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A Modified Bias-Corrected Estimator for the Log-Logistic Shape Parameter

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Abstract. In this article, we propose a modified bias reduced estimator of the shape parameter of the three-parameter log-logistic distribution. The new estimator is based on the tail subsample and allows for the independent estimation of only the shape parameter. Consequently, the new estimator can be used in conjunction with the best linear unbiased estimators for the location and scale parameters. Further, we discuss the choice of the threshold that defines the tail subsample. The theoretical and simulation results demonstrate that the proposed estimator is nearly unbiased and clearly outperforms other approaches, particularly in small and moderate sample sizes. Finally, the performance of the new estimator is illustrated with an application to a real dataset.

Keywords: Bias correction · Log-logistic model · Shape-parameter

1 Introduction

The log-logistic distribution is a flexible heavy tailed model, widely applied in fields such as survival analysis, hydrology, insurance, economics, and reliability. The variable X follows a three parameter log-logistic distribution, with shape parameter $\alpha > 0$, location parameter μ, and scale parameter $\sigma > 0$, if its distribution function (DF) can be expressed as follows

$$F(x) = 1 - \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^{\alpha}} = \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^{-\alpha}}, \quad x > \mu. \tag{1}$$

This model is a particular member of the four-parameter kappa family of distributions [22]. The location parameter defines the lower bound of the support of the distribution. When $\mu = 0$ one obtains the two parameter log-logistic model, which is also known as the Fisk [14] or the Pareto-type III distribution [3]. The scale parameter corresponds to the median of $X - \mu$ and the parameter α controls the tail heaviness.

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The r-th moment about the location parameter, μ, is:

$$E[(X-\mu)^r] = \sigma^r B\left(1-\frac{r}{\alpha}, 1+\frac{r}{\alpha}\right), \quad r<\alpha,$$

where B represents the complete Beta function. This moment exists only if $r < \alpha$. Hence, the shape parameter determines the existence of moments: the mean and the variance only exist if $\alpha > 1$ and $\alpha > 2$, respectively. This moment constraint has important implications for parameter estimation methods based on moments.

Several methods for estimating the parameters of the three-parameter log-logistic model have already been proposed by different researchers (see [1,8,27,28]). Balakrishnan et al. [5] obtained the best linear unbiased estimators (BLUE) for the location and scale parameters in (1), assuming a known shape parameter. Regarding the estimation of the parameter α, Ahsanullah and Alzaatreh [2] observed that the log-logistic model belongs to the class of Pareto-type distributions and derived an estimator, based on the largest order statistics. The selection of the number of largest order statistics is not trivial, as it involves the traditional bias-variance trade-off. More recently, Mateus and Caeiro [23] and Caeiro and Mateus [13] improved the aforementioned shape parameter estimator by proposing alternative bias-reduced estimators. The present work focuses on further reducing the bias in the estimation of the shape parameter of the three-parameter log-logistic distribution.

This paper is structured as follows: Sect. 2 provides a review of several estimators for the parameter α, introduces the proposed bias-reduced estimator and discusses the choice of the sample fraction used in inference. Section 3 presents a Monte Carlo simulation study to evaluate the finite sample performance of the proposed estimator. An application to a real dataset is given in Sect. 4. Finally, Sect. 5 provides a summary and perspectives of future research.

2 Estimation of the Shape Parameter

In the following, let $(X_1, X_2, \ldots, X_n)$ denote a sample of n independent and identically distributed random variables with a common three-parameter log-logistic, with DF in (1). Furthermore, let $(X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n})$ denote the associated sample of order statistics.

2.1 A Review of Several Estimators from the Literature

Ahsanullah and Alzaatreh [2] observed that the log-logistic model exhibits a Pareto-type tail, with a tail index equal to α. More precisely, the survival function $1 - F(x)$ is characterized by:

$$1 - F(x) = x^{-\alpha} l(x), \tag{2}$$

where $\alpha > 0$ and $l(x)$ is slowly-varying, as $x \to \infty$. The parameter $\xi = \alpha^{-1}$, known as the extreme value index, is the primary focus in tail inference. A wide

range of estimators for ξ have been developed in the literature [6,7,18,24]. One of the most widely used estimator for a positive extreme value index was introduced by Hill [21]. This estimator corresponds to the mean of the log-excesses, above the threshold $X_{n-k:n} > 0$,

$$H(k) = \frac{1}{k} \sum_{i=1}^{k} \ln X_{n-i+1:n} - \ln X_{n-k:n}, \qquad 1 \leq k \leq n-1, \tag{3}$$

where k represents the number of the largest order statistics (exceedances) used from the right tail of the distribution.

