

On the Use of Range Volatility Estimators for Rough Fractional Stochastic Volatility Model

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Abstract. Financial models have faced challenges in accurately reproducing the dynamics of volatility. In particular, neither local volatility nor traditional stochastic volatility models are able to simultaneously produce realistic dynamics for the implied volatility surface and achieve a satisfactory fit to its shape. As a result, a new approach has emerged in mathematical finance, called rough volatility models, where the volatility dynamics of financial assets are modeled as a function of fractional Brownian motion with a Hurst exponent H smaller than one half. Rough volatility models are indeed capable of replicating the intended stylized facts using only a limited number of parameters. The Hurst index is estimated using realized volatility measurements as a proxy for the spot volatility. However, several authors argue that this estimator for volatility may be noisy due to the influence of market microstructure noise and that obtaining the necessary data for its calculation is very challenging, as it requires high-frequency data. Alternative proxies, novel in the context of rough volatility estimation, are proposed in this study (This study stems from a master's dissertation archived in the Scientific Repository of the University of Coimbra [1].) for the unobserved volatility process. The chosen proxies are range-based estimators: Open-to-Close, Rogers and Satchell, Parkinson, Garman and Klass, and Meilijson. These proxies will allow us to better determine whether volatility exhibits rough behavior and whether it can continue to reproduce the stylized facts, even in a low-frequency data framework. In addition, we will evaluate which of these estimators are best suited for this purpose.

Keywords: Rough Volatility · Fractional Brownian Motion · Range-based Estimators · Volatility Forecasting

1 Introduction

In financial mathematics, there is a constant effort to predict the uncertainty of future market behavior, despite it being a very challenging task, if not impossible.

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Volatility, in particular, is well-known for its significant repercussions, especially during financial crises, due to its central role in risk management, asset pricing, and portfolio allocation. Despite these challenges, significant achievements have been made in developing volatility models that capture market dynamics with increasing accuracy.

This study focus on a notable class of models called rough volatility models. This class of volatility models has gained prominence due to their superior ability to fit both historical and implied volatility, capable of reproducing stylized facts of financial markets. These models are characterized by their irregular paths, achieved by replacing the standard Brownian motion in volatility models with fractional Brownian motion (fBm), where the Hurst exponent is smaller than one half.

To estimate the Hurst parameter, rough volatility models require proxies for spot volatilities because volatility is not directly observed. For this purpose, most of these models, including the pioneering work of Gatheral *et al.* [2], which introduced the Rough Fractional Stochastic Volatility (RFSV) model, used realized volatility. However, realized volatility requires high- or ultra-high-frequency observations, which are often difficult to obtain. In a low-frequency data context, such approximations are not feasible. On the other hand, some authors claim that this proxy is subject to microstructure effects and turns out to be noisy. This suggests that the rough behavior of volatility could be influenced by the choice of realized volatility as a proxy for spot volatility.

There are several variations within this class of volatility models. In this paper we will focus on estimating the RFSV model using the Hurst parameter estimation method introduced in Gatheral *et al.* [2]. This model is recognized as the pioneering rough volatility model that laid the foundation for subsequent developments in the field, and is notable for its original contributions and wide acceptance in the literature. Furthermore, it is particularly well suited to effectively capture essential stylized facts. We aim to help answer the question: could rough volatility be an artifact of high-frequency return data? To address this question, we propose alternative proxies for the unobserved volatility, named range-based estimators. These estimators are shown to be robust to microstructure noise and efficient even with low-frequency data.

This paper is structured into four sections more, as outlined below. Section 2 covers fundamental concepts required to comprehend rough volatility models, introduces the RFSV model and presents the concepts of realized variance and range-based variance estimators. Section 3 focuses on empirical calculations, diagnostics, and interpretation of the results. In Sect. 4, we forecast volatility and evaluate the accuracy of each model. Lastly, in Sect. 5 the main conclusions of this study are summarized.

2 Methodology

Volatility estimation has been a central topic in financial mathematics literature for the past several decades. It is imperative to acknowledge the pivotal role of

Brownian motion when discussing volatility estimation. Indeed, the estimation of volatility has long been intricately associated with Brownian motion, particularly in Stochastic Volatility (SV) models, due to its ability to capture the stochastic nature of volatility. The Brownian motion is based on a phenomenon observed by Robert Brown in 1827, who noticed a ubiquitous, irregular motion of pollen grains suspended in water. Wiener rigorously formalized Brownian motion, establishing a framework for continuous stochastic processes, which is why Brownian motion is also referred as Wiener process [3].

Another strand of literature that has been heavily used in the field of financial mathematics is Stochastic Volatility models, which treat volatility as a time-varying process, more precisely, a continuous Brownian semi-martingale [2]. Gatheral *et al.* empirically demonstrated that log-volatility behaves essentially as a fractional Brownian motion with a Hurst exponent of approximately 0.1, indicating that volatility is rough [2]. They estimated the Hurst exponent using daily realized variance estimates as proxies for daily spot volatility. This led to the development of the Rough Fractional Stochastic Volatility (RFSV) model, which is characterized by a Hurst exponent $H < 1/2$.

Key concepts required to establish a comprehensive foundation for the RFSV model are detailed next.

2.1 Basic Concepts

Monofractal Scaling Property. The term *fractal* was introduced by Mandelbrot in 1975. A fractal can be defined as a "shape made of parts that are similar to the whole in some way" [4]. This section provides a concise overview of a class of self-affine processes called monofractals. Let $X(t)$ be a random process where $X(0) = 0$. Then $X(t)$ is considered self-affine, with parameter H if it satisfies the scaling rule

$$X(ct) \overset{d}{=} c^H X(t),$$

where $0 < H < 1$ is the Hurst exponent, and c and t are positive [5]. The Hurst exponent, also referred to as the self-affinity index, characterizes the long-term memory properties of time series.

