Dynamics Between Order and Chaos for a Family of Two-Dimensional Cellular Automata

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Abstract. We study a family of finite two-dimensional Boolean cellular automata with a four-neighbour von Neumann neighbourhood and fixed null boundary conditions. With the goal of identifying local rules that generate dynamical regimes intermediate between order and chaos, we present results that allow us to identify rules whose dynamics are extremely simple. We then present computational results showing that the dynamics of another set of local rules is chaotic. This provides a framework for the search for cellular automata whose behaviour can be classified as lying between order and chaos. We conclude by presenting examples of cellular automata whose dynamics should be classified as between order and chaos. Some of these examples show that Wolfram's original idea of classifying dynamical regimes between order and chaos as complex captures only part of what these very simple dynamical systems can exhibit.

Keywords: Finite cellular automata · 2D cellular automata · complex dynamics

1 Dynamics Between Order and Chaos

The study of elementary cellular automata (ECA) occupies a central role in understanding how complex behavior can emerge from extremely simple deterministic rules. Despite their minimal definition, these systems serve as a fundamental testing ground for investigating the transition between order and chaos – a hallmark of complex dynamics in discrete media. SpaceâĂŞtime diagrams generated by ECA vividly demonstrate that structured and disordered patterns can coexist within the same rule, and that transitions between regularity and irregularity naturally arise from local interactions. Such observations reveal that

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even the simplest algorithmic prescriptions are capable of producing emergent structures whose richness far exceeds their computational simplicity.

The search for dynamics between order and chaos need not remain confined to this limited class of elementary automata. The small size of the rule set – comprising only 256 local transition rules – may partially explain the absence of other intermediate dynamical forms in Wolfram's original classification. Moreover, the frequent adoption of periodic boundary conditions further constrains the range of observable behaviors, occasionally suppressing nontrivial regimes. As demonstrated in [2], certain one-dimensional cellular automata display transitions between order and disorder that differ substantially from those reported by Wolfram.

Building on this insight, subsequent studies by various researchers have extended the analysis to two-dimensional lattices and to systems with slightly expanded neighborhoods, see [4], revealing an even broader spectrum of emergent phenomena. This collective body of evidence supports the hypothesis that dimensionality plays a critical role in the development of complexity, suggesting that rich dynamical behavior is not an exclusive feature of simple one-dimensional rules but expands with the spatial structure underlying discrete systems.

In what follows, we turn our attention to the family of two-dimensional peripheral cellular automata with fixed null boundary conditions. By examining their dynamical evolution, we aim to identify regimes that may be interpreted as lying between order and chaos, and to determine the structural features that give rise to such intermediate behaviors. Through this approach, we seek to contribute to a broader understanding of how spatial constraints and boundary conditions influence the emergence of complexity in discrete dynamical systems.

2 A Family of Two-Dimensional Finite Four-Neighbor Cellular Automata

Two-dimensional finite cellular automata (FCA) are discrete computational models defined on a grid of cells, each capable of adopting one of a finite number of possible states. At every time step, the state of each cell is updated synchronously according to a prescribed local rule that depends on the configuration of its neighboring cells. In rectangular lattices, two standard neighborhood schemes have been extensively studied: the von Neumann neighborhood, consisting of the four orthogonally adjacent cells, and the Moore neighborhood, which includes the eight surrounding cells. Even when limiting the analysis to the simplest of these choices, the number of admissible local transition rules is so large that an exhaustive exploration becomes computationally unfeasible, highlighting the need to restrict attention to a simpler family amenable to systematic study.

We consider finite square $n \times n$ Boolean synchronous cellular automata with a four neighbor peripheral neighborhood, that is, a neighborhood of von Neumann type with the center cell not considered. If we denote by

$$\mathsf{C}(t) = c_{ij}(t), \qquad i, j = 1, \cdots, n,$$

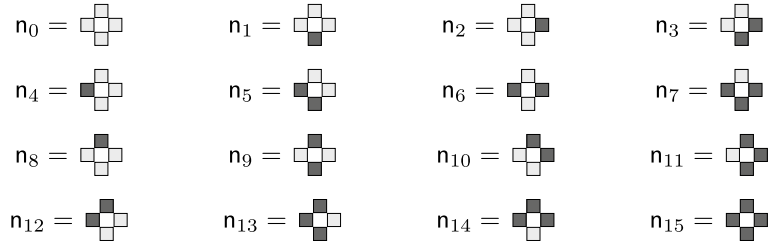
the system state configuration at time t, then the state of the cell c_{ij} at time $t+1$ is given by the value that a Boolean function ϕ of four variables – the local update rule – takes on the 4-tuple consisting of the states of the up, left, right, and down neighbors of that cell at the previous time t:

$$c_{ij}(t+1) \;=\; \phi\left(c_{i-1,j}(t),\, c_{i,j+1}(t),\, c_{i+1,j}(t),\, c_{i,j-1}(t)\right), \quad i,j = 1,\ldots,n.$$

