(t) = \Delta\Big(\lambda_1^\alpha(\lambda_2^\alpha + \sigma^\alpha)\, D_t^\alpha E_1(t) + \lambda_2^\alpha(\lambda_1^\alpha + \sigma^\alpha)\, D_t^\alpha E_2(t) + (\lambda_1^\alpha + \sigma^\alpha)(\lambda_2^\alpha + \sigma^\alpha)\, D_t^\alpha I(t)\Big) =$$
$$\Delta\Big(\lambda_1^\alpha(\lambda_2^\alpha + \sigma^\alpha)\,(p^\alpha\varphi^\alpha\, S\, I - (\lambda_1^\alpha + \sigma^\alpha)\, E_1) + \lambda_2^\alpha(\lambda_1^\alpha + \sigma^\alpha)\,(q^\alpha\varphi^\alpha\, S\, I - (\lambda_2^\alpha + \sigma^\alpha)\, E_2) + (\lambda_1^\alpha + \sigma^\alpha)$$
$$(\lambda_2^\alpha + \sigma^\alpha)\,(\lambda_1^\alpha\, E_1 + \lambda_2^\alpha\, E_2 - (\sigma^\alpha + \gamma^\alpha)\, I)\Big)$$

Utilizing the basic reproduction number and $S = \frac{b^\alpha}{(\sigma^\alpha + \omega^\alpha)}$, we obtain:

$$D_t^\alpha \mathcal{L}(t) = (R_0 - 1)I$$

After analyzing the function $D_t^\alpha \mathcal{L}(t)$, we deduce that $D_t^\alpha \mathcal{L}(t) \leq 0$ along Ξ_0 if and only if $R_0 \leq 1$. If the function $I(t) = 0$, then the maximum value of the operator $D_t^\alpha \mathcal{L}(t)$ is also zero. Consequently, the global stability at Ξ_0 adheres to the LaSalle invariance principle [17] if $R_0 \leq 1$.

5.5 Stability Analysis of the Fractional SE_1E_2IR Model in Endemic Equilibrium Points

This part evaluates the proposed system stability concerning the endemic equilibrium points Ξ^*.

Theorem 5. *The fractional SE_1E_2IR model (3) demonstrates local asymptotic stability at Ξ^* if all eigenvalues of the matrix (20) are negative.*

Proof. The Jacobian matrix of the fractional SE_1E_2IR model (3), for malware propagation in WSNs, at the malware-endemic equilibrium points Ξ^*, can be expressed as follows:

$$\mathcal{J}|_{\Xi^*} = \begin{bmatrix} -(\varphi^\alpha I^* + \sigma^\alpha + \omega^\alpha) & 0 & 0 & -\varphi^\alpha S^* & 0 \\ p^\alpha\varphi^\alpha I^* & -(\lambda_1^\alpha + \sigma^\alpha) & 0 & p^\alpha\varphi^\alpha S^* & 0 \\ q^\alpha\varphi^\alpha I^* & 0 & -(\lambda_2^\alpha + \sigma^\alpha) & q^\alpha\varphi^\alpha S^* & 0 \\ 0 & \lambda_1^\alpha & \lambda_2^\alpha & -(\sigma^\alpha + \gamma^\alpha) & 0 \\ \omega^\alpha & 0 & 0 & \gamma^\alpha & -\sigma^\alpha \end{bmatrix} \tag{20}$$

The characteristic equation of the matrix $\mathcal{J}|_{\Xi^*}$ is:

$$P_j^* = (\eta + \sigma^\alpha)\left(\eta^4 + \xi_1^*\eta^3 + \xi_2^*\eta^2 + (\xi_3^*\eta) + \xi_4^*\right)$$

