The Generalized Jackknife and the Extremal Index Estimation

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Abstract. Extreme observations often occur in clusters over time, reflecting persistence in the underlying stochastic process and inducing serial dependence among exceedances. This phenomenon has important implications for risk assessment. Within the framework of Extreme Value Theory, the extremal index is a key parameter for quantifying the degree of clustering of extreme events. In this work, we investigate bias-reduced estimation of the extremal index using the Generalized Jackknife methodology, applied to two widely used estimators: the upcrossing estimator and the Ferro-Segers estimator. A stability-oriented algorithm for extreme value parameter estimation, previously shown to exhibit strong empirical performance, is adapted to guide the selection of both the threshold level k and the tuning parameter inherent to the Generalized Jackknife scheme. This combined approach aims to enhance robustness and reduce sensitivity to threshold choice. An extensive Monte Carlo simulation study is currently in progress to evaluate the finite-sample performance of the proposed methodology under a range of dependence structures. Particular attention is given to bias reduction and variability. Preliminary results suggest that the Generalized Jackknife approach provides a substantial reduction in bias while maintaining competitive overall performance. An application to environmental data illustrates our procedure.

Keywords: Adaptive algorithm · Bias reduction · Extreme value theory · Extremal index · Generalized Jackknife

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1 Preliminaries on Extreme Value Theory

The statistical study of rare and extreme observations plays a central role in many scientific and applied disciplines. In numerous contexts, the main interest lies in unusually large realizations of a stochastic process, such as extreme precipitation, wind speeds, ocean wave heights, financial losses, or wildfire intensities. In other situations, the focus is on extremely small values, for example in the analysis of drought events, structural fatigue, or material degradation. In all these cases, understanding the probabilistic mechanisms governing extreme outcomes is crucial for risk assessment and mitigation.

Extreme events appear in a wide range of areas, including environmental sciences, finance and insurance, telecommunications, engineering reliability, survival analysis and sports science. A key challenge in the statistical analysis of extremes is the intrinsic scarcity of observations in the tail of a distribution. As a consequence, classical statistical methods often perform inadequately in this setting, which motivates the use of specialized probabilistic models and inferential techniques.

Extreme Value Theory (EVT) provides the mathematical framework for describing the stochastic behaviour of extreme observations. Over the last decades, EVT has developed into a well-established area of probability and statistics, supported by solid theoretical foundations and numerous practical applications. Comprehensive accounts of the theory and its applications can be found in [2,3].

Within the EVT framework, particular attention is devoted to quantities that characterize the tail behaviour of the underlying distribution, since these determine the magnitude and frequency of extreme events. Because the full distribution is typically unknown in practice, inference often relies on semiparametric procedures based on asymptotic results that describe the distributional behaviour in the tail region.

Let $\{X_i\}_{i\geq 1}$ denote a sequence of independent and identically distributed (iid) random variables with cumulative distribution function (cdf) F, and define the associated sequence of partial maxima by $M_n := \max(X_1, \ldots, X_n)$. A fundamental result in classical EVT, originating from the pioneering works of Fréchet, Fisher and Tippett and later formalized by Gnedenko and de Haan, establishes that if normalizing sequences exist such that the distribution of the normalized maxima converges to a non-degenerate limit, then the limiting distribution necessarily belongs to the class of extreme value distributions. The corresponding cdf is given by

$$\mathrm{EV}_\xi(\mathrm{x}) = \begin{cases} \exp[-(1+\xi x)^{-1/\xi}], & 1+\xi x > 0 \text{ if } \xi \neq 0 \\ \exp[-\exp(-x)], & x \in R \qquad \text{if } \xi = 0. \end{cases} \tag{1}$$

This formulation has motivated numerous extensions and refinements in the literature, see [8,15].

The parameter ξ in (1), commonly referred to as the *extreme value index*, characterizes the tail heaviness of the distribution and therefore determines the severity of extreme events.

In many real-world processes, extreme observations tend to occur in clusters rather than appearing as isolated events. Such temporal dependence may substantially increase the impact of extreme episodes, since prolonged periods of large values can intensify the underlying phenomenon (see, for example, [4]). In EVT, this clustering behaviour is summarized by the *extremal index* (EI), typically denoted by θ. This parameter takes values in the interval $[0, 1]$, with smaller values indicating stronger clustering of extremes. The limiting case $\theta = 0$ corresponds to a degenerate situation (see, e.g., [17]) and will not be considered in this work.

Although originally introduced within the context of EVT, the EI has also proved useful in several other areas, particularly in the study of dynamical systems (see, for example, [7,18]).