The multiscaling properties of the self-affine function $X(t)$ can be investigated by calculating the structure function, which is defined as the q-th order moments of the increments $X(t + \Delta) - X(t)$:

$$S_q(\Delta) = \mathbb{E}\left[|X(t + \Delta) - X(t)|^q\right] \sim \Delta^{\zeta(q)},$$

where $q > 0$ and ζ_q is the scale invariant structure function exponent [6].

A self-affine process with a single Hurst exponent has $\zeta(q) = H_q$. These processes are called monofractal or uniscaling because $\zeta(q)$ scales as a linear function of order q [7]. Note that in these processes, H_q changes continuously with q, and H can be easily obtained from the slopes of log-log plots of S_p versus Δ for any q.

Fractional Brownian Motion. In 1923, Wiener defined ordinary Brownian motion W_t, also known as the Wiener process, where $t \in \mathcal{R}$ designates time, as a stochastic process that satisfies the following properties [3]: W_t is a continuous function of time with $W_0 = 0$; the increments $W_{t_2} - W_{t_1}$ are Gaussian with mean zero and variance $|t_2 - t_1|$, which implies that the increments are stationary; the increments over disjoint time intervals are independent for all $t_1 < t_2 < t_3 < t_4 < \ldots < t_m$.

Given the limitations of the standard Brownian motion model in capturing certain dynamics of financial markets, more sophisticated models have emerged. One of the most notable is fractional Brownian motion (fBm). Mandelbrot and Van Ness defined fBm, denoted by $\{W_t^H\}_{t\geq 0}$, in terms of Wiener integrals by

$$W_t^H = \int_0^t (t-s)^{H-1/2} dW_s + \int_{-\infty}^0 (t-s)^{H-1/2} - (-s)^{H-1/2} dW_s,$$

where $W_0^H = 0$ and $\{W_s\}_{s\in\mathcal{R}}$ denotes the standard Brownian motion, with $0 < H < 1$ as the Hurst parameter [8]. Note that if $H = 1/2$, the fractional Brownian motion W_t^H reduces to the standard Brownian motion.

A fBm, characterized as a centered Gaussian process $\{W_t^H\}_{t\geq 0}$ with parameter $0 < H < 1$, exhibits the following covariance function [9]:

$$\mathrm{E}\left[W_t^H W_s^H\right] = \frac{1}{2}\left(t^{2\,H} + s^{2\,H} - |t-s|^{2\,H}\right). \tag{1}$$

The covariance of increments, known as fractional Gaussian noise (fGn), is expressed based on Eq. (1) as follows:

$$\begin{aligned}\mathrm{E}\left[\left(W_{t_1}^H - W_{s_1}^H\right)\left(W_{t_2}^H - W_{s_2}^H\right)\right] = \\ = \frac{1}{2}\left(|t_1 - s_2|^{2\,H} + |t_2 - s_1|^{2\,H} - |t_2 - t_1|^{2\,H} - |s_2 - s_1|^{2\,H}\right).\end{aligned} \tag{2}$$

Based on Eq. (2), it is concluded that the increments are stationary, as they are Gaussian, centered, and share the same covariance structure.

Another significant property of fractional Brownian motion is self-affinity, more specifically, H-self-affinity. It can be observed that, for a fixed $a > 0$, the process $Z_t = W_{at}^H$ has the same covariance structure, and therefore the same distribution, as $a^H W_t^H$.

The fBm for $H \in (0, 1/2)$ is antipersistent, meaning that past increments are negatively correlated. On the other hand, the fBm for $H \in (1/2, 1)$ exhibits persistence, where past increments are positively correlated.

In the case where $H > 1/2$, the increments of the fBm exhibit long memory, in the sense that they display a slow decay in the autocorrelation function [2]:

$$\sum_{k=0}^{+\infty} \mathrm{Cov}\left[W_1^H, W_k^H - W_{k-1}^H\right] = +\infty.$$

Hence, the parameter H characterizes the long-memory properties of the data, as the autocorrelation function of the increments is of order k^{2H-2} as k approaches infinity.

As mentioned previously, the increments of fBm exhibit the properties of self-affinity and stationarity; therefore:

$$\mathbb{E}\left[\left|W_t^H - W_{t+\Delta}^H\right|^q\right] = K_q \Delta^{qH},$$

where K_q is the moment of order q of the absolute value of a standard Gaussian variable.

Realized Variance. Realized volatility gained significant recognition with the pioneering work of Andersen and Bollerslev [10], along with the growing availability of high-frequency data.

For simplicity, in addressing realized volatility, a non-parametric, model-free estimator, we assume a continuous-time diffusive setting, ignoring price jumps, and a frictionless market. Under these assumptions, the asset's logarithmic price process p is a semi-martingale to prevent arbitrage opportunities [11]. Its definition is given by

$$dp_t = \mu_t dt + \sigma_t dW_t, \quad 0 \leq t \leq T,$$

where μ_t is a predictable drift process, $\sigma_t > 0$ denotes a predictable spot volatility process and W_t is a standard Brownian motion. It is assumed that $E(\int_0^t \sigma_s^2 ds) < \infty$ [12]. The continuously compounded return throughout the period from $t-k$ and t, where $0 < k \leq t$, can be expressed as [12]:

$$r(t,k) = p(t) - p(t-k) = \int_{t-k}^{t} \mu_\tau \, d\tau + \int_{t-k}^{t} \sigma_\tau \, dW_\tau.$$

Thus, its quadratic variation (QV) is given by

$$QV(t,k) = \int_{t-k}^{t} \sigma^2(\tau) d\tau. \tag{3}$$

The expression for the realized variance (RV) of the logarithmic price process is given by

$$RV(t,k;n) = \sum_{j=1}^{n \cdot k} r\left(t-k+\frac{j}{n}, \frac{1}{n}\right)^2,$$

where $\{t-k+\frac{j}{n}, j = 1, ..., n \cdot k\}$ is a partition of the interval $[t-k, t]$ [12].

Due to the semimartingale properties, we have

$$RV(t,k;n) \to QV(t,k) \text{ as } n \to \infty.$$

The realized variance provides remarkable measures of actual market volatility [12]. However, although high-frequency data is becoming more accessible, it may not be available for some assets or only available for shorter time periods.