where

$$\begin{cases} \xi_1^* = Tr(A^*) = \lambda_1^\alpha + \lambda_2^\alpha + 4\sigma^\alpha + \gamma^\alpha + \omega^\alpha + \varphi^\alpha I^*, \\ \xi_2^* = (\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \gamma^\alpha) + (2\sigma^\alpha + \lambda_1^\alpha + \omega^\alpha + \varphi^\alpha I^*)(2\sigma^\alpha + \lambda_2^\alpha + \gamma^\alpha)(\lambda_1^\alpha + \sigma^\alpha)(\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*) - \\ p^\alpha\varphi^\alpha S^*\lambda_1^\alpha - q^\alpha\varphi^\alpha S^*\lambda_2^\alpha \\ \xi_3^* = (\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \gamma^\alpha)(2\sigma^\alpha + \lambda_1^\alpha + \omega^\alpha + \varphi^\alpha I^*) + (2\sigma^\alpha + \lambda_2^\alpha + \gamma^\alpha)(\lambda_1^\alpha + \sigma^\alpha)(\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*) + p^\alpha(\varphi^\alpha)^2 \\ S^*I^*\lambda_1^\alpha + q^\alpha(\varphi^\alpha)^2S^*I^*\lambda_2^\alpha - p^\alpha\varphi^\alpha S^*\lambda_1^\alpha(2\sigma^\alpha + \lambda_2^\alpha + \omega^\alpha + \varphi^\alpha I^*) - q^\alpha\varphi^\alpha S^*\lambda_2^\alpha(2\sigma^\alpha + \lambda_1^\alpha + \omega^\alpha + \varphi^\alpha I^*) \\ \xi_4^* = det(A^*) = (\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*)(\lambda_1^\alpha + \sigma^\alpha)(\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \gamma^\alpha) + p^\alpha(\varphi^\alpha)^2S^*I^*\lambda_1^\alpha(\lambda_2^\alpha + \sigma^\alpha) + q^\alpha(\varphi^\alpha)^2S^*I^*\lambda_2^\alpha \\ (\lambda_1^\alpha + \sigma^\alpha) - p^\alpha\varphi^\alpha S^*\lambda_1^\alpha(\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*) - q^\alpha\varphi^\alpha S^*\lambda_2^\alpha(\lambda_1^\alpha + \sigma^\alpha)(\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*) \end{cases}$$

where A^* is the matrix as follows:

$$A^* = \begin{bmatrix} -(\varphi^\alpha I^* + \sigma^\alpha + \omega^\alpha) & 0 & 0 & -\varphi^\alpha S^* \\ p^\alpha\varphi^\alpha I^* & -(\lambda_1^\alpha + \sigma^\alpha) & 0 & p^\alpha\varphi^\alpha S^* \\ q^\alpha\varphi^\alpha I^* & 0 & -(\lambda_2^\alpha + \sigma^\alpha) & q^\alpha\varphi^\alpha S^* \\ 0 & \lambda_1^\alpha & \lambda_2^\alpha & -(\sigma^\alpha + \gamma^\alpha) \end{bmatrix}.$$

All coefficients $\xi_1^*, \xi_2^*, \xi_3^*$, and ξ_4^* of equation $\eta^4 + \xi_1^*\eta^3 + \xi_2^*\eta^2 + (\xi_3^*\eta) + \xi_4^*$ should be positive for satisfying Routh-Hurwitz criteria. It is obvious that $\xi_1^* > 0$, for other coefficients ξ_2^*, ξ_3^*, and ξ_4^*, we have:
$\xi_2^* > 0$ if:

$$(\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \gamma^\alpha) + (2\sigma^\alpha + \lambda_1^\alpha + \omega^\alpha + \varphi^\alpha I^*)(2\sigma^\alpha + \lambda_2^\alpha + \gamma^\alpha)(\lambda_1^\alpha + \sigma^\alpha)(\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*) > p^\alpha\varphi^\alpha S^*\lambda_1^\alpha + q^\alpha\varphi^\alpha S^*\lambda_2^\alpha,$$

$\xi_3^* > 0$ if:

$$(\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \gamma^\alpha)(2\sigma^\alpha + \lambda_1^\alpha + \omega^\alpha + \varphi^\alpha I^*) + (2\sigma^\alpha + \lambda_2^\alpha + \gamma^\alpha)(\lambda_1^\alpha + \sigma^\alpha)(\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*) + p^\alpha(\varphi^\alpha)^2 S^*I^*\lambda_1^\alpha + q^\alpha(\varphi^\alpha)^2S^*I^*\lambda_2^\alpha > p^\alpha\varphi^\alpha S^*\lambda_1^\alpha(2\sigma^\alpha + \lambda_2^\alpha + \omega^\alpha + \varphi^\alpha I^*) + q^\alpha\varphi^\alpha S^*\lambda_2^\alpha(2\sigma^\alpha + \lambda_1^\alpha + \omega^\alpha + \varphi^\alpha I^*),$$