The remainder of the paper is organized as follows. Section 2 presents the theoretical framework for extremes in dependent sequences, introduces the extremal index (EI) as a measure of extremal dependence, reviews existing estimation methods, and describes the Generalized Jackknife estimator considered in this study, together with an automatic stability-based procedure for selecting the tuning parameter δ and the threshold k. Section 3 reports the results of a Monte Carlo simulation study assessing the finite-sample performance of the estimators under temporal dependence. In Sect. 4 we discuss an application to a set of data studied in [2,5] to illustrate our procedure. Finally, a brief conclusion is presented in Sect. 5.

2 Classical and Bias-Reduced Estimators of the Extremal Index

The *extremal index* is a fundamental quantity in the study of extremes for dependent stochastic sequences, as it measures how temporal dependence affects exceedances over high thresholds. In particular, it summarizes how the dependence structure of a stationary process influences both the occurrence rate and the clustering of extreme observations.

Let $\{u_n\}$ be a sequence of increasing thresholds such that, for some constant $\tau > 0$,

$$F(u_n) = 1 - \frac{\tau}{n} + o\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{2}$$

where F denotes the marginal cdf of the process. Within this setting, the EI can be formally defined as follows.

Definition 1 ([17]). *Let $\{X_n\}_{n\geq 1}$ be a strictly stationary sequence with marginal cdf F, and define the partial maxima by*

$$M_n := \max(X_1, \ldots, X_n).$$

The process is said to admit an EI, $\theta \in (0, 1]$ if, for each $\tau > 0$, there exists a sequence of thresholds $\{u_n(\tau)\}_{n\geq 1}$ satisfying

$$n \Pr\{X_1 > u_n(\tau)\} \longrightarrow \tau \quad \text{and} \quad \Pr\{\mathrm{M_n} \leq \mathrm{u_n}(\tau)\} \longrightarrow \mathrm{e}^{-\theta\tau}, \quad \mathrm{n} \to \infty. \tag{3}$$

Several interpretations of the EI have been proposed in the literature. A concise overview of these perspectives is given in [18]. One of the most widely cited interpretations, introduced by [17], links the EI to the clustering of exceedances. Under this viewpoint, θ can be understood as the reciprocal of the limiting mean size of clusters formed by exceedances over high thresholds.

Considerable attention has been devoted to statistical inference for the EI, and a variety of estimation methods have been developed. Early and commonly used approaches include block and runs estimators [16,19,23–25]. These techniques rely on identifying clusters of exceedances through predefined schemes and typically involve tuning parameters whose choice may affect the resulting estimates.

Beyond cluster-based procedures, alternative estimators exploit other probabilistic characterizations of the EI derived from the behaviour of exceedances. Two widely used examples are the upcrossings estimator and the intervals estimator proposed in [6], which are briefly reviewed below.

2.1 The Upcrossings Estimator

Definition 1 also provides the basis for the so-called upcrossings approach to the estimation of the EI, [9–11,19]. The corresponding estimator is defined as

$$\widehat{\theta}^{\mathrm{UC}}(u_n) = \frac{\sum_{i=1}^{n-1} I(X_i \le u_n < X_{i+1})}{\sum_{i=1}^{n} I(X_i > u_n)}. \tag{4}$$

A practical implementation of this estimator was proposed by Gomes *et al.* [12], who suggested selecting the threshold deterministically using order statistics, namely $u = X_{n-k:n}$. This choice leads to the estimator

$$\widehat{\theta}^{\mathrm{UC}}(k) = \frac{1}{k} \sum_{i=1}^{n-1} I(X_i \le X_{n-k:n} < X_{i+1}). \tag{5}$$

For a wide class of dependence structures, such those we are going to consider later in our simulation study, the bias of $\widehat{\theta}^{\mathrm{UC}}(k)$ admits the asymptotic expansion

$$\mathrm{Bias}\left[\widehat{\theta}^{\mathrm{UC}}(k)\right] = \varphi_1(\theta)\frac{k}{n} + \varphi_2(\theta)\frac{1}{k} + o\left(\frac{k}{n}\right) + o\left(\frac{1}{k}\right), \tag{6}$$

as $n \to \infty$, with $k \to \infty$ and $k = o(n)$.

This bias structure naturally motivates the use of the Generalized Jackknife methodology introduced by [14]. Exploiting the expansion in (6), Gomes *et al.* [12] proposed a second–order Generalized Jackknife estimator based on $\widehat{\theta}^{\mathrm{UC}}$ evaluated at three different levels, namely k, $[k/2]+1$ and $[k/4]+1$. The resulting estimator takes the form

$$5\,\widehat{\theta}^{\mathrm{UC}}([k/2]+1) - 2\Big(\widehat{\theta}^{\mathrm{UC}}([k/4]+1) + \widehat{\theta}^{\mathrm{UC}}(k)\Big). \tag{7}$$

Although this estimator substantially reduces bias, it does not improve the mean squared error relative to the original estimator when the threshold level is chosen optimally. To address this limitation, the authors subsequently considered a more general set of evaluation levels, namely k, $[\delta k]+1$ and $[\delta^2 k]+1$, where the parameter δ satisfies $0<\delta<1$.

