Appell-Vietoris Polynomials: Structural Properties and Zero Bounds

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Abstract. The paper focuses on a sequence of real polynomials forming an Appell sequence whose coefficients are built from the well-known Vietoris sequence. A three-term recurrence relation and the corresponding Sturm—Liouville equation are established as well as hypergeometric and Jacobi representations. The distribution of real zeros is characterized via the Hilbert—Klein theory for Jacobi polynomials with arbitrary parameters. Finally, quantitative bounds for all complex zeros are provided by applying variants of the Eneström—Kakeya theorem.

Keywords: Appell polynomials · Vietoris sequence · Jacobi polynomials · Zero bounds · Eneström-Kakeya bounds

1 Introduction

> *This Table has truly admirable and extraordinary properties. Beyond what I have already shown of the mystery of combinations hidden within it, it is known to those skilled in the deeper parts of geometry that the most important secrets of the rest of mathematics lie concealed within it.*
>
> Jacob Bernoulli in *Ars Conjectandi* (1713)

The table that Bernoulli had in mind here is, of course, the table of binomial coefficients, as we call its entries today. Generalizations of Pascal's triangle of various kinds and arising from different areas of theoretical or applied mathematics appear frequently. Nevertheless, further generalizations continue to emerge, since their diversity seems almost inexhaustible and depends largely on the perspective of the researcher.

Perhaps unexpectedly, our perspective on Bernoulli's table was motivated by problems in hypercomplex analysis [8,9]. In fact, while searching for an adequate

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definition of Appell polynomials in several hypercomplex variables with respect to a hypercomplex derivative operator introduced in [11], the dominant role of a monotonically non-increasing sequence of rational numbers became apparent. These numbers are

$$c_k := \frac{1}{2^k}\binom{k}{\lfloor k/2 \rfloor}, \qquad k = 0, 1, 2, \ldots. \tag{1}$$

They form a strictly positive pairwise occurring sequence that first appeared in Vietoris' work on positive trigonometric sums [21]. Since then, the sequence – now known as the *Vietoris sequence* – has been shown to satisfy interesting combinatorial identities and to play a structural role in hypercomplex analysis [2,3,8]. Combining it with the well-known Appell property [1], led to the introduction of the *Appell-Vietoris polynomials* in [5],

$$\mathcal{V}_n(x) = \sum_{k=0}^{n} \binom{n}{k} c_{n-k}\, x^k, \qquad n = 0, 1, 2, \ldots, \tag{2}$$

where qualitative features of these polynomials were highlighted, namely:

(i) for each fixed degree n, the coefficient row $\{\binom{n}{k} c_{n-k}\}_{k=0}^{n}$ is unimodal and log-concave (see [19,22] for definitions); and
(ii) based on inherited structural properties and numerical evidence, we observe (without full proof) that for even n the polynomials are strictly positive on $\mathbb{R}$ (hence have no real zeros), while for odd n they exhibit exactly one sign change with the unique real zero in $(-1, 0)$.

These observations motivate a deeper structural study linking the Appell nature of $\mathcal{V}_n(x)$ to classical special functions and to precise zero distribution results. The paper is organized as follows: Sect. 2 recalls the Vietoris numbers and the definition of the Appell-Vietoris polynomials, including the coefficient triangle and its unimodality/log-concavity. Section 3 establishes the three-term recurrence and the Sturm–Liouville equation, and derives hypergeometric and Jacobi representations together with Rodrigues formula. Using the Hilbert–Klein theory for Jacobi polynomials with arbitrary parameters, Sect. 4 determines the location of the real zeros, and Sect. 5 provides quantitative bounds for all complex zeros based on variants of the Eneström–Kakeya Theorem.

2 Appell-Vietoris Polynomials Revisited

In this section we recall the main definitions and structural properties of the Appell-Vietoris polynomials that will be used throughout the paper.

Let $(a_k(n))_{k\geq 0}$ denote the triangular array given by

$$a_k(n) = \binom{n}{k} c_{n-k}, \;\; n = 0, 1, \ldots; \;\; k = 0, 1, \ldots n, \tag{3}$$

where c_k are the Vietoris numbers defined in (1). Then the Appell-Vietoris polynomials (2) can be rewritten as

$$\mathcal{V}_n(x) = \sum_{k=0}^{n} a_k(n) x^k, \ n = 0, 1, 2 \ldots.$$

By construction, the sequence $(\mathcal{V}_n(x))_{n\geq 0}$ forms an Appell family and therefore satisfies the relation

$$\mathcal{V}_n'(x) = n\mathcal{V}_{n-1}(x), \qquad n \geq 1. \tag{4}$$

Since the polynomials $\mathcal{V}_n(x)$ are defined through the coefficients $a_k(n)$, it is also useful to recall some of their structural properties. The coefficients $a_k(n)$ are positive rationals and form a triangular array whose rows inherit several properties from the Vietoris sequence; see [4,5] for further details.