$\xi_4^* > 0$ if:

$$(\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*)(\lambda_1^\alpha + \sigma^\alpha)(\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \gamma^\alpha) + p^\alpha(\varphi^\alpha)^2S^*I^*\lambda_1^\alpha(\lambda_2^\alpha + \sigma^\alpha) + q^\alpha(\varphi^\alpha)^2S^*I^*\lambda_2^\alpha(\lambda_1^\alpha + \sigma^\alpha) > p^\alpha\varphi^\alpha S^*\lambda_1^\alpha(\lambda_2^\alpha + \sigma^\alpha)(\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*) + q^\alpha\varphi^\alpha S^*\lambda_2^\alpha(\lambda_1^\alpha + \sigma^\alpha)(\sigma^\alpha + \omega^\alpha + \varphi^\alpha I^*).$$

Consequently, The characteristic equation has five roots, $\eta_1, \eta_2, \eta_3, \eta_4$, and η_5, all of which have negative real parts. Therefore, this fact indicates that all five eigenvalues of the matrix $\mathcal{J}|_{\Xi^*}$ include negative real parts, thereby confirming the local stability of the fractional SE_1E_2IR model (3) at the malware-endemic equilibrium points Ξ^*. Given these observations, the proof is considered complete.

Theorem 6. *If $R_0 \geq 1$, then the endemic equilibrium points Ξ^* are globally asymptotically stable.*

Proof. When $R_0 > 1$, suppose $\Psi(x) = x - x^* - x^* ln\frac{x}{x^*}$ and construct the Liapunov function as

$$\mathcal{L}^*(t) = \tau_1\Psi(S(t)) + \tau_2\Psi(E_1(t)) + \tau_3\Psi(E_2(t)) + \tau_4\Psi(I(t)) + \tau_5\Psi(R(t))$$

where $\tau_i > 0, i = 1, 2, ..., 5$. The fractional derivative of $\mathcal{L}^*$ with respect to t is

$$D^\alpha \mathcal{L}^*(t) = D^\alpha \Big(\tau_1 \Psi(S(t)) + \tau_2 \Psi(E_1(t)) + \tau_3 \Psi(E_2(t)) + \tau_4 \Psi(I(t)) + \tau_5 \Psi(R(t))\Big)$$

Utilizing (3), we have

$$D_t^\alpha \mathcal{L}^*(t) \leq \tfrac{S-S^*}{S} D^\alpha S(t) + \tfrac{E_1-E_1^*}{E_1} D^\alpha E_1(t) + \tfrac{E_2-E_2^*}{E_2} D^\alpha E_2(t) + \tfrac{I-I^*}{I} D^\alpha I(t) + \tfrac{R-R^*}{R} D^\alpha R(t)$$

After analyzing the function $D_t^\alpha \mathcal{L}^*(t)$, we deduce that $D_t^\alpha \mathcal{L}^*(t) \leq 0$ along Ξ^* if and only if $R_0 \geq 1$. If the function $S(t) = S^*, E_1(t) = E_1^*, E_2(t) = E_2^*, I(t) = I^*, R(t) = R^*$, then the maximum value of the operator $D_t^\alpha \mathcal{L}^*(t)$ is also zero. Consequently, the global stability at Ξ^* adheres to the LaSalle invariance principle [17] if $R_0 \geq 1$. Therefore, Ξ^* is globally attractive. Combined with the local stability of Ξ^*, it can be seen that Ξ^* is globally asymptotically stable.

6 Simulations and Analysis of the Malware Propagation Model in WSNs

In this section, based on its capabilities, we will use the FABM method to investigate stability and confirm the obtained theoretical results.