This idea leads to a family of estimators, denoted by $\widehat{\Theta}^{\mathrm{GJ}(\delta)}$, constructed from linear combinations of $\widehat{\Theta}^{\mathrm{UC}}$ evaluated at the three levels k, $[\delta k]+1$ and $[\delta^2 k]+1$. In particular, Gomes *et al.* [12] introduced the estimator

$$\widehat{\Theta}^{\mathrm{GJ.UC}(\delta)} = \frac{(\delta^2+1)\,\widehat{\Theta}^{\mathrm{UC}}([\delta k]+1) - \delta\Big(\widehat{\Theta}^{\mathrm{UC}}([\delta^2 k]+1) + \widehat{\Theta}^{\mathrm{UC}}(k)\Big)}{(1-\delta)^2}, \tag{8}$$

which is designed to remove the two leading bias components. Consequently, the estimator is asymptotically unbiased. Furthermore, under suitable regularity conditions, both the upcrossings estimator $\widehat{\Theta}^{\mathrm{UC}}$ and the Generalized Jackknife estimator $\widehat{\Theta}^{\mathrm{GJ.UC}(\delta)}$ are consistent and asymptotically normal for $\theta<1$, see [12,19].

2.2 The Intervals Estimator

An alternative approach relies on the temporal spacing between exceedances of a high threshold. The clustering behaviour of extremes is reflected in the distribution of the time intervals separating consecutive exceedances.

Let $S_1<S_2<\cdots<S_N$ denote the ordered times at which exceedances above a high threshold u occur. The corresponding inter-exceedance times are defined as

$$T_j = S_{j+1}-S_j, \qquad j=1,\ldots,N-1.$$

Ferro and Segers [6] showed that the asymptotic distribution of these intervals contains information about EI. Based on this property, they proposed the *intervals estimator*, defined as

$$\widehat{\Theta}^{\mathrm{FS}} = \frac{2\left(\sum_{j=1}^{N-1}(T_j-1)\right)^2}{(N-1)\sum_{j=1}^{N-1}(T_j-1)(T_j-2)}. \tag{9}$$

Unlike runs-based estimators, this approach avoids explicit declustering and instead relies solely on the spacing between exceedances. Under suitable regularity conditions, the estimator is consistent and asymptotically normal.

The bias structure observed for the UC estimator suggests that similar bias–reduction techniques may also be beneficial for other EI estimators. In particular, the Generalized Jackknife methodology could be naturally extended to the intervals estimator of Ferro and Segers, $\widehat{\Theta}^{\mathrm{FS}}$. This idea is explored in the present work, where a bias-reduced versions of the $\widehat{\Theta}^{\mathrm{FS}}$ estimator are constructed as

$$\widehat{\Theta}^{\mathrm{GJ.FS}(\delta)} = \frac{(\delta^2+1)\,\widehat{\Theta}^{\mathrm{FS}}([\delta k]+1) - \delta\Big(\widehat{\Theta}^{\mathrm{FS}}([\delta^2 k]+1) + \widehat{\Theta}^{\mathrm{FS}}(k)\Big)}{(1-\delta)^2}, \tag{10}$$

and their finite-sample performance are also investigated in a simulation study.

2.3 Computational Algorithm for Parameter Selection

The practical implementation of the bias–reduced estimator $\widehat{\Theta}^{GJ.\bullet(\delta)}$, where $\bullet$ denotes an EI estimator, requires the specification of two parameters: the number of upper order statistics k and the tuning parameter δ. The performance of the estimator may depend substantially on these choices, making their selection an important aspect of the estimation procedure.

To address this issue, we adopt a stability–based strategy inspired by the algorithmic approach proposed by [13] and later refined by [20]. This methodology exploits regions of stability in the sequence of estimates as a function of k, allowing the identification of suitable parameter values. Similar ideas have also been considered in the context of extremal index estimation; see [22].

The procedure described below provides a data–driven method for selecting both the tuning parameter δ and the threshold level k.