Table 1 displays the first rows of the triangle (3) along with the corresponding Appell-Vietoris polynomials $\mathcal{V}_n(x), \ n = 0, \ldots, 6$.

Table 1. Triangle $a_k(n)$: first rows and corresponding Appell-Vietoris polynomials

n	$a_k(n)$	$\mathcal{V}_n(x)$
0	1	1
1	$\frac{1}{2} \quad 1$	$\frac{1}{2} + \mathrm{x}$
2	$\frac{1}{2} \quad 1 \quad 1$	$\frac{1}{2} + \mathrm{x} + x^2$
3	$\frac{3}{8} \quad \frac{3}{2} \quad \frac{3}{2} \quad 1$	$\frac{3}{8} + \frac{3}{2}\,\mathrm{x} + \frac{3}{2}x^2 + x^3$
4	$\frac{3}{8} \quad \frac{3}{2} \quad 3 \quad 2 \quad 1$	$\frac{3}{8} + \frac{3}{2}x + 3x^2 + 2x^3 + x^4$
5	$\frac{5}{16} \quad \frac{15}{8} \quad \frac{15}{4} \quad 5 \quad \frac{5}{2} \quad 1$	$\frac{5}{16} + \frac{15}{8}\,\mathrm{x} + \frac{15}{4}x^2 + 5x^3 + \frac{5x^4}{2} + x^5$
6	$\frac{5}{16} \quad \frac{15}{8} \quad \frac{45}{8} \quad \frac{15}{2} \quad \frac{15}{2} \quad 3 \quad 1$	$\frac{5}{16} + \frac{15}{8}\,\mathrm{x} + \frac{45}{8}x^2 + \frac{15}{2}x^3 + \frac{15}{2}x^4 + 3x^5 + x^6$

Furthermore, for each fixed $n \geq 1$, the finite sequence

$$a_0(n), a_1(n), \ldots, a_n(n)$$

is both log-concave and unimodal. More precisely, it satisfies the log-concavity property

$$(a_k(n))^2 \geq a_{k-1}(n) a_{k+1}(n), \ k = 1, \ldots, n-1,$$

and the unimodality condition

$$a_0(n) \leq a_1(n) \leq \cdots \leq a_{s-1}(n) \leq a_s(n) \geq a_{s+1}(n) \geq \cdots \geq a_n(n), \tag{5}$$

where $s = \lfloor \frac{n+1}{2} \rfloor$ (cf. [5, Th. 2]).

These properties will play a significant role in Sect. 5, where we study bounds for the zeros of the Appell-Vietoris polynomials.

3 Analytic Framework: Recurrence, Sturm–Liouville Equation, and Jacobi Polynomials

Polynomial sequences often satisfy recurrence relations with polynomial coefficients. They are important tools to derive several properties of the considered polynomial sequence and provide an easy way to construct explicitly the sequence.

Theorem 1. *The Appell–Vietoris polynomials satisfy, for $n \geq 2$, the recurrence relation*

$$(n+1)\mathcal{V}_n(x) = (1+2nx)\mathcal{V}_{n-1}(x) + (n-1)(1-x^2)\mathcal{V}_{n-2}(x), \tag{6}$$

with initial conditions $\mathcal{V}_0(x) := 1$ and $\mathcal{V}_1(x) := x + \frac{1}{2}$.