Since we have a finite system, we know we have to choose some boundary conditions: in this case we choose fix null boundary conditions. Hence, we have

$$\begin{aligned} c_{0k}(t) \;&=\; c_{n+1,k}(t) \;=\; 0 \qquad k = 1,\ldots,n \\ c_{k0}(t) \;&=\; c_{k,n+1}(t) \;=\; 0 \end{aligned}$$

For convenience, we will follow the usual procedure of associating an integer code number $\mathcal{N}_\phi$ with each automaton ϕ. The numbering scheme considered here is the one used in [1] and is defined as follows: to each neighborhood state $(\mathsf{u},\, \mathsf{l},\, \mathsf{r},\, \mathsf{d})$, with $\mathsf{u},\mathsf{l},\mathsf{r},\mathsf{d} \in \{0,1\}$, we associate the integer number whose binary representation is $(\mathsf{u}\,\mathsf{l}\,\mathsf{r}\,\mathsf{d})_2$. Hence, we have 16 different neighborhood states $\mathsf{n}_k;\ k = 0,\ldots,15$, ordered according to these numbers. The graphical representation of such distinct neighborhood states can be given as



The integer code $\mathcal{N}_\phi$, corresponding to the rule ϕ, is then given by

$$\mathcal{N}_\phi = \sum_{k=0}^{15} \phi(\mathsf{n}_k)\, 2^k.$$

In what follows, we will indistinctly refer to a cellular automaton either by the associated boolean function ϕ or the integer code $\mathcal{N}_\phi$. Let us illustrate with an example the concepts of Wolfram integer code and local transition rule.

Example 1. Consider the transition rule $\mathcal{N}_\phi = 38\,617 = (1001011011010111)_2$, whose graphical representation can be given as For a system comprising 5×5 elements, we computed the system configurations at time instances $T = 1,\ldots,24$, starting from a randomly chosen configuration, as shown in Table 1. Upon examining the set of configurations, we can readily identify the dynamics attractor, a cycle with a period of 8, emerging after a transient of 15 time steps.

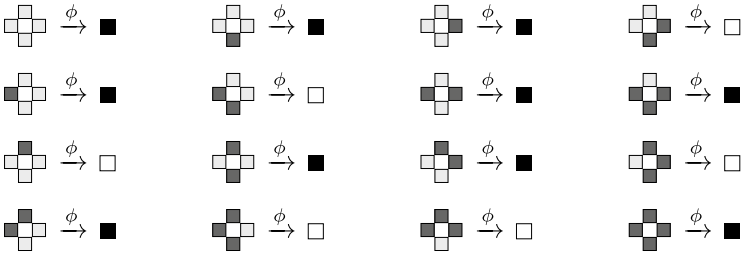
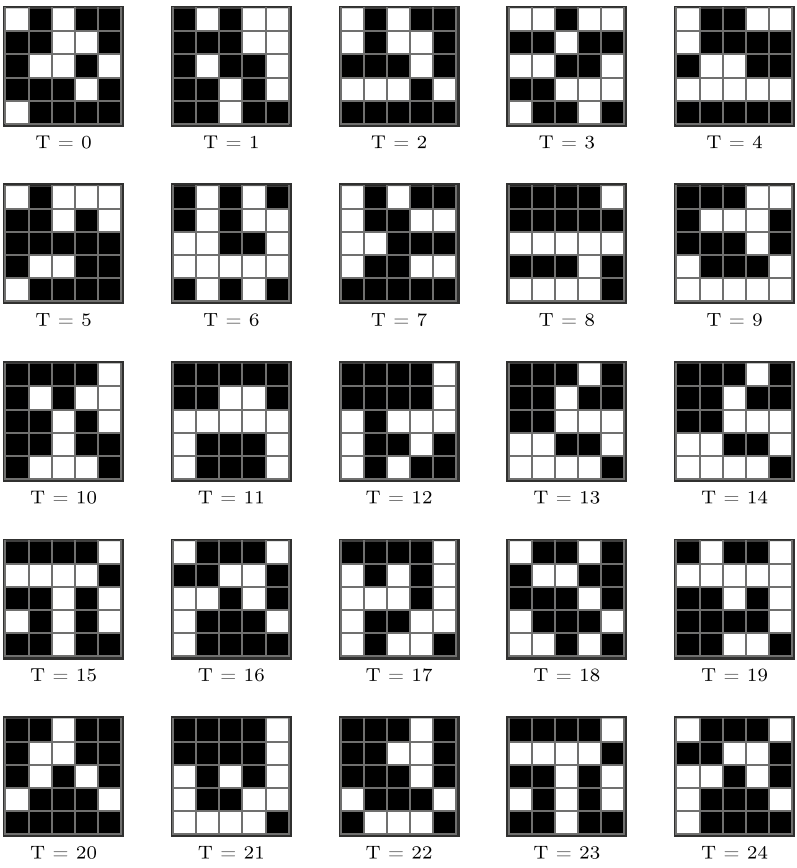