Case 1. Assume that $b = 0.4$, $\beta = 0.001$, $\lambda_1 = 0.002$, $\lambda_2 = 0.001$, $\sigma = 0.001$, $\gamma = 0.003$, $\omega = 0.004$, $r = 1$, $L = 12$, and $p = 0.7$. The values of $R_0 = 0.2691$ for $\alpha = 1$ and $R_0 = 0.1019$ for $\alpha = 0.80$ can be computed using (18). When $R_0 < 1$, the various sensor nodes are displayed in Fig. 2. The number of Susceptible nodes decreases as the number of nodes in other categories rises in WSN, according to the analysis of Fig. 2 for case 1. However, when $R_0 < 1$, the number of I, E_1, E_2, and R nodes gradually starts to decline and gets closer to zero. The Infected nodes in the WSN eventually vanish. As a result, the malware is eliminated from the WSN automatically. Based on Fig. 2, all solutions converge to Ξ_0 as $t \to \infty$ for case 1. Therefore, according to the graphical method, this fulfills Theorems 3 and 4 and shows that for $R_0 < 1$, the WSN will be both locally and globally stable.

Case 2. Let us to consider $b = 0.4$, $\beta = 0.002$, $\lambda_1 = 0.002$, $\lambda_2 = 0.001$, $\sigma = 0.0001$, $\gamma = 0.003$, $\omega = 0.004$, $r = 1.5$, $L = 10$, and $p = 0.7$. The values of $R_0 = 4.1795$ for $\alpha = 1$ and $R_0 = 1.4695$ for $\alpha = 0.80$ can both be calculated using (18). The various sensor nodes are shown in Fig. 3 when $R_0 > 1$. The malware will consistently be present in WSN for case 2, as seen in Fig. 3. The solutions $R(t), I(t), E_1(t)$, and $E_2(t)$ for the case 2 are clearly visible in Fig. 3. These solutions move away from Ξ_0 as $t \to \infty$. Consequently, the graphical method complies with Theorem 3, which states that the WSN will become unstable when $R_0 > 1$. Under these circumstances, malware will propagate within the WSN;

therefore, malware continuously persists in the system. Theorems 5 and 6 are validated by this result. It is observed that the number of Infected nodes rises quickly in the early stages, then begins to decrease before stabilizing. Malware is still spreading among network nodes, demonstrating how infection and recovery rates affect malware propagation in WSNs. This is also consistent with the main conclusion drawn from Theorems 5 and 6, which states that the system is finally at the endemic equilibrium point Ξ^*.

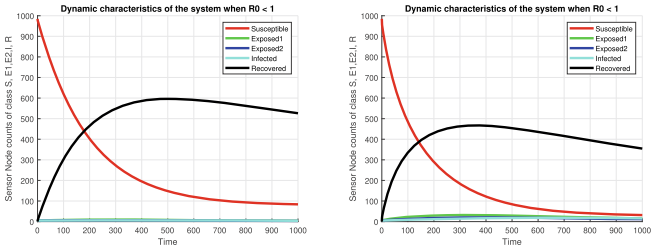


Fig. 2. Status Variables when $R_0 < 1$ (Left: $\alpha = 1$, and Right: $\alpha = 0.80$).

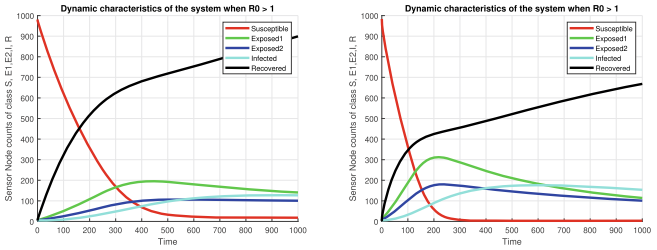


Fig. 3. Status variables when $R_0 > 1$ (Left: $\alpha = 1$, and Right: $\alpha = 0.80$).