Range-Based Variance Estimators. Traditional estimators based solely on daily closing prices are often imprecise, which led to the development of alternative methods to improve accuracy. One notable approach is range-based volatility estimators, which estimate the volatility of a financial asset by considering the range of prices within a given time period, including the high, low, open, and close prices [13].

Let's assume that the price P follows a geometric Brownian motion and the log-price $p = \ln(P)$ follows a Brownian motion with zero drift and a diffusion parameter σ [14], i.e.,

$$dp_t = \sigma dB_t. \tag{4}$$

Let O_t represent the opening price, C_t the closing price, H_t the highest price, and L_t the lowest price of the day t [14]. The returns for open-to-close, open-to-high, and open-to-low are calculated by

$$c_t = \ln(C_t) - \ln(O_t),$$

$$h_t = \ln(H_t) - \ln(O_t),$$

$$l_t = \ln(L_t) - \ln(O_t).$$

The daily return c is a random variable drawn from a normal distribution with zero mean and variance σ^2. Since c^2 is an unbiased estimator of σ^2, the first range-based variance estimator [14], which in this work we will refer to as Open-to-Close variance for identification purposes, is given by

$$\widehat{\sigma_s^2} = c_t^2. \tag{5}$$

Since this estimator is highly noisy, Parkinson proposes the following estimator for variance [15]:

$$\widehat{\sigma_P^2} = \frac{(h_t - l_t)^2}{4\ln 2}. \tag{6}$$

This estimator, now known as the Parkinson estimator, was the first to formalize the use of the range for volatility estimation [14].

Inspired by Parkinson's work, Garman and Klass [13] introduced an estimator that incorporates not only the high-low range but also the closing price, resulting in a more efficient estimator:

$$\widehat{\sigma_{GK}^2} = 0.511(h_t - l_t)^2 - 0.019\left(c_t(h_t + l_t) - 2h_t l_t\right) - 0.383c_t^2.$$

Garman and Klass also proposed a simplified minimum variance analytical estimator, given by:

$$\widehat{\sigma_{GK}^2} = 0.5(h_t - l_t)^2 - (2\ln 2 - 1)c_t^2. \tag{7}$$

Meilijson introduced a different estimator with a smaller variance than the Garman and Klass estimator [16]. This estimator is formulated as

$$\widehat{\sigma_M^2} = 0.274\sigma_1^2 + 0.16\sigma_s^2 + 0.365\sigma_3^2 + 0.2\sigma_4^2 \tag{8}$$

knowing that

$$\sigma_1^2 = 2\left[(h'_t - c'_t)^2 + (l'_t)^2\right],$$
$$\sigma_3^2 = 2\,(h'_t - c'_t - l'_t)\,c'_t,$$
$$\sigma_4^2 = -\frac{(h'_t - c'_t)\,l'_t}{2\ln 2 - 5/4},$$

where if $c_t > 0$, then $c'_t = c_t$, $h'_t = h_t$, and $l'_t = l_t$; and if $c_t < 0$, then $c'_t = -c_t$, $h'_t = -l_t$, and $l'_t = -h_t$. This condition implies that two different formulas are applied depending on whether c_t is positive or negative.

Rogers and Satchell proposed the following estimator that accommodates arbitrary drift [17]:

$$\widehat{\sigma^2_{RS}} = h_t(h_t - c_t) + l_t(l_t - c_t). \tag{9}$$

There are other estimators, but in this work, we address these due to their significant contributions to the literature and the notorious practical applications in improving volatility estimation accuracy.

2.2 Rough Fractional Stochastic Volatility

Background and Motivation. One of the most recognized stochastic volatility (SV) models is the Hull and White model which provides an option pricing framework where the volatility of the underlying asset is time-varying and correlated with a specific risk parameter [18]. Moreover, the long-memory property of the volatility process has been widely recognized as a stylized fact [19]. Comte and Renault expanded the Hull and White model to capture volatility persistence [20]. They proposed a continuous-time long-memory SV model, called Fractional Stochastic Volatility (FSV), by replacing the standard Brownian motion in the original model with a fBm with a Hurst exponent $H > 1/2$.

The concept of long memory in volatility, initially described as a slow decay of autocorrelation, slower than exponential decay, has become more precise, meaning that the autocorrelation function is not integrable or, more specifically, that it decays as a power law with an exponent less than 1 [2].

Another challenge that volatility models face is accurately replicating the volatility skew observed in the market [21]. Let $\sigma_{\mathrm{BS}}(k,\tau)$ denote the implied volatility, with log-moneyness k defined as $k = \ln\left(\frac{K}{S}\right)$, where K is the strike price, S is the current price of the underlying asset, and τ represents the time to maturity. The term structure of at-the-money (ATM) volatility skew is defined by [2]:

$$\psi(\tau) = \left|\frac{\partial}{\partial k}\sigma_{\mathrm{BS}}(k,\tau)\right|_{k=0}.$$

Fukasawa demonstrated that a SV model, in which volatility is driven by a fBm with Hurst exponent H, generates an ATM volatility skew of the form $\psi(\tau) \sim \tau^{H-1/2}$, at least for a small time to expiration τ [22]. To produce a volatility skew consistent with the market observations, the value of H must be

close to zero, as the skew term structure in financial markets behaves approximately as $\tau^{-1/2}$ for small τ. Gatheral *et al.* calculated non-parametric estimates of the S&P 500 index ATM volatility skews for a single day and verified that these estimates followed a power-law pattern fitting the form $\psi(\tau) = A\tau^{-0.4}$ [2].

Motivated by these observations, Gatheral *et al.* developed the Rough FSV (RFSV) model, considered a particular case of the FSV model that captures the roughness of the volatility process, rather than focusing on traditional long-memory behavior. The RFSV model, characterized by a Hurst exponent $H < 1/2$, reflects the irregular nature of the volatility process [2]. Their findings suggest that the dynamics of log-volatility are close to that of a fBm due to the Gaussianity and scaling properties of the increments of log-volatility, at any reasonable time scale. They estimated the Hurst parameter H to be approximately between 0.06 and 0.20, with a typical value around 0.1.