Proof. Define

$$\mathcal{W}_n(x) := (1+2nx)\mathcal{V}_{n-1}(x) + (n-1)(1-x^2)\mathcal{V}_{n-2}(x).$$

We show that $\mathcal{W}_n(x) = (n+1)\mathcal{V}_n(x)$ by comparing coefficients. Write

$$\mathcal{W}_n(x) = \sum_{k=0}^{n} A_{k,n} x^k.$$

Using the explicit representation of $\mathcal{V}_m(x)$ (and the convention $\binom{m}{r} = 0$ for $r < 0$ or $r > m$, and $c_{-j} = 0$, $j \geq 1$), a direct computation yields

$$A_{k,n} = (2n+1-k)\tbinom{n-1}{k-1}c_{n-k} + \tbinom{n-1}{k}c_{n-1-k} + (n-1)\tbinom{n-2}{k}c_{n-2-k}.$$

Now use the recurrence (see [4])

$$(j+2)c_{j+1} = c_j + jc_{j-1}, \qquad j \geq 1, \qquad c_0 = 1,\ c_1 = \tfrac{1}{2}, \tag{7}$$

with $j = n-1-k$, i.e.

$$c_{n-1-k} = (n+1-k)c_{n-k} - (n-1-k)c_{n-2-k}.$$

Substituting into $A_{k,n}$ and grouping the terms in c_{n-k} and c_{n-k-2} gives

$$A_{k,n} = \Big((2n+1-k)\tbinom{n-1}{k-1} + (n+1-k)\tbinom{n-1}{k}\Big)c_{n-k} + \Big((n-1)\tbinom{n-2}{k} - (n-1-k)\tbinom{n-1}{k}\Big)c_{n-2-k}.$$

Using $\binom{n-1}{k} = \binom{n-2}{k} + \binom{n-2}{k-1}$, the contributions to the coefficient of c_{n-k-2} cancel pairwise, hence this term disappears. Finally, applying the standard binomial identities $\binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$, $\binom{n-1}{k} = \frac{n-k}{n}\binom{n}{k}$, $\binom{n-2}{k-2} = \frac{k(k-1)}{n(n-1)}\binom{n}{k}$, one obtains

$$A_{k,n} = (n+1)\binom{n}{k}c_{n-k}.$$

Therefore $\mathcal{W}_n(x) = (n+1)\mathcal{V}_n(x)$, corresponding to (6). □

Special-value evaluations of the Appell—Vietoris polynomials reveal connections with classical number sequences, and these links play a central role in Sect. 4. In particular, evaluating at $x = 0$ recovers the Vietoris numbers:

$$\mathcal{V}_n(0) = c_n, \tag{8}$$

which can be given explicitly by (1) or, equivalently, by the recurrence obtained from Theorem 1,

$$(n+1)c_n = c_{n-1} + (n-1)c_{n-2}, \quad n \geq 2, \qquad c_0 = 1,\ c_1 = \tfrac{1}{2},$$

(see also [6, Th. 6]). The special value (8) illustrates one of several connections between the Appell—Vietoris polynomials and classical number sequences. Additional links of this kind will be derived in the corollary that follows.

Corollary 1. *Let $\mathcal{C}_n$ and R_n denote the Catalan numbers and the Riordan numbers, respectively. Then*

$$\mathcal{V}_n(-1) = (-2)^{-n}\mathcal{C}_n \qquad \text{and} \qquad \mathcal{V}_n\big(-\tfrac{1}{2}\big) = (-2)^{-n}R_n.$$

Proof. Evaluating the recurrence relation (6) at the special points $x = -1$ and $x = -\frac{1}{2}$ and applying the specified normalization conditions yields

1. The sequence $(\mathcal{C}_n)_{n\geq 0}$ satisfies

$$\mathcal{C}_n = \frac{4n-2}{n+1}\,\mathcal{C}_{n-1}, \qquad n \geq 1, \qquad \mathcal{C}_0 = 1,$$

   and therefore coincides with the Catalan numbers (see, e.g., [13, Sect. 5]).
2. The sequence $(R_n)_{n\geq 0}$ satisfies

$$(n+1)R_n = (n-1)\big(2R_{n-1} + 3R_{n-2}\big), \quad n \geq 2, \quad R_0 = 1,\ R_1 = 0, \tag{9}$$

   which reproduces the Riordan numbers (see e.g. [7], Th. 4.1).

□

Another consequence of the recursion (6), together with the Appell property of the polynomials $\mathcal{V}_n(x)$, is the following result.