6.1 Analyzing the Effect of Fractional Derivative α

Using the following parameter values, we examine how the fractional derivative order α affects the system when $R_0 < 1$. Let us to consider $b = 0.4$, $\beta = 0.001$, $\lambda_1 = 0.002$, $\lambda_2 = 0.001$, $\sigma = 0.001$, $\gamma = 0.003$, $\omega = 0.004$, $r = 1$, $L = 12$, and $p = 0.7$ (Case 1). The dynamic relation between time and S, E_1, E_2, I, and R nodes is depicted in Fig. 4. Figure 4 makes it clear that, for a range of α values, the status variables connected to different nodes in WSN decrease over time. Furthermore, the image makes it clear that, in contrast to a lesser value of α, the malware leaves the WSN very quickly as α converges to 1. Observing the

relation between time and Recovered nodes shows that recovery speed reduces as α gets closer to 1. As illustrated in Fig. 4, all solutions converge to Ξ_0 as $t \to \infty$ when α converges to 1. The WSN will, therefore, show both local and global asymptotic stability for a range of values of α when $R_0 < 1$, according to the graphic approach, which concludes that the Theorems 3 and 4 are satisfied. Additionally, Fig. 4 clarifies that the FABM method yields positive answers and converges toward its accurate steady state.

We also investigate how the system is affected when $R_0 > 1$ by the fractional derivative order α. The parameter values used in this investigation are $b = 0.4$, $\beta = 0.002$, $\lambda_1 = 0.002$, $\lambda_2 = 0.001$, $\sigma = 0.0001$, $\gamma = 0.003$, $\omega = 0.004$, $r = 1.5$, $L = 10$, and $p = 0.7$ (Case 2). The dynamic relationship between time and S, E_1, E_2, I, and R nodes is shown in Fig. 5. For different values of α, Fig. 5 shows that the number of S nodes in WSN declines with time. However, given varying values of α, the number of E_1, E_2, I, and R nodes increases over time. Moreover, Fig. 5 clarifies that the malware stays on the system indefinitely as α gets closer to 1. The malware will spread throughout the WSN in this situation. The number of Recovered nodes rises as α gets closer to 1, according to the correlation between time and the number of Recovered nodes. As seen in Fig. 5, all solutions that do not converge to Ξ_0 as $t \to \infty$ do so when α goes to 1. As $t \to \infty$ for varying values of α, the four solutions $R(t), I(t)$, $E_1(t)$, and $E_2(t)$ move away from Ξ_0. The graphic approach thus leads to the conclusion that Theorem 3 is satisfied, meaning that when $R_0 > 1$, the WSN will become unstable for different values of α. In this situation, malware spreads in the WSN, where it continuously persists. This result validates Theorems 5 and 6. In this case, the FABM technique also converges toward its accurate steady state and provides positive solutions, as seen in Fig. 5.

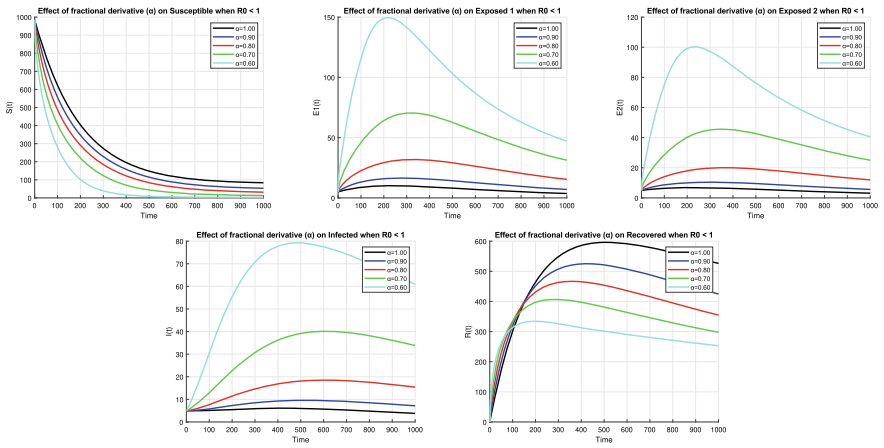


Fig. 4. Analyzing the effect of α when $R_0 < 1$.