The Scaling Property and Normality. This section details the method proposed in [2] to estimate the Hurst exponent H of the RFSV model.

Assume access to discrete observations of the volatility process. For $q \geq 0$, the q-order moment of a log-volatility increment is defined as:

$$m(q, \Delta) = \frac{1}{N} \sum_{k=1}^{N} \left| \log\left(\sigma_{k\Delta}\right) - \log\left(\sigma_{(k-1)\Delta}\right) \right|^q,$$

where $\sigma_0, \sigma_\Delta, \ldots, \sigma_{k\Delta}, \ldots, \quad k \in \{0, N\}$, are discrete observations of the volatility process and $N = \lfloor T/\Delta \rfloor$ represents the total number of observations in the time interval $[0, T]$.

The fact that the Hurst parameter is considered a smoothness parameter arises from the following assumption that for some $s_q > 0$ and $b_q > 0$,

$$N^{qs_q} m(q, \Delta) \to b_q \tag{10}$$

as Δ tends to zero [2]. Rosenbaum demonstrated that s_q can be interpreted as the regularity of the volatility when measured in l_q norm [23]. If the logarithm of the volatility follows a fBm with Hurst parameter H, Eq. (10) hold in probability with $s_q = H$.

It is not possible to compute $m(q, \Delta)$ precisely, since the volatility process is not directly observable. Therefore, a proxy for spot volatility is required. Gatheral *et al.* applied realized volatility for this purpose. Assuming that the increments of the log-volatility process are stationary and that the law of large numbers applies, $m(q, \Delta)$ is an unbiased estimator of $\mathbb{E}\left[\left|\log\left(\sigma_\Delta\right) - \log\left(\sigma_0\right)\right|^q\right]$, and can consequently be viewed as its empirical counterpart. As a result, Gatheral *et al.* empirically discovered the following monofractal scaling relationship:

$$\mathbb{E}\left[\left|\log\left(\sigma_\Delta\right) - \log\left(\sigma_0\right)\right|^q\right] = b_q \Delta^{\zeta_q}, \tag{11}$$

where b_q denotes a constant and $\zeta_q = qs_q$ is the slope.

Additionally, Gatheral *et al.* made the important observation that the smoothness parameter s_q does not seem to depend on q. In fact, Gatheral *et al.*

revealed a linear relationship $\zeta_q \sim Hq$ by plotting ζ_q against q, with estimated H ranging between 0.06 and 0.20, leading to the conclusion that $H < 1/2$.

Andersen *et al.* reported in the early 2000s the well-established stylized fact that the distribution of the log-volatility process is very close to a Gaussian distribution [19]. Gatheral *et al.* confirmed that, for any Δ, the empirical distribution of log-volatility increments closely resembles a Gaussian distribution and that the 1-day fit of the normal density, scaled by Δ^H, aligns even more closely with the Gaussian distribution [2].

Model Specification. The increments of the log-volatility show a consistent scaling property with a constant smoothness parameter, and their distribution closely resembles a Gaussian distribution. Thus, a simple model is first proposed by [2]:

$$\log \sigma_{t+\Delta} - \log \sigma_t = v \left(W^H_{t+\Delta} - W^H_t \right), \tag{12}$$

where $t \geq 0$, W^H is a fBm with Hurst parameter H and v is a positive constant. The volatility can be expressed in exponential form:

$$\sigma_t = \sigma \exp\left(v W^H_t \right), \tag{13}$$

where σ is a positive constant.

This model does not exhibit the required stationarity, which can complicate both the mathematical analysis and the modeling process. To address this issue, the log-volatility is instead modeled as a fractional Ornstein-Uhlenbeck (fOU) process with a very long reversion timescale. The fOU process is an extension of the classical stationary Ornstein-Uhlenbeck process, and it also has a stationary solution [24].

The fOU process X_t is defined through the stochastic differential equation:

$$\mathrm{d}X_t = v \, \mathrm{d}W^H_t - \alpha \left(X_t - m \right) \mathrm{d}t, \tag{14}$$

and given by its solution:

$$X_t = v \int_{-\infty}^{t} e^{-\alpha(t-s)} \mathrm{d}W^H_t + m, \tag{15}$$

where $m \in \mathcal{R}$ is the long-term mean and $\alpha \in \mathcal{R}^+$ is the velocity of mean reversion.

Thus, the RFSV model is expressed as:

$$\sigma_t = \exp\left(X_t \right), \tag{16}$$

where $t \in [0, T]$ and X_t defined by Eq. (15) and $H < 1/2$.

Gatheral *et al.* verified that log-volatility modeled as a fOU behaves like a fBm as α tends to zero, through the following propositions [2]:

$$\mathbb{E}\left[\sup_{t \in [0,T]} \left| X^\alpha_t - X^\alpha_0 - v W^H_t \right| \right] \to 0, \tag{17}$$

and

$$\mathbb{E}\left[\left|X^{\alpha}_{t+\Delta} - X^{\alpha}_{t}\right|^{q}\right] \to v^{q} K_{q} \Delta^{qH}. \tag{18}$$

From Eq. (17), it is observed that if $\alpha \ll 1/T$, i.e., α is much less than $1/T$, the log-volatility process can be assumed to behave as a fBm within the interval of interest $[0, T]$. Additionally, from Eq. (18), the scaling property of a fBm is approximately reproduced by the fOU process for very small α.

As mentioned earlier, the RFSV model is considered a particular case of the FSV model. The Stochastic Differential Equation (SDE) of the RFSV model, as expressed in Eq. (16), is similar to that of the FSV model, with the fundamental difference being the range of the Hurst parameter H, where $H > 1/2$ in the FSV model and $H < 1/2$ in RFSV model.