This estimator is only invariant under scale transformations. In the case of data from a model with a location parameter, such as the three-parameter log-logistic distribution, different locations lead to different bias patterns (see [4,16,20] for further details). This problem led Ahsanullah and Alzaatreh [2] to use a particular case of the location-invariant Hill-type estimator introduced in [15],

$$\hat{H}(k) = \frac{1}{k} \sum_{i=1}^{k} \ln \frac{X_{n-i+1:n} - X_{1:n}}{X_{n-k:n} - X_{1:n}}, \quad 1 \leq k \leq n-2. \tag{4}$$

Thus, this approach yielded the following scale- and location-invariant estimator for the shape parameter:

$$\hat{\alpha}^{LIH}(k) = \frac{1}{\hat{H}(k)}. \tag{5}$$

The choice of the threshold k is critical and involves the well known trade-off between the bias and the variance of the estimator ([9,11,17,26]). Ahsanullah and Alzaatreh [2] suggested $k = \lfloor n/10 \rfloor$, if $n > 100$, where $k = \lfloor x \rfloor$ designate the floor function. It is known that, to minimize the mean square error (MSE), k should be intermediate, that is, the threshold should satisfies the following conditions:

$$k = k_n \to \infty \quad \text{and} \quad \frac{k}{n} \to 0, \quad \text{as} \qquad n \to \infty. \tag{6}$$

Mateus and Caeiro [23] obtained the distributional behaviour

$$\hat{\alpha}^{LIH}(k) \overset{d}{=} \alpha \left(1 - \frac{Z_k}{\sqrt{k}} + O_p\left(\frac{\sqrt{k}}{n}\right) - \frac{k}{2n}(1 + o_p(1)) \right), \tag{7}$$

where Z_k represents an asymptotic standard normal random variable, and found that the optimal threshold, k_0^{LIH}, which minimizes the asymptotic MSE (AMSE) of $\hat{\alpha}^{LIH}(k)$, is

$$k_0^{LIH} = (2n^2)^{1/3}.$$

Since the location-invariant Hill-type, in (4), is strongly affected by bias, other alternative bias reduced estimators appeared in the literature. Mateus and Caeiro [23] adapted the generalized Jackknife (GJ) estimator for the extreme

value index, introduced in [19], thereby deriving the following location-invariant GJ estimator for the shape parameter α:

$$\hat{\alpha}^{GJ}(k) = \frac{\hat{H}(k)}{\hat{M}(k) - (\hat{H}(k))^2}, \quad 1 \leq k \leq n-2, \tag{8}$$

with

$$\hat{M}(k) = \frac{1}{k}\sum_{i=1}^{k}\left(\ln \frac{X_{n-i+1:n} - X_{1:n}}{X_{n-k:n} - X_{1:n}}\right)^2. \tag{9}$$

While this estimator has a null dominant term of asymptotic bias, it exhibits a larger asymptotic variance than the location-invariant Hill-type estimator $\hat{\alpha}^{LIH}(k)$, in (4). More recently, Caeiro and Mateus [13] eliminated the main term of asymptotic bias of $\hat{H}(k)$, in (4), without increasing its asymptotic variance. They derived the following bias-reduced (BR) estimator for the shape parameter α:

$$\hat{\alpha}^{BR}(k) = \frac{1}{\hat{H}^{BR}(k)}, \quad 1 \leq k \leq n-2, \tag{10}$$

where

$$\hat{H}^{BR}(k) = \hat{H}(k) \times \left(1 - \frac{k}{2n}\right), \quad 1 \leq k \leq n-2. \tag{11}$$

This estimator is obtained by estimating the main element of asymptotic bias of $\hat{H}(k)$, which is $\alpha^{-1}\frac{k}{2n}$, by $\hat{H}(k)\frac{k}{2n}$, and subsequently subtracting it from the original estimator, that is,

$$\hat{H}^{BR}(k) = \hat{H}(k) - \hat{H}(k)\frac{k}{2n}.$$

Since only the dominant bias term is removed, the $\hat{H}^{BR}(k)$ estimator tends to exhibit a positive bias for large values of k.