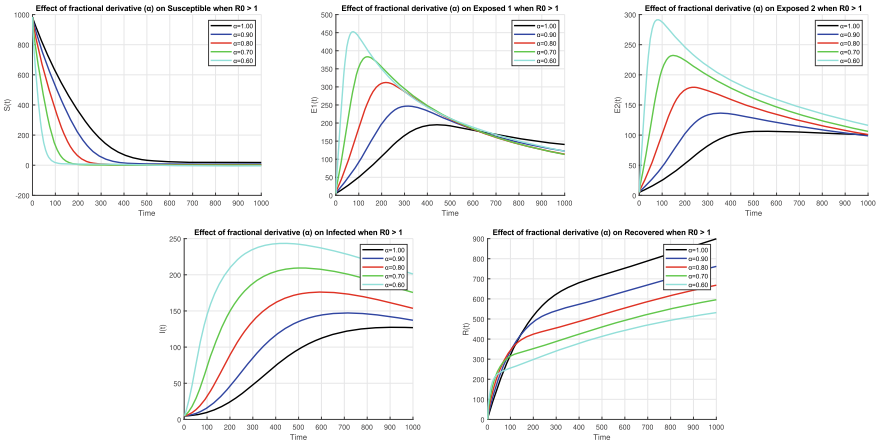


Fig. 5. Analyzing the effect of α when $R_0 > 1$.

7 Conclusion

This work illustrated the applicability of fractional derivatives in investigating the propagation dynamics of multi-malware infections in WSNs using epidemiological modeling. The equilibrium points for a malware-free state have been identified, and R_0 has been calculated to assess the system dynamics and analyze the stability requirements of the system. The analysis has demonstrated that when $R_0 < 1$, the WSN will be stabilized in a malware-free equilibrium; however, if $R_0 > 1$, malware will continue to exist in the WSN. The network's stability has been analyzed under various situations, with the corresponding theorems and their proofs provided. The FABM approach was employed to confirm the theoretical findings and identify the problem of malware attacks. Simulation results indicated that the proposed fractional-order propagation model outperforms the integer-order propagation one.

Acknowledgements. This work was partially supported by SERICS (PE00000014) under PNRR MUR program funded by EU-NGEU.

References

1. Awasthi, S., et al.: An epidemic model for the investigation of multi-malware attack in wireless sensor network. IET Commun. **17**(11), 1274–1287 (2023). https://doi.org/10.1049/cmu2.12622
2. Brauer, F., Castillo-Chavez, C., Feng, Z.: Mathematical Models in Epidemiology. TAM, vol. 69. Springer, New York (2019). https://doi.org/10.1007/978-1-4939-9828-9