Theorem 2 (Differential equation). *The Appell-Vietoris polynomial $\mathcal{V}_n(x)$ is a solution of the second order linear homogeneous differential equation*

$$(1-x^2)y''(x) + (1+2nx)y'(x) - n(n+1)y(x) = 0, \tag{10}$$

which can be rewritten, for $x \in (-1,1)$, in the Sturm–Liouville form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = 0, \tag{11}$$

where

$$p(x) = (1-x)^{-\frac{1}{2}-n}(1+x)^{\frac{1}{2}-n} \tag{12}$$

and

$$q(x) = -n(n+1)(1-x)^{-\frac{3}{2}-n}(1+x)^{-\frac{1}{2}-n}. \tag{13}$$

Proof. To prove that $\mathcal{V}_n(x)$ is a solution of (10) it is sufficient to use the recurrence relation (6) together with the Appell property (4). The Sturm–Liouville form can be obtained by using standard techniques: normalise (10) to $y''(x) + P(x)\,y'(x) + Q(x)\,y(x) = 0$ with $P(x) = (1 + 2nx)/(1 - x^2)$ and $Q(x) = -n(n+1)/(1-x^2)$; multiply the resulting equation by the integrating factor $p(x) = \exp\left(\int P(x)\,dx\right) = (1-x)^{-1/2-n}(1+x)^{1/2-n}$, which is strictly positive on $(-1, 1)$ to obtain (11) with (12), (13). □

The next result shows that the Appell—Vietoris polynomials belong to a certain subclass of Jacobi polynomials. Jacobi polynomials can be defined in several ways; for our purposes, here we define Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ as solutions of the differential equation

$$\left(1 - x^2\right) y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta + 1)y = 0, \tag{14}$$

where α, β are arbitrary complex parameters. The cases where $\alpha > -1$ and $\beta > -1$ are the most common in virtue of their classical orthogonality. For these restrictions on the parameters, it is well known that the polynomial solutions of (14) are of the form $\kappa P_n^{(\alpha,\beta)}$, where κ is a constant (cf. [20, Th. 4.2.2]). In the general case of arbitrary parameters, Theorem 4.23.2 in [20] establishes that the differential equation has two linearly independent polynomial solutions only in the following two scenarios:

(1) α, β negative integers, $\alpha \geq -n,\ \beta \geq -n,\ \alpha + \beta \leq -n - 1,\ n \geq 1$;

(2) α, β negative integers, $\alpha < -n,\ \beta < -n,\ n \geq 0$.

The following result shows the relationship between Appell-Vietoris polynomials $\mathcal{V}_n(x)$ and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, with parameters α and β as negative half-integers that vary linearly with the degree n. This dependence is illustrated by adopting henceforth the notation (α_n, β_n) for the parameters.

Theorem 3 (Jacobi polynomials). *The Appell-Vietoris polynomials can be expressed in terms of Jacobi polynomials as*

$$\mathcal{V}_n(x) = \frac{(-2)^n}{n+1} P_n^{(\alpha_n,\beta_n)}(x), \tag{15}$$

where

$$\alpha_n = -n - \tfrac{3}{2} \qquad \textit{and} \qquad \beta_n = -n - \tfrac{1}{2}. \tag{16}$$

Proof. Observe that (16) leads to

$$\alpha_n + \beta_n = -2n - 2 \qquad \text{and} \qquad \beta_n - \alpha_n = 1,$$

and (14) reduces to (10). This means that both $\mathcal{V}_n(x)$ and $P_n^{(\alpha_n,\beta_n)}(x)$ are polynomial solutions of (14). Since α_n and β_n are not negative integers, we are not in either of the exceptional cases (1)–(2) above. It follows that (14) admits at

most one polynomial solution (up to a multiplicative constant), and thus there exists a constant κ_n such that

$$\mathcal{V}_n(x) = \kappa_n P_n^{(\alpha_n,\beta_n)}(x).$$

To determine κ_n, we use the values at $x = 1$. From the recurrence relation (6) we obtain

$$\mathcal{V}_n(1) = \frac{(2n+1)!!}{(n+1)!},$$

and from [20, p. 65] we have

$$P_n^{(\alpha_n,\beta_n)}(1) = \binom{n+\alpha_n}{n} = \binom{-\frac{3}{2}}{n} = (-1)^n \frac{(2n+1)!!}{2^n\, n!}.$$

Therefore,

$$\kappa_n = \frac{\mathcal{V}_n(1)}{P_n^{(\alpha_n,\beta_n)}(1)} = (-1)^n \frac{2^n}{n+1},$$

which yields (15). □

The following results are straightforward, given the relationship between Appell-Vietoris polynomials and Jacobi polynomials as outlined in Theorem 3.