1. For a given sample $(x_1, \ldots, x_n)$, compute the estimator $\widehat{\Theta}^{\mathrm{GJ}.\bullet}(k)$, as those defined in defined in (8) and (10), for all $k = 1, \ldots, n-1$ and for a grid of values $\delta \in (0, 1)$.
2. For each k, round the estimates to zero decimal places and define

$$a_k^{\widehat{\Theta}^{GJ.\bullet(\delta)}}(0) = \mathrm{round}\Big(\widehat{\Theta}^{GJ.\bullet(\delta)}(k), 0\Big).$$

3. For each δ, identify consecutive values of k for which $a_k^{\widehat{\Theta}^{GJ.\bullet(\delta)}}(0)$ remains constant. Let $k_{\min}^{\widehat{\Theta}^{GJ.\bullet(\delta)}}$ and $k_{\max}^{\widehat{\Theta}^{GJ.\bullet(\delta)}}$ denote the endpoints of the longest such sequence, with length

$$l_{\widehat{\Theta}^{GJ.\bullet(\delta)}} = k_{\max}^{\widehat{\Theta}^{GJ.\bullet(\delta)}} - k_{\min}^{\widehat{\Theta}^{GJ.\bullet(\delta)}}.$$

4. Determine the smallest integer $j_0 \geq 0$ such that the rounded values

$$a_k^{\widehat{\Theta}^{GJ.\bullet(\delta)}}(j) = \mathrm{round}\Big(\widehat{\Theta}^{GJ.\bullet(\delta)}(k), j\Big)$$

   are not all identical when $j = j_0$.
5. Using the sequence $a_k^{\widehat{\Theta}^{GJ.\bullet(\delta)}}(j_0)$, identify the longest run of equal values and denote its endpoints by $k_{\min}^{\widehat{\Theta}^{GJ.\bullet(\delta)}}$ and $k_{\max}^{\widehat{\Theta}^{GJ.\bullet(\delta)}}$.

6. For $k_{\min}^{\widehat{\Theta}^{GJ.\bullet(\delta)}} \leq k \leq k_{\max}^{\widehat{\Theta}^{GJ.\bullet(\delta)}}$, recompute the estimates rounded to $j_0 + 2$ decimal places and determine their modal value. Let $\mathcal{K}_{\widehat{\Theta}^{GJ.\bullet(\delta)}}$ denote the set of indices attaining this value.

7. Define

$$\hat{k}_{\widehat{\Theta}^{GJ.\bullet(\delta)}} = \max \mathcal{K}_{\widehat{\Theta}^{GJ.\bullet(\delta)}}.$$

8. Choose δ_0 as the value of δ that maximizes $l_{\widehat{\Theta}^{GJ.\bullet(\delta)}}$, for all values of δ considered.

9. The final estimate of the extremal index is given by

$$\widehat{\Theta}^{GJ.\bullet(\delta_0)}\left(\hat{k}_{\widehat{\Theta}^{GJ.\bullet(\delta_0)}}\right).$$

The algorithm therefore identifies stability regions in the sequence of estimates and uses this information to determine suitable values for both the tuning parameter δ and the threshold level k.

3 Monte Carlo Study of Extremal Index Estimators Under Temporal Dependence

Monte Carlo simulation experiments are a standard tool for evaluating the performance of statistical estimators in finite samples. Such experiments allow full control over the data-generating mechanism and the true parameter values, making it possible to analyse the behaviour of competing estimators under well-defined stochastic settings. All simulations were conducted using the *R* software environment. In this section we examine the empirical performance of the EI estimators considered in this work. In particular, the study includes both the bias-reduced estimators based on the Generalized Jackknife methodology. The comparison is carried out through Monte Carlo experiments under dependent stochastic models with known extremal properties.

The simulated data are generated from first-order max-autoregressive processes, commonly referred to as ARMAX models. These processes are frequently used in extreme value analysis because they possess analytically tractable extremal characteristics while allowing the strength of extremal dependence to be controlled through a single parameter.

Max-Autoregressive Process I (ARMAX I) Let $\{Z_i\}_{i\geq 1}$ be independent random variables with unit Fréchet distribution. For a parameter $0 < \theta \leq 1$, define the stochastic process $\{X_i\}_{i\geq 1}$ recursively as

$$X_1 = Z_1, \qquad X_i = \max\big\{(1-\theta)X_{i-1},\, \theta Z_i\big\}, \quad i \geq 2.$$

This construction yields a strictly stationary process with unit Fréchet marginal distribution, corresponding to an extreme value index $\xi = 1$. Moreover, for threshold sequences of the form $u_n = nx$, with $0 < x < \infty$, the distribution of the sample maximum

$$M_n = \max(X_1, \ldots, X_n)$$

satisfies

$$P\{M_n \leq u_n\} \longrightarrow \exp\left(-\frac{\theta}{x}\right), \qquad n \to \infty.$$

The EI associated with this process is equal to θ, [2].