Fukasawa demonstrated that a volatility process driven by a fBm leads to a term structure of ATM volatility skew of the form $\psi(\tau) \sim \tau^{H-1/2}$, at least for small τ [22]. This implies that in the FSV model, where $H > 1/2$, the ATM volatility skew increases with time to expiration τ for small values of τ. However, this is inconsistent with empirical observations, where the ATM volatility skew typically shows a decreasing pattern as τ increases. In contrast, the RFSV model, with $H < 1/2$, aligns with the observed term structure of ATM volatility skew [2,22]. Furthermore, in the RFSV model, $\alpha \ll 1/T$, indicating that the dynamics of the volatility process exhibit a behavior approximated to that of a fBm, as referenced in Eqs. (17) and (18).

Gatheral *et al.* challenge the stylized fact that volatility exhibits long memory. The classical concept of long memory is characterized by an autocorrelation function that decays as a power law. However, Gatheral *et al.* demonstrated that empirically, and within the RFSV model, the autocovariance function does not decay as a power law [2]. To further test this concept, Gatheral *et al.* applied some standard statistical procedures commonly used in financial econometrics to both the data and the RFSV model simulations to identify long memory [2]. Notably, the results confirm the fragility of long-memory estimation processes, as they falsely indicated that the RFSV model exhibits long memory, highlighting that traditional methods may mistakenly identify long memory. A possible justification for these spurious results may be the strict modeling assumptions of these procedures (See Sect. 4 of [2]).

The innovative aspect of our approach lies in the proxies used for spot volatility. Gatheral *et al.* calculate the Hurst exponent H using realized volatility. In our study, however, we will calculate the Hurst exponent using range-based volatility estimators.

3 Empirical Study

3.1 Data and Preliminary Analysis

In order to estimate the parameters of the RFSV model, it is necessary to use a proxy for spot volatility values, as it is not possible to observe the volatility process directly. Gatheral *et al.* used daily realized volatility estimates [2].

We propose an alternative approach to estimating the parameters of the RFSV model by using range-based volatility estimators as proxies for spot volatility values, and evaluate and compare their performance. The range-based estimators include Open-to-Close volatility, Garman and Klass volatility, Parkinson volatility, Rogers and Satchell volatility and Meilijson volatility.

The dataset consists of nine stock indices, including the Standard & Poor's 500 (S&P500), Financial Times Stock Exchange 100 (FTSE100), Nikkei 225 (N225), Deutscher Aktienindex (DAX), All Ordinaries (AO), Cotation Assistée en Continu (CAC40), Swiss Market Index (SMI), Índice Bursátil Español 35 (IBEX35), and Nifty 50 (N50). The period covered in this analysis spans from April 19, 2001, to September 26, 2016.[1]

The daily realized variance estimates are available from the Oxford-Man Institute of Quantitative Finance Realized Library, with realized variance computed using 5-minute intraday intervals. The range-based estimators are based on daily price data, specifically the open, close, high, and low prices that were collected for that time period. The number of observations[2] is slightly higher for the range-based estimators than for the realized variance, with a notable difference in the Nifty 50 index, as shown in Table 1.

Table 1. Number of observations for each index.

Estimators/Index	SP500	CAC40	SMI	IBEX35	N50	FTSE100	N225	DAX	AO
Range-based estimators	3884	3951	3894	3923	3842	3898	3784	3933	3925
Realized volatility	3863	3935	3864	3904	3509	3877	3736	3908	3856

The statistics of each estimator are evaluated to better understand its behavior. These include the mean, median, maximum, minimum, standard deviation, skewness, excess kurtosis, and finally, autocorrelation (with one lag). The statistics for the Meilijson estimator and the realized volatility are presented in Tables 2 and 3, respectively. From these, we can draw interesting insights. The median values are lower than the means, suggesting a right-skewed distribution, also known as a positively skewed distribution of the data. Excess kurtosis values are notably high, indicating the presence of heavy tails in the volatility distribution. The autocorrelation (with one lag) is consistently positive across all estimators and indices, which indicates volatility clustering, another well-known stylized fact in financial markets [25]. Similar results are obtained for the remaining estimators. The normality test Jarque-Bera was performed, and for all estimators, the null hypothesis of normality was rejected with a p-value

[1] During this period, the indices included the following numbers of companies: S&P500 had 500 companies, FTSE100 had 100, Nikkei225 had 225, DAX had 30, All Ordinaries had 500, CAC40 included 40, SMI had 20, IBEX35 had 35, and Nifty 50 had 50 companies.

[2] The term "observations" refers to the number of daily variance values. We are not referring to the high-frequency data used to compute realized variance, which is based on intraday information.

Table 2. Descriptive statistics of the daily Meilijson volatility.

Statistic/Index	SP500	CAC40	SMI	IBEX35	N50	FTSE100	N225	DAX	AO
Mean	0.0077	0.0098	0.0079	0.0106	0.0107	0.0084	0.0082	0.0108	0.0055
Median	0.0063	0.0082	0.0064	0.0092	0.0088	0.0069	0.0072	0.0088	0.0047
Maximum	0.0854	0.0670	0.0877	0.0534	0.1256	0.0586	0.1066	0.0846	0.0466
Minimum	0.0012	0.0012	0.0014	0.0016	0.0017	0.0014	0.0010	0.0011	0.0008
Standard Dev.	0.0057	0.0063	0.0055	0.0062	0.0076	0.0057	0.0052	0.0074	0.0034
Skewness	3.6567	2.4387	3.3786	1.8840	4.5152	2.8019	4.3101	2.5514	2.8668
Excess Kurtosis	21.7898	7.0062	19.2253	3.0057	37.7419	10.2025	44.5314	8.0326	13.7861
Autocorrelation	0.6906	0.6841	0.6857	0.6817	0.6055	0.6047	0.5355	0.7174	0.6047

Table 3. Descriptive statistics of the daily realized volatility.