Table 1. The dynamics observed for transition rule $\mathcal{N}_\phi = 38\,617$, starting from a randomly chosen initial configuration, for $T = 0$ to 24. Upon careful observation, we can easily discern the emergence of a final cycle with a period of 8, occurring after a transient of 15 time steps.



Since our aim is to study exhaustively the dominant dynamics of the family of cellular automata described above, it is essential to reduce their number. Taking

advantage of the results presented in [3], and taking into account the differences in the choice of boundary conditions – the conjugacy symmetry is no longer valid, since we are considering systems with fix null boundary conditions – we can establish that the following result holds.

Proposition 1. *There are 9 616 dynamically nonequivalent square Boolean peripheral cellular automata.*

We believe that this number may be sufficiently small to allow a detailed study of the dynamics of this type of two-dimensional cellular automata, in a manner similar to what has been done for the one-dimensional elementary cellular automata.

Next, we will identify the cellular automata for which a homogeneous configuration is dynamically relevant.

3 Homogeneous Cellular Automata

Wolfram's classification of one-dimensional elementary cellular automata, based on an analysis of the patterns produced by their temporal evolution, places in Class I those automata whose behavior is characterized by convergence to homogeneous final states [1]. The special configurations referred to by Wolfram are precisely the homogeneous ones, that is, those in which all cells of the system are in either the zero state or the one state. Next, we identify the subset of local transition rules whose dynamics are characterized by convergence to homogeneous final configurations.

Proposition 2. *Consider the cellular automaton*

$$\mathcal{N}_\phi = (d_{15}\, d_{14}\, d_{13}\, d_{12}\, d_{11}\, d_{10}\, d_9\, d_8\, d_7\, d_6\, d_5\, d_4\, d_3\, d_2\, d_1\, d_0)_2.$$

If at least one of the following conditions holds:

$d_0 = d_1 = d_2 = d_3 = 0$;
$d_0 = d_1 = d_4 = d_5 = 0$;
$d_0 = d_4 = d_8 = d_{12} = 0$;
$d_0 = d_2 = d_8 = d_{10} = 0$,

then its dynamics will have as fixed point the homogeneous null configuration.

Proof. Consider a system with $n \times n$ elements and let $\mathsf{C}^T = (c_{ij}^T)$ denote the configuration of the system at time T. Each of the conditions listed previously corresponds to evaluating the system's dynamics from one of its four corners. Here we describe the reasoning for the upper-left corner under the first condition; the remaining three cases follow analogously.

Assume that $d_0 = d_1 = d_2 = d_3 = 0$. This ensures that, no matter the initial configuration chosen, $c_{11}^T = 0$, for $T > 0$. Likewise, we obtain $c_{12}^T = c_{21}^T = 0$, for $T > 1$. Iterating the argument yields $c_{1n}^T = c_{n1}^T = 0$, for $T > n-1$. Consequently, the same reasoning allows us to conclude that, after a sufficiently large time T, $\mathsf{C}^T = (c_{ij}^T)$ will be the homogeneous zero configuration, as required. □

The next two results characterise those cellular automata whose dynamics eventually settle either into a fixed point consisting of the homogeneous configuration 1, or into a 2-cycle consisting of both homogeneous configurations.