2.2 An Improved Bias-Reduced Estimator

To further decrease the effect of bias, we now propose an improved bias-reduced (IBR) estimator for the parameter α. The estimator is constructed by using bias correction techniques similar to the one used in [10,12]. More precisely, since $e^x - 1 = x + o(x)$, as $x \to 0$ and $e^x - 1 > x$, for $x > 0$, the main bias term of $\hat{H}(k)$ is estimated by $\hat{H}(k)(\exp\left(\frac{k}{2n}\right) - 1)$ and subsequently subtracted from the $\hat{H}(k)$ estimator, leading to the improved bias reduced (IBR) estimator:

$$\hat{\alpha}^{IBR}(k) = \frac{1}{\hat{H}^{IBR}(k)}, \quad 1 \leq k \leq n-2 \tag{12}$$

where

$$\hat{H}^{IBR}(k) = \hat{H}(k) \times \left[2 - \exp\left(\frac{k}{2n}\right)\right], \quad 1 \leq k \leq n-2. \tag{13}$$

For a detailed analysis of the asymptotic properties of the aforementioned estimators, see Caeiro and Mateus [13]. Furthermore, the asymptotic normality of (13) follows from the results established in the same paper.

2.3 Threshold Selection

For small and moderate sample sizes, the determination of the optimal threshold should not be made based on asymptotic results. Here we apply the algorithm proposed in [23]:

Algorithm [Empirical Threshold Selection]:

1. Generate 5000 samples of size n from a log-logistic distribution, with n taking values between 50 and 2000, with step 50.
2. Let $\hat{\alpha}(k,i,n)$ denote the estimates based on the i-th sample of size n. For each sample size,
   - compute $\hat{\alpha}(k,i,n)$, $k = 1,2,\ldots,n-2$, $i = 1,\ldots,5000$.
   - compute the empirical Mean Squared Error, as a function of k.
   - Determine the value $\hat{k}_0(n)$ that minimizes the empirical Mean Squared Error.
3. Finally, perform a Power Regression with $\hat{k}_0(n)$ the response variable and the sample size as the predictor variable. The regression coefficients are the vector values (a_1, a_2). The empirical threshold is then given by

$$\hat{k}_E = \hat{k}_E(n) = \lfloor a_1 n^{a_2} \rfloor, \tag{14}$$

where $\lfloor x \rfloor$ is the floor function (the integer part of x).

Remark: The agreement between the empirical thresholds and the power regression line is dependent on the parameters (sample size, step and replicates), defined in Step 1.

After applying this Algorithm to samples of the log-logistic model, with parameters $(\alpha,\mu,\sigma) = (1,1,1)$, the following empirical thresholds, for the estimators $\hat{\alpha}^{GJ}(k)$, $\hat{\alpha}^{BR}(k)$ and $\hat{\alpha}^{IBR}(k)$, we obtained:

$$k_E^{GJ} = \lfloor 1.11 n^{0.92} \rfloor, \tag{15}$$

$$\hat{k}_E^{BR} = \lfloor 1.09 n^{0.86} \rfloor, \tag{16}$$

and

$$\hat{k}_E^{IBR} = \lfloor 0.79 n^{0.99} \rfloor. \tag{17}$$

Figure 1 displays the empirical optimal sample fractions and their corresponding fitted regression curves, for the reduced bias estimators of α. Clearly, there is a good alignment between the empirical data and the fitted curves.