Max-Autoregressive Process II (ARMAX II) Let $\{Z_i\}_{i\geq 1}$ be independent random variables with unit Fréchet distribution and let X_0 be a random variable with cumulative distribution function

$$H_0(x) = \exp\big(-x^{-1}(\beta^{-1}-1)\big).$$

For a parameter $\beta \in (0,1)$, define the stochastic process

$$X_i = \beta \max\{X_{i-1}, Z_i\}, \qquad i \geq 1.$$

The EI of this process is given by $\theta = 1-\beta$, [1].

The EI is estimated using the algorithm described in Sect. 2.3, based on samples of size $n = 500,\ 1000,\ 2000$ generated from both ARMAX I and ARMAX II models. The simulation study considers several values of EI together with different choices of the tuning parameter δ. For each model, Tables 1 and 2 report the corresponding estimates obtained using the two estimators, $\widehat{\Theta}^{\mathrm{UC}}$ and $\widehat{\Theta}^{\mathrm{FS}}$, together with their bias-reduced versions for some values of δ. For each configuration, the value of k associated with the longest stability region is also presented for every value of δ considered, along with the corresponding estimate of θ. As an initial assessment of the results, the combinations of δ, k, and the resulting estimates that are closest to the true value of θ are highlighted in **bold**.

These preliminary simulation results suggest that, for smaller sample sizes estimates closer to the true parameter value tend to be obtained for relatively small values of δ. As the sample size increases, however, the values of δ that yield estimates nearest to the true parameter value also tend to increase.

Table 1. EI estimates selecting the parameter δ and the threshold level k, using algorithm described in Sect. 2.3 for ARMAX I process

	$n = 500$											
	$\theta = 0.25$				$\theta = 0.50$				$\theta = 0.75$			
	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$
	207	0.1683	340	0.1878	248	0.2691	323	0.3353	161	0.5864	237	0.8244
δ	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS.}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$
0.15	412	0.1722	292	0.0774	**425**	**0.4915**	385	0.4779	**237**	**0.7942**	265	0.6053
0.20	433	0.2106	403	0.2022	293	0.4339	342	0.3923	426	0.7037	**432**	**0.7165**
0.25	**489**	**0.2445**	304	0.1781	290	0.4868	**472**	**0.4857**	458	0.7971	347	0.5937
0.30	390	0.2032	453	0.2943	375	0.5705	336	0.5764	429	0.8097	317	0.5025
0.35	422	0.2297	386	0.2866	298	0.6030	317	0.5949	422	0.8938	343	0.9617
0.40	369	0.2042	470	0.3132	378	0.3893	409	0.0614	487	0.9767	287	0.8856
0.45	314	0.2131	461	0.3122	453	0.4053	349	0.1067	269	0.8259	266	0.5729
0.50	399	0.2894	**434**	**0.2281**	353	0.3238	285	0.1001	474	0.9326	406	1.0000
	$n = 1000$											
	$\theta = 0.25$				$\theta = 0.50$				$\theta = 0.75$			
	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$
	825	0.0847	411	0.2709	338	0.3864	567	0.4193	354	0.5127	404	0.7889
δ	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS.}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$
0.15	**754**	**0.2289**	592	0.1984	651	0.5934	335	0.6225	849	0.7904	400	0.9193
0.20	983	0.2760	928	0.3362	526	0.5973	**869**	**0.5060**	**871**	**0.7635**	653	0.8026
0.25	870	0.2775	**640**	**0.3264**	**895**	**0.5081**	972	0.5797	987	0.7321	850	0.9083
0.30	924	0.3092	871	0.4687	757	0.4613	742	0.5979	897	0.7312	658	0.8749
0.35	870	0.3135	980	0.4137	507	0.4795	530	0.5221	923	0.7098	671	0.9558
0.40	656	0.3048	913	0.3979	958	0.6133	934	0.6118	779	0.6833	589	1.0000
0.45	649	0.3244	647	0.4343	910	0.5839	637	0.4900	750	0.7298	**440**	**0.7940**
0.50	732	0.1891	576	0.5207	752	0.6146	693	0.6128	660	0.7189	788	1.0000
	$n = 2000$											
	$\theta = 0.25$				$\theta = 0.50$				$\theta = 0.75$			
	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$
	640	0.2197	714	0.2607	491	0.4024	1389	0.3905	265	0.6955	745	0.7637
δ	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS.}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$
0.15	**1972**	**0.2727**	1838	0.2739	1985	0.5331	1758	0.5998	1939	0.8024	1328	0.7138
0.20	1523	0.2756	**1437**	**0.2634**	1988	0.4994	1724	0.5658	1843	0.7841	**1695**	**0.7636**
0.25	1738	0.3180	690	0.2154	1718	0.5061	1654	0.4139	1986	0.8013	1870	0.8905
0.30	1034	0.2665	1845	0.4282	**1324**	**0.5005**	1455	0.4216	1845	0.7982	1140	0.7766
0.35	1705	0.3239	1795	0.4331	1528	0.5214	1815	0.5910	1827	0.7951	942	0.7919
0.40	1638	0.3347	1388	0.4882	1633	0.5321	**1651**	**0.5004**	**1589**	**0.7765**	1462	1.0000
0.45	1840	0.3262	1780	0.4253	1880	0.5994	1762	0.6204	1607	0.8407	1396	0.9129
0.50	1886	0.3209	1839	0.3809	1689	0.5955	1287	0.4938	1357	0.7844	1234	0.8835