Lemma 1. *The cellular automaton* $\mathcal{N}_\phi = 65\,535 = (1111111111111111)_2$ *is the only local transition rule whose dynamics have as fixed point the homogeneous one configuration.*

Lemma 2. *There are no cellular automata whose dynamics have as attractor the homogeneous 2-cycle.*

From these results, we may conclude that there are 1 706 dynamically nonequivalent square Boolean peripheral homogeneous cellular automata with fixed null boundary conditions whose dynamics end in a fixed point consisting of a homogeneous configuration.

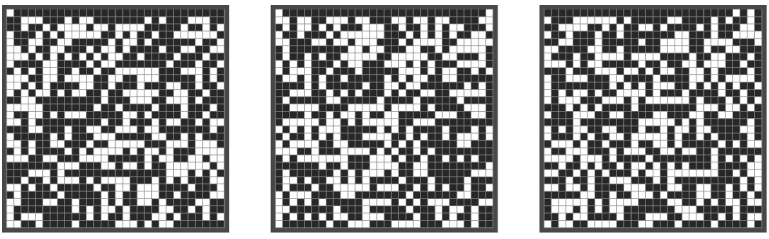
As the preceding results indicate, the adoption of fixed null boundary conditions for cellular automata yields a number of local transition rules with trivial homogeneous dynamics that is greater than one might initially expect. This observation naturally raises the question of whether there exist cellular automata whose dynamics genuinely display chaotic behavior.

In the following, we examine in detail the typical dynamics of a cellular automaton and show that they exhibit features commonly associated with chaos.

4 An Example of a Chaotic Cellular Automaton

We have already discussed Wolfram's classification of cellular automata in the context of trivial homogeneous Class I dynamics. We now show that there exist square Boolean peripheral cellular automata with fixed null boundary conditions whose dynamics exhibit characteristics akin to those associated with chaotic Class III cellular automata.

Table 2. Configurations of the system during its evolution from a randomly chosen initial configuration on a 30×30 lattice for the cellular automaton $\mathcal{N}_\phi = 38\,617$. The sequence illustrates the temporal development of the dynamics and suggests the absence of any clear tendency toward regularity.



T = 10 000 000 T = 20 000 000 T = 30 000 000

Consider the cellular automaton $\mathcal{N}_\phi = 38\,617$ introduced in Example 1. We argue that its dynamics are associated with chaotic behavior. To substantiate this claim, we first present several configurations of the system at different time steps, showing that no tendency toward regularity can be discerned. In other words, one does not observe a growth of local patterns in the images corresponding to the system's configurations at large times (Table 2).
Another way to identify chaotic dynamics is to study how the period of their attractors varies as the number of elements in the system increases. In what follows, we present the results of a statistical analysis of the period of the final cycle of the dynamics, obtained from 1 000 randomly selected initial configurations, for systems of varying sizes. The results are summarized in the following tables.

from 1 000 initial 6×6 configurations

period	n.configs	period	n.configs	period	n.configs
400	500	3 600	12	23 600	415
472	68	4 248	5		

from 1 000 initial 7×7 configurations

period	n.configs	period	n.configs	period	n.configs
96	134	11 208	11	68 182	4
894	41	14 304	145	98 070	9
934	76	28 954	1	119 428	1
1 636	17	31 290	10	171 780	1
3 360	12	39 264	97	417 498	144
3 576	8	44 832	170	731 292	50
7 008	2	65 262	4	764 012	62
9 816	1				

from 1 000 initial 8×8 configurations

period	n.configs	period	n.configs	period	n.configs
8 496	366	1 231 920	23	6 380 496	453
13 518	141	1 960 110	16		

from 1 000 initial 9×9 configurations

period	n.configs	period	n.configs	period	n.configs
522	83	64 214	155	16 759 854	243
58 240	74	15 200 640	167	1 869 911 680	269
60 428	1	15 771 708	7	9 700 808 980	2

If we focus on the growth of the largest attractor periods as a function of the number of elements in the system, we claim that the data obtained allow us to conclude that the cellular automaton $\mathcal{N}_\phi = 38\,617$ belongs to Wolfram's Class III.