Statistic/Index	SP500	CAC40	SMI	IBEX35	N50	FTSE100	N225	DAX	AO
Mean	0.0086	0.0105	0.0081	0.0109	0.0105	0.0077	0.0093	0.0114	0.0061
Median	0.0071	0.0090	0.0066	0.0100	0.0088	0.0063	0.0082	0.0095	0.0052
Maximum	0.0965	0.0675	0.0529	0.0690	0.1005	0.0571	0.0619	0.0802	0.0394
Minimum	0.0015	0.0020	0.0026	0.0020	0.0016	0.0020	0.0025	0.0021	0.0011
Standard Dev.	0.0060	0.0061	0.0048	0.0059	0.0066	0.0050	0.0051	0.0070	0.0036
Skewness	3.5936	2.5244	2.7264	2.1519	3.6433	2.7175	3.1835	2.7694	2.6560
Excess Kurtosis	21.2837	8.4299	8.4967	7.2521	22.5662	9.3577	15.3124	10.3073	8.8260
Autocorrelation	0.8306	0.8183	0.8588	0.7970	0.7946	0.8405	0.7431	0.8422	0.7059

smaller than 2×10^{-16}, as expected due to the presence of skewness, heavy tails, and excess kurtosis.

3.2 Parameters Estimations

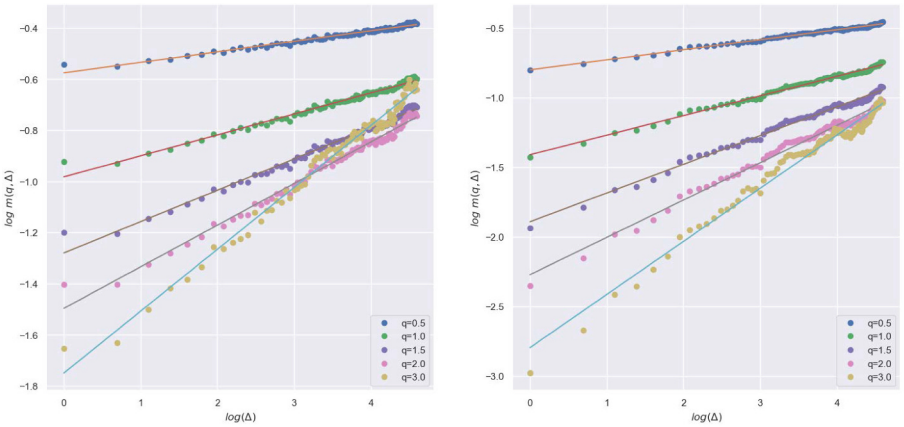
The parameters of the RFSV model are estimated by showing that log-volatility exhibits the scaling property (12) and follows a distribution similar to the normal distribution, using range-based estimators as proxies.

The computation of $m(q, \Delta)$ based on non-observable volatility is not possible in practice, so we use as proxies the volatility estimators calculated. We compute $m(q, \Delta)$ for different values of q. For illustration purposes, considering the Meilijson estimator and the realized variance, in Fig. 1, we observe that the values of $\log m(q, \Delta)$, plotted against $\log \Delta$ for each q, are closely clustered around their respective linear regression lines. The regression lines with lower slopes exhibit a better fit, where ζ_q represents the slope. Additionally, we note that the plots show quite different slopes for each estimator, however, all seem to exhibit the monofractal scaling property described in Eq. (11), leading to the estimation of H, as will be shown in the next step. Table 4 shows the slope ζ_q values. The Open-to-Close estimator shows the lowest values, which corresponds to lines almost horizontal in a plot $\log m(q, \Delta)$ against $\log \Delta$. The realized estimator shows higher values, indicating that it is more sensitive to changes in the time scale.

We can estimate parameter ν. Noting that $m(2, \Delta)$ can be seen as the empirical counterpart of $E\left[\left|\log(\sigma_{t+\Delta}) - \log(\sigma_t)\right|^2\right]$, the parameter ν is obtained from

Table 4. Values of ζ_q for each estimator for the S&P 500 index.

Estimators/q	0.5	1	1.5	2	3
Open-to-Close	0.010	0.0188	0.0261	0.0325	0.0432
Garman and Klass	0.0397	0.0798	0.1199	0.1600	0.2397
Parkinson	0.0406	0.0811	0.1213	0.1612	0.2400
Rogers and Satchell	0.0254	0.0481	0.0680	0.0851	0.1133
Meilijson	0.0412	0.0820	0.1225	0.1626	0.2419
Realized estimator	0.0720	0.1410	0.2063	0.2682	0.3818



(a) Meilijson estimator. (b) Realized estimator.

Fig. 1. $\log m(q, \Delta)$ as a function of $\log \Delta$, S&P 500.

Eq. (18), where, setting $q = 2$, we deduce that

$$m(2, \Delta) \approx \nu^2 \Delta^{2H}.$$

By analyzing Table 5, we observe that the Hurst exponent values vary across the proxies used, yet they consistently remain below 1/2. The parameter ν, often referred to as the "volatility of volatility," appears notably higher when employing the Open-to-Close estimator, indicating larger fluctuations in the variance process. This, combined with the lower value of H, suggests a model exhibiting more irregular behavior. Conversely, when realized volatility is used as the proxy, the model shows a higher H and a smaller ν, implying smoother swings in the variance process and a less irregular volatility trajectory.

Additionally, we observe that the SMI index shows the highest H in our sample, while the All Ordinaries index presents the lowest H, except for the Open-to-Close and Garman and Klass estimators. Under the Open-to-Close estimator, the All Ordinaries index still holds the lowest H value, while the S&P 500 index displays the highest. For the Garman and Klass estimator, the SMI retains the highest H.

Table 5. Estimated H for all indices.