Table 2. EI estimates selecting the parameter δ and the threshold level k, using algorithm described in Sect. 2.3 for ARMAX II process

	$n = 500$											
	$\theta = 0.25$				$\theta = 0.50$				$\theta = 0.75$			
	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$
	333	0.1287	448	0.1224	153	0.4026	348	0.4169	253	0.3858	248	0.5799
δ	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS.}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$
0.15	**463**	**0.2372**	474	0.2889	277	0.3323	342	0.3985	345	0.8769	465	0.9407
0.20	451	0.2734	378	0.3484	241	0.3068	405	0.5560	461	0.7810	224	0.9358
0.25	474	0.2808	**487**	**0.2263**	**410**	**0.4667**	**446**	**0.5205**	369	0.7922	**385**	**0.7412**
0.30	488	0.3074	487	0.1822	418	0.5973	316	0.5445	274	0.8195	388	1.0000
0.35	410	0.2654	298	0.3184	371	0.5745	303	0.5953	**329**	**0.7255**	170	0.8275
0.40	343	0.2833	464	0.1082	369	0.6429	340	0.6253	306	0.7120	334	0.8150
0.45	270	0.2869	273	0.0780	282	0.5792	462	0.7093	413	0.6139	393	0.7866
0.50	454	0.3156	282	0.0344	347	0.7789	364	0.4394	468	0.6857	359	0.6233
	$n = 1000$											
	$\theta = 0.25$				$\theta = 0.50$				$\theta = 0.75$			
	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$
	462	0.1901	653	0.1696	389	0.3589	406	0.4889	647	0.2978	459	0.6431
δ_0	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS.}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$
0.15	**652**	**0.2654**	732	0.4049	**941**	**0.5029**	**602**	**0.5054**	553	0.7123	627	0.6782
0.20	983	0.2735	893	0.3974	705	0.5154	453	0.4158	**848**	**0.7702**	**605**	**0.7220**
0.25	834	0.2668	890	0.3881	985	0.5290	918	0.5993	934	0.7784	864	0.6409
0.30	915	0.2815	858	0.3879	970	0.5079	893	0.7049	965	0.7931	708	0.5778
0.35	986	0.3212	721	0.3789	818	0.5162	850	0.7012	892	0.8285	786	0.7090
0.40	953	0.3301	873	0.3894	915	0.5673	557	0.6783	786	0.8023	590	0.6863
0.45	459	0.1989	741	0.4207	969	0.5903	879	0.7272	730	0.7841	611	0.7846
0.50	841	0.3291	**816**	**0.3167**	948	0.5848	689	0.4659	922	0.8224	545	0.6008
	$n = 2000$											
	$\theta = 0.25$				$\theta = 0.50$				$\theta = 0.75$			
	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$	$\hat{k}$	$\widehat{\theta}^{UC}$	$\hat{k}$	$\widehat{\theta}^{FS}$
	924	0.1676	1319	0.1579	556	0.3896	1141	0.4291	521	0.5594	799	0.6726
δ	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS.}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$	$\hat{k}$	$\widehat{\theta}^{GJ.UC}$	$\hat{k}$	$\widehat{\theta}^{GJ.FS}$
0.15	**1985**	**0.2382**	1365	0.2981	1746	0.4695	1450	0.3937	1232	0.7985	785	0.9122
0.20	1853	0.2257	**1743**	**0.2064**	1988	0.5229	913	0.3188	1688	0.7031	1448	0.8022
0.25	1335	0.2025	1862	0.2033	1886	0.5233	718	0.2974	1840	0.6780	1286	0.7281
0.30	1585	0.2295	1735	0.1948	**1655**	**0.5116**	1692	0.5986	1802	0.7015	**1431**	**0.7349**
0.35	1393	0.2295	1744	0.1791	1329	0.5126	1733	0.5847	1111	0.5899	1584	0.7700
0.40	1766	0.2686	1563	0.1664	1191	0.5238	1389	0.6278	**1770**	**0.7874**	1170	0.7278
0.45	1060	0.2239	1794	0.1722	1694	0.5270	**1701**	**0.5061**	1525	0.7951	1339	0.8071
0.50	1798	0.3222	1453	0.1299	1499	0.4933	1454	0.4775	1372	0.7903	918	0.8348