3 Monte Carlo Simulation Study

In this section, we employ the Monte Carlo simulation method to analyse the finite-sample performance of the proposed bias-reduced estimator for the shape parameter α, in (12), against the estimators defined in (5), (8) and (10). The simulation study is described as follows:

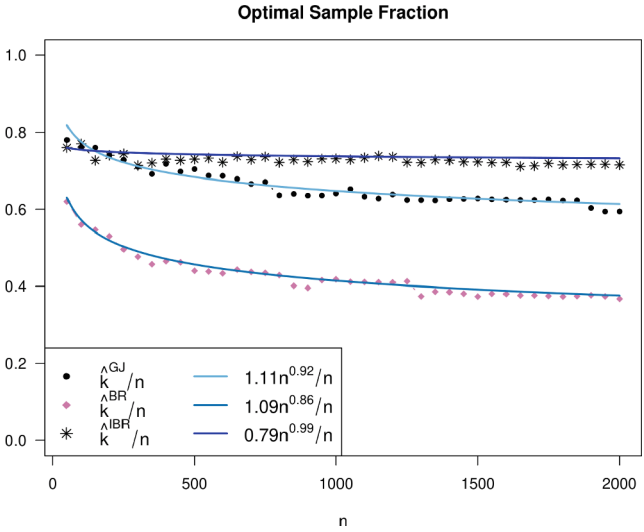

Fig. 1. Empirical optimal sample fraction for $\hat{\alpha}^{GJ}(k)$, $\hat{\alpha}^{BR}(k)$ and $\hat{\alpha}^{IBR}(k)$.

1. For each of the parameter combinations listed below, we generated 5000 samples from the three-parameter log-logistic model using the inverse transformation technique. The quantile function of the log-logistic model is specified by
$$Q(u) = \mu + \sigma \left(\frac{u}{1-u} \right)^{1/\alpha}, \tag{18}$$
with $0 < u < 1$.
2. Let $\hat{\alpha}^M(k)$, with $M \in \{LIH, GJ, BR, IBR\}$, generically represent any of the aforementioned estimators under study, for the parameter α. For each generated sample, the estimates $\hat{\alpha}_i^M(k)$, $i = 1, 2, \ldots, 5000$, were calculated for every k. We assess the estimators accuracy and efficiency using three standard statistical measures: Bias, Standard Deviation (SD), and Root Mean Squared Error (RMSE). These metrics are defined as follows:
$$Bias(\hat{\alpha}^M(k)) = \bar{\alpha}^M(k) - \alpha, \tag{19}$$
$$SD(\hat{\alpha}^M(k)) = \frac{1}{5000} \sum_{i=1}^{5000} (\hat{\alpha}_i^M(k) - \bar{\alpha}^M(k))^2 \tag{20}$$
$$RMSE(\hat{\alpha}^M(k)) = \sqrt{\frac{1}{5000} \sum_{i=1}^{5000} (\hat{\alpha}_i^M(k) - \alpha)^2}, \tag{21}$$
with
$$\bar{\alpha}^M(k) = \frac{1}{5000} \sum_{i=1}^{5000} \hat{\alpha}_i^M(k), \tag{22}$$

and $1 \leq k \leq n-2$.

3. We also determined the simulated metrics at their optimal level. For simplicity of notation, hereafter we shall denote

$$\hat{\alpha}_0^{LIH} = \hat{\alpha}^H(k_0^{LIH}),$$

$$\hat{\alpha}_E^{GJ} = \hat{\alpha}^{GJ}(\hat{k}_E^{GJ}),$$

$$\hat{\alpha}_E^{RB} = \hat{\alpha}^{RB}(\hat{k}_E^{RB}),$$

$$\hat{\alpha}_E^{IRB} = \hat{\alpha}^{IRB}(\hat{k}_E^{IRB}).$$

4. We used samples of size n, with $n \in \{25, 50, 100, 250, 500, 1000, 2500, 5000\}$, across the following four scenarios:
   - Scenario I: $(\alpha, \mu, \sigma) = (0.5, 1.5, 1)$;
   - Scenario II: $(\alpha, \mu, \sigma) = (0.75, 1.5, 1)$;
   - Scenario III: $(\alpha, \mu, \sigma) = (1.5, 1.5, 1)$;
5. All numerical computations and statistical evaluations were conducted in R programming environment [25].