Corollary 2 (Rodrigues formula). *For every $n \geq 0$,*

$$\mathcal{V}_n(x) = \frac{1}{(n+1)!}(1-x)^{\frac{3}{2}+n}(1+x)^{\frac{1}{2}+n}\frac{d^n}{dx^n}\left((1-x)^{-\frac{3}{2}}(1+x)^{-\frac{1}{2}}\right).$$

Corollary 3 (Connection to hypergeometric function). *For every $n \geq 0$,*

$$\mathcal{V}_n(x) = \frac{(2n+1)!!}{(n+1)!}\ {}_2F_1\left(-n, -(n+1); -(n+\tfrac{1}{2}); \tfrac{1-x}{2}\right).$$

4 Real Zeros of Appell-Vietoris Polynomials

The coefficients of the Appell-Vietoris polynomials are all strictly positive; therefore, none of their real zeros can be positive. In [5] it was observed that $\mathcal{V}_n(x)$ exhibit specific behavior regarding their real zeros: they have no real zeros when n is even, and exactly one real zero when n is odd.

Relying on Theorem 3, it is enough to study the real zeros of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, for $\alpha_n = -n - \frac{3}{2}$ and $\beta_n = -n - \frac{1}{2}$. In the case of the classical Jacobi polynomials with parameters $\alpha > -1$ and $\beta > -1$, their orthogonality implies that all the zeros are real and simple and lie in the interval $(-1, 1)$. For more general parameters this statement is no longer valid: the classical orthogonality fails and the zeros can be complex and multiple. However, known results on the number of real zeros of the corresponding Jacobi polynomials, developed by Hilbert, Klein and Stieltjes (mentioned by Szegö in [20], Th.

6.72), can be applied. We start by considering the symbol used by F. Klein in [12]

$$E(u) = \begin{cases} 0, & \text{if } u \leq 0 \\ \lfloor u \rfloor, & \text{if } u > 0, u \text{ is not integer} \\ u-1, & \text{if } u = 1, 2, 3, \ldots \end{cases}.$$

Taking into account that $-1, 1$ and ∞ are not zeros of $\mathcal{V}_n(x)$ and, consequently, of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, with $\alpha_n = -n - \frac{3}{2}$, $\beta_n = -n - \frac{1}{2}$, the following theorem can be applied to establish the number and location of the real zeros of the Appell-Vietoris polynomials.

Theorem 4. [20, Th. 6.72] *Let α, β be arbitrary real values, and set*

$$X = X(\alpha,\beta) = E\{\tfrac{1}{2}(|2n+\alpha+\beta+1| - |\alpha| - |\beta| + 1)\},$$
$$Y = Y(\alpha,\beta) = E\{\tfrac{1}{2}(-|2n+\alpha+\beta+1| + |\alpha| - |\beta| + 1)\},$$
$$Z = Z(\alpha,\beta) = E\{\tfrac{1}{2}(-|2n+\alpha+\beta+1| - |\alpha| + |\beta| + 1)\}.$$

Denoting by N_1, N_2 and N_3 the number of zeros of $P_n^{(\alpha,\beta)}(x)$ in $(-1,1)$, $(-\infty,-1)$, and $(1,+\infty)$, respectively, these values are given by

$$N_1 = N_1(\alpha,\beta) = \begin{cases} 2\lfloor \frac{X+1}{2} \rfloor, & \text{if } (-1)^n \binom{n+\alpha}{n}\binom{n+\beta}{n} > 0 \\ 2\lfloor \frac{X}{2} \rfloor + 1, & \text{if } (-1)^n \binom{n+\alpha}{n}\binom{n+\beta}{n} < 0 \end{cases},$$
$$N_2 = N_2(\alpha,\beta) = \begin{cases} 2\lfloor \frac{Y+1}{2} \rfloor, & \text{if } \binom{2n+\alpha+\beta}{n}\binom{n+\beta}{n} > 0 \\ 2\lfloor \frac{Y}{2} \rfloor + 1, & \text{if } \binom{2n+\alpha+\beta}{n}\binom{n+\beta}{n} < 0 \end{cases},$$
$$N_3 = N_3(\alpha,\beta) = \begin{cases} 2\lfloor \frac{Z+1}{2} \rfloor, & \text{if } \binom{2n+\alpha+\beta}{n}\binom{n+\alpha}{n} > 0 \\ 2\lfloor \frac{Z}{2} \rfloor + 1, & \text{if } \binom{2n+\alpha+\beta}{n}\binom{n+\alpha}{n} < 0 \end{cases}.$$

We can now state the following theorem concerning the real zeros of the Appell-Vietoris polynomials.