4 An Application to Real Data

In this section we apply our procedure to the series of daily maximum temperatures (in degrees Celsius) recorded in Uccle, Belgium, during the month of July over the period 1901–1999. The dataset contains $n = 3051$ observations and is available in the `exdex` package in `R` under the name `Uccle`. Estimates are computed over a wide range of threshold values k, allowing the identification of stable regions in the estimation of the EI. The tunparameter δ is selected through the proposed stability-based algorithm, yielding the value $\delta = 0.2$. Conditional on this choice, the algorithm identifies the optimal threshold $k = 2973$ for $\widehat{\Theta}^{GJ_UC}$ estimator, corresponding to a pronounced stability region and producing the estimate $\widehat{\theta}^{GJ_UC} = 0.4804$. Similarly for the $\widehat{\Theta}^{GJ_FS}$ estimator, the optimal threshold is found to be $k = 2915$, leading to the estimate $\widehat{\theta}^{GJ_FS} = 0.4782$. Figure 1 illustrates the procedure considered here and also displays the corresponding extremal index estimates.

This dataset was previously investigated by [2] in the context of extremal dependence analysis using parametric approaches. The estimated values of the extremal index were found to lie approximately between 0.49 and 0.56. Later [5] considered this dataset to illustrate the comparison of some estimation methods. The values obtained with our procedure agree with the values presented in those papers.

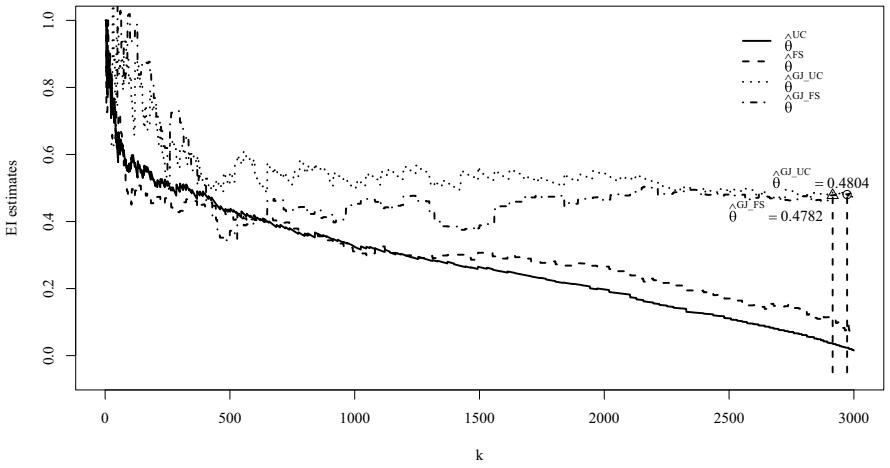

Fig. 1. Adaptive choice of the level k with the optimal value of δ ($\delta = 0.2$) for estimating θ for the daily maximum temperatures (in degrees Celsius) in Uccle, during the month of July, over the period 1901–1999.

5 Concluding Remarks

This work aims to improve the practical estimation of the parameter θ, with particular emphasis on bias reduction and on the data-driven selection of tuning parameters that strongly influence finite-sample performance.

In this context, we investigated the use of the Generalized Jackknife applied to two EI estimators. A stability-based procedure for the automatic calibration of both the tuning parameter δ and the threshold level k used in the estimation of the EI is proposed. The main idea is to identify regions where the estimator exhibits stable behaviour, providing a principled criterion for selecting the relevant tuning parameters.

A preliminary version of this procedure was presented in [21], where only samples of size $n = 1000$ were considered for the UC estimator.

In the present work, the performance of the proposed approach was further examined through simulation experiments under scenarios with temporal dependence. The results suggest that the stability-based calibration provides a systematic and effective strategy for determining the tuning parameters required by the estimator. Further simulation studies involving a wider range of sample sizes, dependence structures, and parameter configurations are currently under development, with the aim of providing a more comprehensive assessment of the robustness and applicability of the proposed methodology. An application to a real dataset already used in previous studies illustrates the good performance of our methodology.