H	Index	ν
0.0164	S&P 500	1.4961
0.0104	CAC 40	1.6064
0.0148	SMI	1.5173
0.0102	IBEX 35	1.5782
0.0106	Nifty 50	1.5534
0.0146	FTSE 100	1.5281
0.0105	Nikkei 225	1.5986
0.0128	DAX	1.5766
0.0089	All Ordinaries	1.5230

Open-to-Close estimator

H	Index	ν
0.0791	S&P 500	0.4792
0.0786	CAC 40	0.4529
0.0937	SMI	0.4171
0.0750	IBEX 35	0.4329
0.0697	Nifty 50	0.4725
0.0795	FTSE 100	0.4505
0.0630	Nikkei 225	0.4957
0.0880	DAX	0.4442
0.0542	All Ordinaries	0.4814

Garman and Klass estimator

H	Index	ν
0.0418	S&P 500	0.6770
0.0600	CAC 40	0.5455
0.0727	SMI	0.5135
0.0547	IBEX 35	0.5323
0.0523	Nifty 50	0.5383
0.0463	FTSE 100	0.6381
0.0339	Nikkei 225	0.6570
0.0662	DAX	0.5474
0.0142	All Ordinaries	0.8731

Rogers and Satchell estimator

H	Index	ν
0.0799	S&P 500	0.5029
0.0741	CAC 40	0.4852
0.0868	SMI	0.4537
0.0688	IBEX 35	0.4700
0.0658	Nifty 50	0.5075
0.0851	FTSE 100	0.4593
0.0619	Nikkei 225	0.5301
0.0850	DAX	0.4734
0.0546	All Ordinaries	0.5158

Parkinson estimator

H	Index	ν
0.0813	S&P 500	0.4732
0.0800	CAC 40	0.4492
0.0946	SMI	0.4140
0.0762	IBEX 35	0.4294
0.0689	Nifty 50	0.4735
0.0820	FTSE 100	0.4435
0.0654	Nikkei 225	0.4865
0.0889	DAX	0.4394
0.0575	All Ordinaries	0.4749

Meilijson estimator

H	Index	ν
0.1335	S&P 500	0.3236
0.1325	CAC 40	0.2933
0.1817	SMI	0.2219
0.1271	IBEX 35	0.2868
0.1181	Nifty 50	0.3119
0.1449	FTSE 100	0.2688
0.1165	Nikkei 225	0.3214
0.1518	DAX	0.2751
0.0880	All Ordinaries	0.3504

Realized estimator

4 Forecasting Volatility Using RFSV

We would like to find out which of the estimators is predicted most accurately by the RFSV model. For this purpose, we compare the variance and volatility forecasts of several indexes obtained using the RFSV model, with the parameters estimated for each estimator.

The log-volatility process can be approximated as a fBm, over any reasonable timescale of interest, as shown by Eq. (17) and Eq. (18). Therefore, we obtain an approximate non-stationary model for the increments of the log-volatility in Eq. (12), which can be rewritten as:

$$\log \sigma_t^2 \approx 2\nu W_t^H + C,$$

for some constant C [2].

The prediction method is based on the following formula:

$$\mathbb{E}\left[W_{t+\Delta}^H \mid \mathcal{F}_t\right] = \frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{W_s^H}{(t-s+\Delta)(t-s)^{H+1/2}}ds,$$

where W^H denotes a fBm with $H < 1/2$ and $\mathcal{F}_t$ the filtration it generates [2].

By substituting the fBm with the log-volatility, we have the prediction formula for the log-variance:

$$\mathbb{E}\left[\log\sigma_{t+\Delta}^2 \mid \mathcal{F}_t\right] = \frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{\log\sigma_s^2}{(t-s+\Delta)(t-s)^{H+1/2}}ds, \quad (19)$$

where $\Delta = 1, 5, 20$ [2]. Note that we want the formula for the RFSV predictor of the variance. Nuzman and Poor showed that $W_{t+\Delta}^H$is conditionally Gaussian with the following conditional variance [26]:

$$\mathrm{Var}[W_{t+\Delta}^H|\mathcal{F}_t] = c\Delta^{2H}$$

where

$$c = \frac{\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}.$$

Thus, we easily arrive at the formula for the RFSV predictor of the variance [2]:

$$\mathbb{E}\left[\sigma_{t+\Delta}^2 \mid \mathcal{F}_t\right] = \exp\left\{\mathbb{E}\left[\log\sigma_{t+\Delta}^2 \mid \mathcal{F}_t\right] + 2c\nu^2 2H\right\},$$

where $\mathbb{E}\left[\log\sigma_{t+\Delta}^2 \mid \mathcal{F}_t\right]$ is defined by Eq. (19) and approximated through a Riemann sum.

We need a reliable statistic to compare the forecast quality of the estimators. For this purpose, we compute the following ratio P, which compares the mean squared error of our predictor with the approximate variance of the log-variance:

$$P = \frac{\sum_{k=500}^{N-\Delta}\left(\log(\sigma_{k+\Delta}^2) - \widehat{\log(\sigma_{k+\Delta}^2)}\right)^2}{\sum_{k=500}^{N-\Delta}\left(\log(\sigma_{k+\Delta}^2) - \mathcal{E}[\log(\sigma_{t+\Delta}^2)]\right)^2},$$

where $\mathcal{E}[\log(\sigma_{t+\Delta}^2)]$ represents the empirical mean of the log-variance over the entire time period and $\widehat{\log(\sigma_{k+\Delta}^2)}$ is the predictor of $\log(\sigma_{k+\Delta}^2)$ [2].

The ratio P measures the proportion of the total variance of $\log(\sigma_{k+\Delta}^2)$ that is not captured by the predicted values $\widehat{\log(\sigma_{k+\Delta}^2)}$. Ideally, a P value close to zero indicates better predictive performance, as it implies that most of the variance is captured by the model. Conversely, if $P = 1$, the mean squared error of the predictor matches the empirical variance of the log-volatility, indicating that the predictor does not improve upon a simple average. If P is greater than 1, it means that the model performs worse than simply using the expected mean.

Table 6 display the ratio P. It is clear that realized volatility, as a proxy, is more accurately captured in its behavior by the RFSV model compared to

Table 6. Ratio P for different volatility estimators proxies.