Theorem 5. *The Appell-Vietoris polynomials $\mathcal{V}_n(x)$ have no real zeros when n is even, and exactly one real zero when n is odd, located in the interval $[-\frac{1}{2}, 0)$.*

Proof. In a first stage, due to relation (15), we count the number of real zeros in the interval $(-1,1)$ of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, with $\alpha_n = -n - \frac{3}{2}$ and $\beta_n = -n - \frac{1}{2}$, applying Theorem 4. For that purpose, we compute the numbers N_1, N_2 and N_3 for this special class of Jacobi polynomials:

1. $X = E(-n) = 0$ and

$$(-1)^n \binom{n+\alpha_n}{n}\binom{n+\beta_n}{n} = (-1)^n \binom{-3/2}{n}\binom{-1/2}{n} = (-1)^n \times \text{positive factor};$$

therefore

$$N_1 = \begin{cases} 2\lfloor \frac{X+1}{2} \rfloor = 0, & \text{if } n \text{ is even} \\ 2\lfloor \frac{X}{2} \rfloor + 1 = 1, & \text{if } n \text{ is odd} \end{cases}.$$

2. $Y = E(1/2) = 0$ and

$$\binom{2n+\alpha_n+\beta_n}{n}\binom{n+\beta_n}{n} = \binom{-2}{n}\binom{-1/2}{n} > 0\,;$$

therefore

$$N_2 = 2\lfloor \tfrac{Y+1}{2} \rfloor = 0\,.$$

3. $Z = E(-1/2) = 0$ and

$$\binom{2n+\alpha_n+\beta_n}{n}\binom{n+\alpha_n}{n} = \binom{-2}{n}\binom{-3/2}{n} > 0\,;$$

therefore

$$N_3 = 2\lfloor \tfrac{Z+1}{2} \rfloor = 0\,.$$

We have just proved that the Appell—Vietoris polynomials $\mathcal{V}_n(x)$ possess no real zeros when n is even, and exactly one real zero when n is odd, which lies in the interval $(-1, 1)$. Moreover, this interval can be further narrowed. Taking into account the connection between $\mathcal{V}_n(-\frac{1}{2})$ and the Riordan numbers established in Corollary 1, together with the nonnegativity of the Riordan sequence (cf. (9)), it follows that $\mathcal{V}_n(-\frac{1}{2}) \leq 0$ for odd n, with equality only when $n = 1$. Since $\mathcal{V}_n(0) = c_n > 0$ (cf. (8)), there is a change of sign, and hence the unique real zero for odd n lies in the interval $[-\frac{1}{2}, 0)$. □

To illustrate Theorem 5, we present in Fig. 1 the graphs of the first Appell-Vietoris polynomials, grouped by the parity of their degrees.

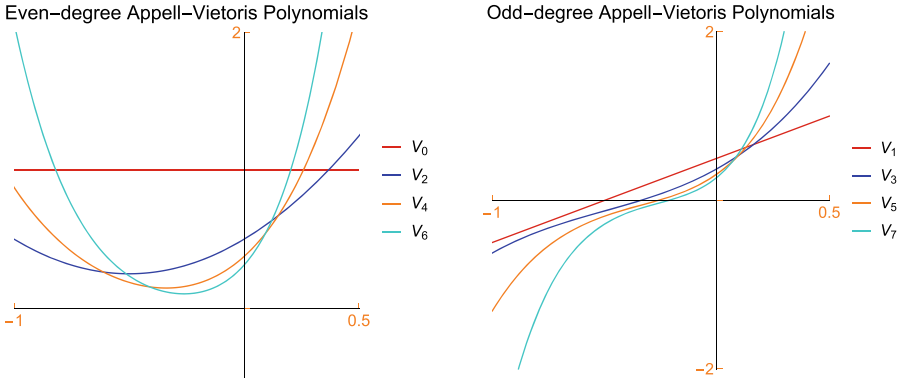

Fig. 1. First Appell-Vietoris Polynomials

5 Eneström-Kakeya Bounds

The literature provides numerous results on the localization of polynomial zeros, including Gershgorin-type bounds, root-radius estimates, and refinements of the classical Cauchy and Fujiwara inequalities [15,17]. In this work, we derive bounds for the zeros of the Appell-Vietoris polynomials $\mathcal{V}_n(x)$ by applying two variants of the Eneström–Kakeya Theorem and some of its generalizations.

We first recall the following result from [10, Th. 4.1], which applies to polynomials with positive coefficients forming an unimodal sequence.