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References

1. Alpuim, M.T.: An extremal Markovian sequence. J. Appl. Probab. **26**(2), 219–232 (1989). https://doi.org/10.2307/3214030
2. Beirlant, J., Goegebeur, Y., Segers, J., Teugels, J.L.: Statistics of Extremes: Theory and Applications. Wiley (2006). https://doi.org/10.1002/SERIES1345
3. Embrechts, P., Klüppelberg, C., Mikosch, T.: Modelling Extremal Events. AM, vol. 33. Springer, Heidelberg (1997). https://doi.org/10.1007/978-3-642-33483-2
4. Faranda, D., Messori, G., Yiou, P., Thao, S., Pons, F., Dubrulle, B.: Dynamical footprints of hurricanes in the tropical dynamics. Chaos Interdisc. J. Nonl. Sci. **33**(1) (2023). https://doi.org/10.1063/5.0093732
5. Ferreira, M.: Heuristic tools for the estimation of the extremal index: a comparison of methods. REVSTAT-Stat. J. **16**(1), 115–136 (2018)
6. Ferro, C.A.T., Segers, J.: Inference for clusters of extreme values. J. R. Stat. Soc. Ser. B Stat Methodol. **65**(2), 545–556 (2003)

7. Freitas, A.C.M., Freitas, J.M., Soares, J.V.: Rare events for product fractal sets. J. Phys. A: Math. Theor. **54**(34), 345202 (2021). https://doi.org/10.1088/1751-8121/ac16c6
8. Gnedenko, B.: Sur la distribution limite du terme maximum d'une serie aleatoire. Ann. Math. **3**(44), 423–453 (1943). https://doi.org/10.2307/1968974
9. Gomes, M.I.: Statistical inference in an extremal Markovian model. In: Compstat: Proceedings in Computational Statistics, 9th Symposium held at Dubrovnik, Yugoslavia, pp. 257–262. Springer (1990). https://doi.org/10.1007/978-3-642-50096-1-39
10. Gomes, M.I.: In: I Congresso Ibero-Americano de Esdadistica e Investigacion Operativa, pp. 209–220 (1992). Modelos extremais em esquemas de dependência
11. Gomes, M.I.: On the estimation of parameters of rare events in environmental time series. Stat. Environ. **2**, 225–241 (1993)
12. Gomes, M.I., Hall, A., Miranda, M.C.: Subsampling techniques and the jackknife methodology in the estimation of the extremal index. Comput. Stat. Data Anal. **52**(4), 2022–2041 (2008). https://doi.org/10.1016/j.csda.2007.06.023
13. Gomes, M.I., Henriques-Rodrigues, L., Fraga Alves, M.I., Manjunath, B.G.: Adaptive port–MVRB estimation: an empirical comparison of two heuristic algorithms. J. Stat. Comput. Simul. **83**(6), 1129–1144 (2013). https://doi.org/10.1080/00949655.2011.652113
14. Gray, H.L., Schucany, W.R.: The generalized jackknife statistic. M. Dekker, New York (1972). https://doi.org/10.1080/00224065.1973.11980580
15. de Haan, L.: On Regular Variation and Its Application to the Weak Convergence of Sample Extremes, vol. 32. Centrum Voor Wiskunde en Informatica (1970)
16. Hsing, T.: Extremal index estimation for a weakly dependent stationary sequence. Ann. Stat., 2043–2071 (1993). https://doi.org/10.1214/aos/1176349409
17. Leadbetter, M.R., Lindgren, G., Rootzen, H.: Extremes and Related Properties of Random Sequences and Processes. Springer Series in Statistics. Springer (1983). https://doi.org/10.1007/978-1-4612-5449-2
18. Moloney, N.R., Faranda, D., Sato, Y.: An overview of the extremal index. Chaos Interdisc. J. Nonl. Sci. **29**(2) (2019). https://doi.org/10.1063/1.5079656
19. Nandagopalan, S.: Multivariate Extremes and Estimation of the Extremal Index. The University of North Carolina at Chapel Hill (1990)
20. Neves, M.M., Gomes, M.I., Figueiredo, F., Gomes, D.P.: Modeling extreme events: sample fraction adaptive choice in parameter estimation. J. Stat. Theory Pract. **9**(1), 184–199 (2014). https://doi.org/10.1080/15598608.2014.890984
21. Prata Gomes, D., Neves, M.M.: Computational Methods for Extremal Index Estimation. Accepted
22. Prata Gomes, D., Neves, M.M.: Resampling procedures for a more reliable extremal index estimation. In: Data Analysis and Applications 4, chap. 6, pp. 89–101. Wiley (2020). https://doi.org/10.1002/9781119721611.ch6
23. Robert, C.Y., Segers, J., Ferro, C.A.: A sliding blocks estimator for the extremal index. Electron. J. Stat. **3**, 993–1020 (2009). https://doi.org/10.1214/08-EJS345
24. Smith, R.L., Weissman, I.: Estimating the extremal index. J. Royal Stat. Soc. Ser. B (Methodol. **56**(3), 515–528 (1994). https://doi.org/10.1111/j.2517-6161.1994.tb01997.x
25. Weissman, I., Novak, S.Y.: On blocks and runs estimators of the extremal index. J. Stat. Plan. Inference **66**(2), 281–288 (1998). https://doi.org/10.1016/S0378-3758(97)00095-5