3. Conte, D., Farsimadan, E., Moradi, L., Palmieri, F., Paternoster, B.: Time-delay fractional optimal control problems: a survey based on methodology. In: Abdel Wahab, M. (ed.) Proceedings of the 8th International Conference on Fracture, Fatigue and Wear, pp. 325–337. Springer Singapore, Singapore (2021). https://doi.org/10.1007/978-981-15-9893-7_23
4. Conte, D., Farsimadan, E., Moradi, L., Palmieri, F., Paternoster, B.: A galerkin approach for fractional delay differential equations using hybrid chelyshkov basis functions. In: Gervasi, O. (ed.) Computational Science and Its Applications – ICCSA 2022, pp. 130–142. Springer International Publishing, Cham (2022). https://doi.org/10.1007/978-3-031-10522-7_10
5. Conte, D., Farsimadan, E., Moradi, L., Palmieri, F., Paternoster, B.: Numerical solution of delay volterra functional integral equations with variable bounds. Appl. Numer. Math. **200**, 148–161 (2024). https://doi.org/10.1016/j.apnum.2023.05.002
6. De Santis, A., Farsimadan, E., Moradi, L., Palmieri, F.: Generalized defensive modeling of fake news propagation in social networks using fractional differential equations. IEEE Trans. Comput. Soc. Syst. **12**(2), 622–634 (2025). https://doi.org/10.1109/TCSS.2024.3492097
7. Dong, N.P., Giang, N.L., Long, H.V.: Interconnected takagi-sugeno system and fractional sirs malware propagation model for stabilization of wireless sensor networks. Inf. Sci. **670**, 120620 (2024). https://doi.org/10.1016/j.ins.2024.120620
8. Farsimadan, E., Moradi, L., Conte, D., Paternoster, B., Palmieri, F.: Comparison between protein-protein interaction networks $cd4^+t$ and $cd8^+t$ and a numerical approach for fractional HIV infection of $cd4^+t$ cells. In: Gervasi, O. (ed.) Computational Science and Its Applications – ICCSA 2021, pp. 78–94. Springer International Publishing, Cham (2021). https://doi.org/10.1007/978-3-030-86653-2_6
9. Farsimadan, E., Moradi, L., Palmieri, F.: A general study on the malware propagation models in wireless sensor networks. In: Gervasi, O. (ed.) Computational Science and Its Applications – ICCSA 2024 Workshops, pp. 83–99. Springer Nature Switzerland, Cham (2024). https://doi.org/10.1007/978-3-031-65223-3_6
10. Frutos-Bernal, E., Rodríguez-Rosa, M., Anciones-Polo, M., Martín-del Rey, A.: Analyzing malware propagation on wireless sensor networks: a new approach using queueing theory and HJ-biplot with a sirs model. Mathematics **12**(1) (2024). https://doi.org/10.3390/math12010135
11. Geng, L., Zheng, H., Qiao, G., Geng, L., Wang, K.: Online public opinion dissemination model and simulation under media intervention from different perspectives. Chaos, Solitons Fractals **166**, 112959 (2023). https://doi.org/10.1016/j.chaos.2022.112959
12. Graef, J.R., Kong, L., Ledoan, A., Wang, M.: Stability analysis of a fractional online social network model. Math. Comput. Simul. **178**, 625–645 (2020). https://doi.org/10.1016/j.matcom.2020.07.012
13. Lin, W.: Global existence theory and chaos control of fractional differential equations. J. Math. Anal. Appl. **332**(1), 709–726 (2007). https://doi.org/10.1016/j.jmaa.2006.10.040
14. Mishra, J., Srivastava, P.K., Srivastava, V.: Securing wireless sensor network against multi-malware threats: a comprehensive modeling and analysis approach. SN Comput. Sci. **6**(3), 236 (2025)
15. Moradi, L., Farsimadan, E., D'Angelo, G., Carpentieri, B., Palmieri, F.: Epidemic analysis of the propagation of multiple malware infectious in wireless sensor networks. In: Barolli, L. (ed.) Advanced Information Networking and Applications, pp. 140–148. Springer, Cham (2025). https://doi.org/10.1007/978-3-031-87784-1_13

16. Odibat, Z.M., Shawagfeh, N.T.: Generalized Taylor's formula. Appl. Math. Comput. **186**(1), 286–293 (2007). https://doi.org/10.1016/j.amc.2006.07.102
17. LaSalle, J.P.: The Stability of Dynamical Systems. SIAM (1976). https://doi.org/10.1137/1.9781611970432
18. Martín-del Rey, A.: A novel model for malware propagation on wireless sensor networks. Math. Biosci. Eng. **21**(3), 3967–3998 (2024). https://doi.org/10.3934/mbe.2024176
19. Shi, X., Luo, A., Chen, X., Huang, Y., Huang, C., Yin, X.: The dynamical behaviors of a fractional-order malware propagation model in information networks. Mathematics **12**(23) (2024). https://doi.org/10.3390/math12233814
20. Verma, C., Gupta, C.P.: Effect of vaccination on stability of wireless sensor network against malware attack: an epidemiological model. SN Comput. Sci. **5**(2), 240 (2024)
21. Zhou, Y., Liu, B.T., Zhou, K., Shen, S.F.: Malware propagation model of fractional order, optimal control strategy and simulations. Front. Phys. **11** (2023). https://doi.org/10.3389/fphy.2023.1201053
22. Zhou, Y., Wang, Y., Zhou, K., Shen, S.F., Ma, W.X.: Dynamical behaviors of an epidemic model for malware propagation in wireless sensor networks. Front. Phys. **11** (2023). https://doi.org/10.3389/fphy.2023.1198410