Index/Estimator	Open-to-close	Garman and Klass	Parkinson	Rogers and Satchell	Meilijson	Realized
S&P 500 $\Delta = 1$	1.9556	0.5643	0.5895	0.8863	**0.5439**	**0.3009**
S&P 500 $\Delta = 5$	1.9644	0.6661	0.6828	0.9457	**0.6628**	**0.4732**
S&P 500 $\Delta = 20$	2.1029	**0.8539**	0.8904	1.0644	0.8560	**0.7082**
CAC 40 $\Delta = 1$	2.2324	0.5450	0.6017	0.7102	**0.5427**	**0.3044**
CAC 40 $\Delta = 5$	2.2206	**0.6417**	0.6905	0.7793	0.6459	**0.4454**
CAC 40 $\Delta = 20$	2.3097	**0.8171**	0.8674	0.9304	0.8176	**0.6622**
SMI $\Delta = 1$	2.0751	**0.5332**	0.5965	0.7063	0.5338	**0.2221**
SMI $\Delta = 5$	2.0965	**0.6340**	0.6903	0.7744	0.6384	**0.3628**
SMI $\Delta = 20$	2.2118	**0.8350**	0.8952	0.9408	0.8382	**0.6046**
IBEX 35 $\Delta = 1$	2.1293	0.4643	0.5236	0.6244	**0.4604**	**0.2402**
IBEX 35 $\Delta = 5$	2.1158	0.5585	0.6124	0.6824	**0.5567**	**0.3669**
IBEX 35 $\Delta = 20$	2.1828	**0.6889**	0.7415	0.7866	0.6890	**0.5297**
Nifty 50 $\Delta = 1$	2.0751	0.5348	0.5730	0.6573	**0.5335**	**0.3030**
Nifty 50 $\Delta = 5$	2.0457	**0.6433**	0.6691	0.7400	0.6443	**0.4644**
Nifty 50 $\Delta = 20$	2.1232	0.8071	0.8340	0.8817	**0.8059**	**0.6514**
FTSE 100 $\Delta = 1$	2.0342	0.5369	0.5530	0.8165	**0.5265**	**0.2359**
FTSE 100 $\Delta = 5$	2.0463	0.6195	0.6355	0.8556	**0.6160**	**0.3614**
FTSE 100 $\Delta = 20$	2.1588	0.7881	0.8287	0.9853	**0.7846**	**0.5664**
Nikkei $\Delta = 1$	2.1918	0.6995	0.7222	0.9982	**0.6781**	**0.3975**
Nikkei $\Delta = 5$	2.1860	0.8294	0.8489	1.0489	**0.8140**	**0.5941**
Nikkei $\Delta = 20$	2.2485	**0.9941**	1.0186	1.1547	0.9953	**0.8417**
DAX $\Delta = 1$	2.1563	0.5449	0.6066	0.7057	**0.5428**	**0.2983**
DAX $\Delta = 5$	2.1357	**0.6443**	0.6805	0.7990	0.6484	**0.4543**
DAX $\Delta = 20$	2.2702	**0.8445**	0.8938	0.9727	0.8452	**0.7012**
All Ordinaries $\Delta = 1$	2.0559	0.5926	0.6319	1.1734	**0.5807**	**0.3850**
All Ordinaries $\Delta = 5$	2.0217	0.6530	0.6753	1.1629	**0.6441**	**0.4519**
All Ordinaries $\Delta = 20$	2.0694	0.7517	0.7907	1.1587	**0.7514**	**0.5959**

Note: Best and second-best predictions are in boldface.

range-based estimators. Among the range-based volatility estimators, the Meilijson estimator performs better, followed by the Garman-Klass. The high values observed in the Open-to-Close estimator show its significant level of noise. We also note that the predictive accuracy deteriorates with the forecast horizon, as expected, and that as Δ increases, the gap between the realized variance and the other estimators decreases.

Regarding the indices that are better predicted by these models, those depend on the value of Δ. For $\Delta = 1$, the best-performing index was the SMI for most estimators except for the Open-to-Close, where it was the S&P 500. Con-

versely, the worst-performing index varies more according to the estimator used as a proxy. For the Garman and Klass and Parkinson estimators, the worst-performing index was the IBEX 35, whereas for the Realized Variance and Rogers and Satchell estimators, it was the All Ordinaries. Additionally, for the Meilijson estimator, the worst was the Nikkei, and for the Open-to-Close, it was the CAC 40.

In general, as Δ increases, the indices that yield the best results remain consistent across all estimators, as do those yielding the worst results, except for the Open-to-Close estimator. For $\Delta = 5$, the best index for most estimators was the IBEX 35, except for the Open-to-Close, where it was the Nifty 50. The worst-performing index was the Nikkei, except for the Open-to-Close, where it was the S&P 500. Finally, for $\Delta = 20$, the best-performing index was the IBEX 35, and the worst was the Nikkei, except for the Open-to-Close estimator, where the best index was the S&P 500 and the worst was the CAC 40.

5 Conclusion

The RFSV model, known for its consistency with some financial stylized facts, has relatively few parameters. Nevertheless, the estimation of these parameters typically relies on the use of high-frequency realized variance, which can be costly and unavailable for some assets. To mitigate this, range-based volatility estimators are proposed that use daily open, high, low and close prices. Range-based proxies were found to be efficient in estimating the RFSV model parameters. We verified that the logarithm of range-based volatilities behaves like a fractional Brownian motion with a Hurst exponent less than 1/2. Furthermore, we observe that all Hurst exponents estimated using range-based estimators are smaller than those obtained from realized variance. This suggests that the RFSV model estimated with range proxies seems to produce values of H consistent with those obtained from the ATM volatility skew.

This study supports the idea that volatility is indeed rough, demonstrating this behavior even in low-frequency data. Furthermore, the RFSV model can be still useful, when calculated using range-based proxies. We believe that the Open-to-Close estimator should indeed not be used for this purpose. However, although realized variance shows better results, the Meilijson, Garman and Klass, and Parkinson estimators, in that order, perform relatively well and could serve as alternatives if high-frequency data are not accessible.

It is crucial for future research to investigate the performance of these range-based volatility estimators as proxies for the RFSV model when applied to other methods of estimating the Hurst exponent. This includes examining smoother versions of these estimators that incorporate more recent past observations alongside current ones, to better understand their limitations and potential improvements. We note a clear relationship between the indices and the quality of the results, which may be influenced by factors such as market capitalization, transaction volume, or market efficiency. We suggest that financial analysts further investigate these aspects to draw more conclusive insights.

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