Theorem 6. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with positive real coefficients. If*

$$a_0 \le a_1 \le \cdots \le a_k \ge a_{k+1} \ge \cdots \ge a_n,$$

for some $k \in \{0, \ldots, n\}$, then all zeros of p satisfy

$$|z| \le \frac{2a_k}{a_n} - 1.$$

Since, for each degree n, the coefficient sequences (3) of the Appell-Vietoris polynomials $\mathcal{V}_n(x)$ are positive and unimodal (see (5)), we obtain the following corollary.

Corollary 4. *All zeros of $\mathcal{V}_n(x)$ lie in the disk $|x| \le R^{(n)}$, where*

$$R^{(n)} = 2\binom{n}{s} c_{n-s} - 1, \tag{17}$$

with $s = \lfloor (n+1)/2 \rfloor$ and c_{n-s} denoting the Vietoris numbers defined in (1).

Table 2 reports, for each degree $n = 0, \ldots, 10$, the bound $R^{(n)}$ given by (17) and the quantity

$$z_{\max}^{(n)} = \max_{0 \le i \le n} |z_i^{(n)}|,$$

where $z_i^{(n)}$ are the zeros of $\mathcal{V}_n(x)$. We note that these bounds are very conservative and become increasingly loose as n grows, in contrast with the observed distribution of the zeros.

In order to obtain sharper bounds for the class of polynomials under consideration, we apply now the following form of the Eneström-Kakeya Theorem [10, Th. 1.4].

Theorem 7. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with positive real coefficients. All zeros of p lie within the annulus*

$$\min_{0 \le k \le n-1} \frac{a_k}{a_{k+1}} \le |z| \le \max_{0 \le k \le n-1} \frac{a_k}{a_{k+1}}. \tag{18}$$

Applying this result to the Appell-Vietoris polynomials yields the following corollary.

Table 2. Bounds and zeros of $\mathcal{V}_n(x)$ obtained from Corollary 4.

n	$z_{\max}^{(n)}$	$R^{(n)}$	n	$z_{\max}^{(n)}$	$R^{(n)}$
1	0.5	1.	6	2.05	14.
2	0.71	1.	7	2.37	25.25
3	1.05	2.	8	2.69	51.5
4	1.39	5.	9	3.01	93.5
5	1.72	9.	10	3.32	156.5

Corollary 5. *Let $\mathcal{V}_n(x)$ denote the Appell-Vietoris polynomial of fixed degree n. All zeros of $\mathcal{V}_n(x)$ lie in the annulus*

$$R_1^{(n)} \leq |x| \leq R_2^{(n)},$$

where

$$R_1^{(n)} = \begin{cases} \dfrac{1}{n}, & \text{if } n \text{ is even} \\ \dfrac{1}{n+1}, & \text{if } n \text{ is odd} \end{cases} \qquad \text{and} \qquad R_2^{(n)} = \frac{n}{2}. \tag{19}$$

Proof. Define, for $k = 0, \ldots, n-1$, the finite sequence

$$b_k(n) = \frac{a_k(n)}{a_{k+1}(n)},$$

where $a_k(n)$ are the Appell-Vietoris coefficients (3). Observing (1), the relation between consecutive terms of the Vietoris sequence is given by

$$c_k = \begin{cases} \frac{k}{k+1} c_{k-1}, & \text{if } k \text{ odd} \\ c_{k-1}, & \text{if } k > 1 \text{ even} \\ 1 & \text{if } k = 0 \end{cases},$$

and, consequently

$$b_k(n) = \begin{cases} \dfrac{k+1}{n-k}, & \text{if } n-k \text{ is even} \\ \dfrac{k+1}{n-k+1}, & \text{if } n-k \text{ is odd} \end{cases}.$$

Since

$$\frac{b_{k+1}(n)}{b_k(n)} = \begin{cases} \dfrac{k+2}{k+1} > 1, & \text{if } n-k \text{ is even} \\ \dfrac{(k+2)(n-k+1)}{(k+1)(n-k-1)} > 1, & \text{if } n-k \text{ is odd} \end{cases},$$

it follows at once that $(b_k(n))_k$ is a strictly increasing sequence, and therefore

$$b_0 \leq b_k \leq b_{n-1}.$$

Since $b_0 = \frac{1}{n}$, for n even, $b_0 = \frac{1}{n+1}$ for n odd and $b_{n-1} = \frac{n}{2}$, the conclusion follows directly from (18). □

Table 3 reports, for several values of n, the bounds $R_1^{(n)}$ and $R_2^{(n)}$ given by (19) together with the moduli of the smallest and largest zeros. Figure 2 provides a geometric illustration of these annular bounds, showing the zeros and the corresponding Eneström–Kakeya annuli for degrees $n = 2, \ldots, 6$.