The simulated metrics for all values of k are presented in Figs. 2, 3, 4, and 5 for $n = 500$. Similar patterns were observed for the other sample sizes considered in this study. Since the simulated bias is nearly identical to the Mbias across all parameter combinations, we report only the simulated bias. This bias is generally positive for $\hat{\alpha}^{GJ}(k)$ and non-positive for the remaining estimators. Notably, $\hat{\alpha}^{IBR}(k)$ yields a near-zero absolute bias over a wide range of k-values. Regarding the standard deviation (SD), for a fixed k, we observe:

$$SD(\hat{\alpha}^{LIH}(k)) \leq SD(\hat{\alpha}^{BR}(k)) \leq SD(\hat{\alpha}^{IBR}(k)) \leq SD(\hat{\alpha}^{GJ}(k)).$$

If different values of k are allowed for each estimator, these rankings may vary. Furthermore, Fig. 5 shows that the proposed estimator, $\hat{\alpha}^{IBR}(k)$, can achieve the lowest RMSE. Table 1 presents the aforementioned metrics, computed at the optimal level for each estimator. For each metric and each sample size, the best results of the simulation are highlighted in bold. The results indicate that the proposed estimator $\hat{\alpha}^{IBR}(k)$ consistently outperforms the considered alternatives across all criteria.

4 Application to Real Data

In this section, we present an application using a real-world dataset to illustrate the behaviour of the estimators for the log-logistic shape parameter. This dataset, originally studied by Ahmad et al. [1], consists of the annual maximum flow (in m^3/s) for the River Spey at Kinrara from 1952 to 1982. The observations are as follows: 89.8, 109.1, 202.2, 146.3, 212.3, 116.7, 109.1, 80.7, 127.4, 138.8, 283.5, 85.6, 105.5, 118.0, 387.8, 80.7, 165.7, 111.6, 134.4, 131.5, 102.0, 104.3, 242.5, 214.8, 144.6, 114.2, 98.3, 102.8, 104.3, 196.2, 143.7.

Table 1. Simulated Bias, SD and RMSE, for the estimators at their optimal thresholds.