As a final remark in this section, we note that the data above reveal something notably different from what is known about Class III elementary cellular automata: some dynamics have small attractor periods. In other words, this cellular automaton exhibits a kind of coexistence of dynamics. This is more apparent in the data obtained for systems with 9×9 elements, but it is also evident in the other cases. The nature and implications of this coexistence of dynamics merit closer examination in future work.

5 Ordered and Chaotic Cellular Automata

Following Wolfram's approach, we now aim to identify the cellular automata belonging to Class II – those whose typical dynamics display very short transients and small attractor periods. To this end, we computed the evolution of 1 000 random 30×30 initial configurations and selected the cellular automata whose dynamics reached a final state within 100 time steps. According to this criterion, 1 761 cellular automata were classified as belonging to Class II. The Wolfram integer code of those cellular automata as well as the homogeneous cellular automata belonging to Class I, will be provided in an appendix file.

Having identified both Class I and Class II cellular automata, we next focused on detecting those belonging to Class III, characterized by chaotic dynamics and long-lived, irregular transients. For this purpose, we computed the evolution of 1 000 random 8×8 initial configurations and selected the cellular automata whose dynamics never reached a final state within 1 000 time steps. According to this criterion, 321 cellular automata were classified as belonging to Class III. Similarly, an appendix will include the Wolfram integer codes corresponding to these cellular automata.

We acknowledge that the criterion adopted in our search for Class III cellular automata may seem rather stringent, particularly in view of our observation that some cellular automata exhibit a coexistence of dynamical behaviours. Given the analysis presented above, it is therefore unsurprising that cellular automaton 28 617 fails to satisfy this criterion. Thus, although this cellular automaton was used as an example of chaotic behaviour, it will not be listed among the chaotic cellular automata of this family. Nevertheless, we deem it appropriate to maintain this more restrictive criterion until the mechanisms underlying such coexistence are better understood.

In conclusion, we say that there are 7 114 dynamically nonequivalent square Boolean peripheral cellular automata with fixed null boundary conditions that may exhibit dynamics between order and chaos. The Wolfram integer code of those cellular automata will be provided in an appendix file.

Our proposal to study this family of square cellular automata with fixed null boundary conditions, aimed at identifying dynamics that lie between order and chaos, would be meaningless if their existence were not already established. We now proceed to present an example of a cellular automaton with these characteristics, whose dynamics are clearly positioned between order and disorder.

6 Cellular Automata Between Order and Chaos

The search for cellular automata whose typical dynamics exhibit behaviors clearly lying between order and disorder is extremely difficult, since a priori there is no criterion comparable to those used to identify local transition rules belonging either to Class II or to Class III. In this sense, it is necessary to carry out an exhaustive investigation of the typical dynamics within the set of cellular automata identified as having the potential to exhibit this type of behavior.

Considering that what we are proposing is only the beginning of a study that requires more time – it is clearly a work in progress – we would like to introduce a class of cellular automata whose typical dynamics display unique characteristics that set them apart from both Classes II and III.

Consider the cellular automaton $\mathcal{N}_\phi = 1\,241 = (0000010011011001)_2$. We argue that its typical dynamics simultaneously exhibit features of both order and disorder. Let us consider a system of 30×30 elements with an initial configuration chosen at random. After 2 000 time steps, the dynamics appear as an information flow from left to right, in which small geometric structures are translated nearly perfectly from the left side of the system to the right, as illustrated by the configurations shown in Table 3.