Table 3. Bounds and zeros of $\mathcal{V}_n(x)$ obtained from Corollary 5.

n	$R_1^{(n)}$	$z_{\min}^{(n)}$	$z_{\max}^{(n)}$	$R_2^{(n)}$
1	0.5	0.5	0.5	0.5
2	0.5	0.707	0.707	1.
3	0.25	0.339	1.052	1.5
4	0.25	0.440	1.391	2.
5	0.167	0.261	1.723	2.5
6	0.167	0.328	2.049	3.
7	0.125	0.214	2.371	3.5
8	0.125	0.265	2.689	4.
9	0.1	0.183	3.006	4.5
10	0.1	0.224	3.321	5.
⋮	⋮	⋮	⋮	⋮
20.	0.05	0.129	6.438	10.

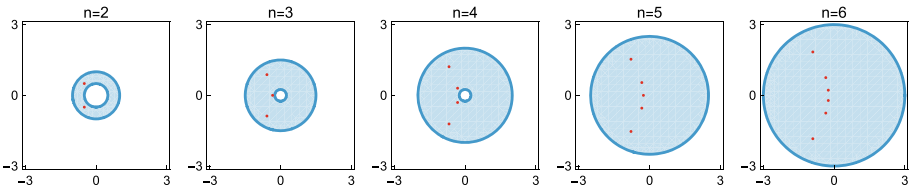

Fig. 2. Bounds for the zeros of the Appell-Vietoris polynomials of degrees $n = 2, \ldots, 6$.

Several results from the literature were tested for the localization of the zeros of the Appell—Vietoris polynomials, with Corollary 5 consistently yielding the sharpest bounds. Nevertheless, Corollary 4 has been retained as a complementary reference: it allows the exploitation of the unimodality property of the coefficients and serves as a baseline for comparison, highlighting the improvement achieved by the sharper classical Eneström–Kakeya estimates. Together, these results provide a comprehensive picture of the zero distribution and illustrate the relative effectiveness of different bounding approaches available in the literature.

6 Concluding Remarks

In this paper, we revisited some fundamental properties of the coefficients of the Appell-Vietoris polynomials that were used to derive structural properties as a three-term recurrence relation and a Sturm–Liouville differential equation for this family. We subsequently proved that the Appell-Vietoris polynomials form a subclass of Jacobi polynomials with negative, non-integer parameters depending linearly on the degree. Polynomials of this type play an important role in Chebyshev spectral methods for boundary-value problems (cf. [18]).

The last two sections were dedicated to the zeros of the Appell-Vietoris polynomials. Their real zeros were counted and localized by combining known results for Jacobi polynomials with specific properties of the Appell-Vietoris family. We also carried out a comparative study of bounds for the full zero set, relying on the Eneström–Kakeya estimates. Other approaches could in principle be pursued, for instance by invoking results on the zero distribution of hypergeometric polynomials or of Jacobi polynomials with varying parameters; see Corollary 3 and Theorem 3, respectively. Concerning the hypergeometric route, we conducted an extensive literature search but, to the best of our knowledge, the specific parameter configuration arising here has not been treated, even after applying standard linear and quadratic transformations. Regarding Jacobi polynomials with varying parameters the zero distribution has been investigated by some authors under conditions on the parameter sequences. In particular, Finkelshtein et al. [16] analyse the zero behaviour for the class of Jacobi polynomials in the parameter range where classical orthogonality fails. They considered $P_n^{(\alpha_n,\beta_n)}(x)$ such that

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = A \qquad \text{and} \qquad \lim_{n\to\infty} \frac{\beta_n}{n} = B, \qquad A, B \in \mathbb{R},$$

and distinguish five cases according to the values of A and B. The case of the Jacobi polynomials considered in the present work, namely $A + B = -2$, corresponds to a transition between these five cases and is not covered by their analysis. As Kuijlaars et al. observed (cf. [14, p. 198]), this case "presents additional difficulties".

The results obtained, together with our numerical experiments, indicate that the zero sets exhibit a specific underlying structure. A more detailed analysis of their zero distribution will be pursued in future work, where other possible approaches will be explored.

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