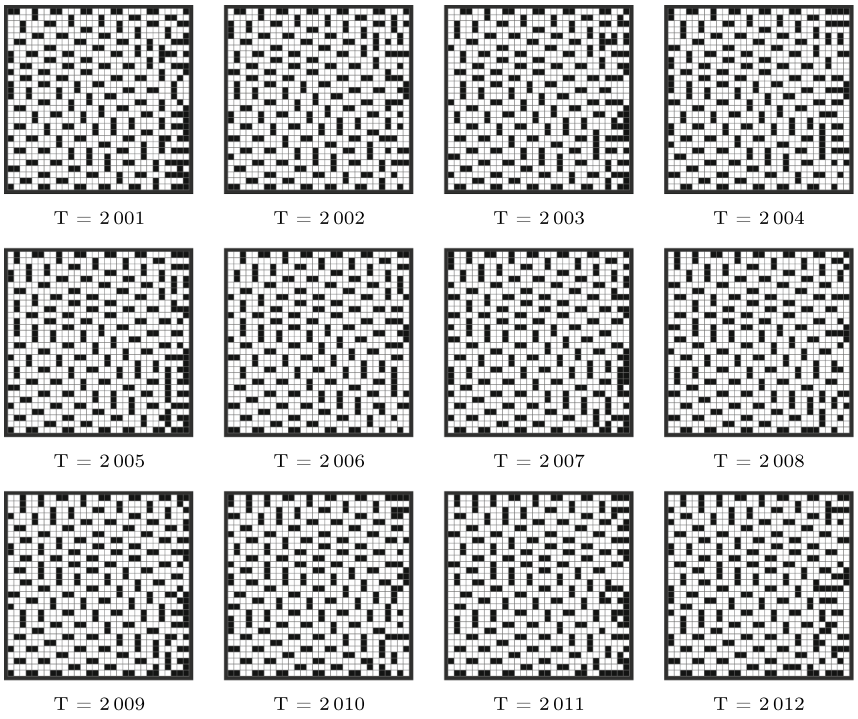
The dynamic behavior illustrated by the configurations presented above is noteworthy in two respects. First, it involves an apparent information flow in a system with fixed null boundary conditions, which is highly counterintuitive, since the boundary states are fixed and therefore cannot act as sources of incoming information. Second – and equally importantly – the translation of small geometric structures is evident yet imperfect, ultimately leading to an attractor with an exceptionally long period of 28 764. Furthermore, since, in this particular case, the transient phase has length 1 361, it follows that all configurations shown above already belong to the final cycle.

In this section, we have proposed the identification of cellular automata with dynamics between order and chaos through the empirical observation of selected configurations. Measures aimed at detecting Wolfram's Class IV cellular automata have been introduced in the past, but only after the characteristic features shared by such dynamics had been recognized. Similarly, we believe that a parameter measuring the notion of flow may be introduced for the cellular automata considered here. Nevertheless, such a parameter is likely to be meaningful only within this particular class of systems, whose dynamics lie between order and chaos. For this reason, in our opinion, any attempt to classify these systems should first rely on a careful empirical analysis of their temporal evolution, from which the common dynamical features of this intermediate class may then be extracted.

7 Conclusions

In this work, we investigated a class of finite two-dimensional Boolean cellular automata defined by a four-neighbour von Neumann neighbourhood and fixed

Table 3. Configurations of the system during its evolution from a randomly chosen initial configuration on a 30×30 lattice, for the cellular automaton 1 241. The sequence illustrates the temporal development of the dynamics and suggests that, after some instants of time, there is a flow of information from the left to the right.



null boundary conditions. Although this setting gives rise to a large number of possible local rules, namely 65 536, the finite character of the system makes a direct computational exploration feasible. This allowed us to perform a systematic characterization of the dynamical behaviours generated within this family, rather than focusing only on isolated examples.

Our exploration revealed a broad range of behaviours. Some local rules give rise to remarkably simple dynamics, rapidly converging to fixed points or short-period attractors, whereas others produce evolutions that are clearly chaotic in character. Between these two extremes, we identified rules whose dynamics display more subtle and structured forms of evolution, suggesting the presence of intermediate regimes between order and chaos. In this sense, the present study provides not only a catalogue of representative behaviours, but also a practical route for locating and analysing cellular automata whose dynamics do not fit neatly into the usual ordered/chaotic dichotomy.

The examples discussed throughout the paper show that the notion of complexity underlying Wolfram's classification captures only part of the phenomenology exhibited by these systems. In particular, the two-dimensional character of the configurations plays a decisive role in the emergence of spatial struc-

tures, apparent information flows, coexistence of dynamical behaviours, and long-period attractors.

One of the main contributions of this work is therefore to show that finite two-dimensional cellular automata with simple local rules and fixed null boundary conditions already provide a rich enough framework for studying the transition between order and chaos. The results presented here indicate that such transitions can be explored empirically through the temporal evolution of configurations, and that this empirical analysis may serve as a first step towards the definition of more refined quantitative measures, possibly tailored to specific families of cellular automata.

Overall, our results reinforce the view that finite two-dimensional cellular automata constitute an effective and insightful setting for studying discrete dynamical systems. At the same time, they suggest that the classification of their behaviours requires criteria sensitive not only to convergence or apparent randomness, but also to spatial organization, propagation phenomena, and the coexistence of distinct dynamical regimes.

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