	25	50	100	250	500	1000	2500	5000
	$(\alpha, \mu, \sigma) = (0.5, 1.5, 1)$							
$Bias(\hat{\alpha}_0^{LIH})$	–0.0875	-0.0750	–0.0588	–0.0445	–0.0367	–0.0298	–0.0227	–0.0178
$Bias(\hat{\alpha}_E^{GJ})$	0.1685	0.0962	0.0584	0.0349	0.0239	0.0173	0.0122	0.0098
$Bias(\hat{\alpha}_E^{BR})$	–0.0370	–0.0270	–0.0234	–0.0165	–0.0138	–0.0108	–0.0082	-0.0067
$Bias(\hat{\alpha}_E^{IBR})$	**0.0126**	**0.0067**	**0.0010**	**–0.0016**	**–0.0020**	**–0.0018**	**–0.0016**	**–0.0012**
$SD(\hat{\alpha}_0^{LIH})$	0.1317	0.1015	0.0811	0.0625	0.0502	0.0406	0.0307	0.0248
$SD(\hat{\alpha}_E^{GJ})$	0.2424	0.1542	0.1057	0.0693	0.0507	0.0368	0.0245	0.0178
$SD(\hat{\alpha}_E^{BR})$	**0.1074**	0.0779	0.0566	0.0393	0.0294	0.0221	0.0149	0.0111
$SD(\hat{\alpha}_E^{IRB})$	0.1141	**0.0769**	**0.0538**	**0.0340**	**0.0243**	**0.0169**	**0.0107**	**0.0076**
$RMSE(\hat{\alpha}_0^{LIH})$	0.1580	0.1262	0.1002	0.0768	0.0622	0.0503	0.0382	0.0305
$RMSE(\hat{\alpha}_E^{GJ})$	0.2952	0.1818	0.1207	0.0776	0.0561	0.0407	0.0274	0.0203
$RMSE(\hat{\alpha}_E^{BR})$	**0.1136**	0.0825	0.0612	0.0426	0.0325	0.0246	0.0170	0.0129
$RMSE(\hat{\alpha}_E^{IBR})$	0.1148	**0.0772**	**0.0538**	**0.0341**	**0.0244**	**0.0170**	**0.0108**	**0.0077**
	$(\alpha, \mu, \sigma) = (0.75, 1.5, 1)$							
$Bias(\hat{\alpha}_0^{LIH})$	–0.1337	–0.1132	–0.0884	–0.0668	–0.0551	–0.0446	–0.0341	–0.0268
$Bias(\hat{\alpha}_E^{GJ})$	0.2640	0.1473	0.0884	0.0526	0.0359	0.0259	0.0183	0.0147
$Bias(\hat{\alpha}_E^{BR})$	-0.0646	–0.0434	–0.0360	–0.0249	–0.0207	–0.0162	–0.0123	–0.0100
$Bias(\hat{\alpha}_E^{IBR})$	**0.0040**	**0.0044**	**-0.0008**	**–0.0031**	**–0.0034**	**–0.0028**	**-0.0025**	**–0.0019**
$SD(\hat{\alpha}_0^{LIH})$	0.1965	0.1521	0.1216	0.0938	0.0752	0.0609	0.0461	0.0372
$SD(\hat{\alpha}_E^{GJ})$	0.3692	0.2319	0.1586	0.1040	0.0761	0.0553	0.0368	0.0267
$SD(\hat{\alpha}_E^{BR})$	**0.1592**	0.1164	0.0848	0.0589	0.0442	0.0332	0.0224	0.0166
$SD(\hat{\alpha}_E^{IBR})$	0.1684	**0.1147**	**0.0805**	**0.0510**	**0.0364**	**0.0254**	**0.0160**	**0.0114**
$RMSE(\hat{\alpha}_0^{LIH})$	0.2376	0.1896	0.1503	0.1152	0.0932	0.0755	0.0573	0.0458
$RMSE(\hat{\alpha}_E^{GJ})$	0.4539	0.2747	0.1815	0.1166	0.0841	0.0610	0.0411	0.0304
$RMSE(\hat{\alpha}_E^{BR})$	0.1718	0.1242	0.0921	0.0640	0.0488	0.0369	0.0255	0.0194
$RMSE(\hat{\alpha}_E^{IBR})$	**0.1685**	**0.1148**	**0.0805**	**0.0511**	**0.0366**	**0.0255**	**0.0162**	**0.0116**
	$(\alpha, \mu, \sigma) = (1.5, 1.5, 1)$							
$Bias(\hat{\alpha}_0^{LIH})$	–0.3258	–0.2586	–0.1935	–0.1408	–0.1140	–0.0913	–0.0691	–0.0540
$Bias(\hat{\alpha}_E^{GJ})$	0.5120	0.2834	0.1712	0.1025	0.0702	0.0508	0.0362	0.0290
$Bias(\hat{\alpha}_E^{RB})$	-0.2454	–0.1532	–0.1096	-0.0675	–0.0517	-0.0383	–0.0274	–0.0216
$Bias(\hat{\alpha}_E^{IRB})$	**-0.1470**	**–0.0858**	**–0.0628**	**–0.0392**	**–0.0279**	**–0.0186**	**–0.0120**	**–0.0081**
$SD(\hat{\alpha}_0^{LIH})$	0.3728	0.2952	0.2398	0.1864	0.1499	0.1216	0.0921	0.0744
$SD(\hat{\alpha}_E^{GJ})$	0.7303	0.4534	0.3121	0.2062	0.1512	0.1101	0.0734	0.0533
$SD(\hat{\alpha}_E^{BR})$	**0.3017**	0.2259	0.1670	0.1171	0.0879	0.0662	0.0447	0.0331
$SD(\hat{\alpha}_E^{IBR})$	0.3201	**0.2249**	**0.1606**	**0.1026**	**0.0731**	**0.0511**	**0.0322**	**0.0230**
$RMSE(\hat{\alpha}_0^{LIH})$	0.4951	0.3924	0.3081	0.2336	0.1883	0.1521	0.1152	0.0919
$RMSE(\hat{\alpha}_E^{GJ})$	0.8918	0.5347	0.3559	0.2302	0.1667	0.1213	0.0818	0.0606
$RMSE(\hat{\alpha}_E^{BR})$	0.3889	0.2729	0.1997	0.1352	0.1020	0.0764	0.0524	0.0396
$RMSE(\hat{\alpha}_E^{IBR})$	**0.3522**	**0.2407**	**0.1724**	**0.1098**	**0.0783**	**0.0544**	**0.0343**	**0.0244**

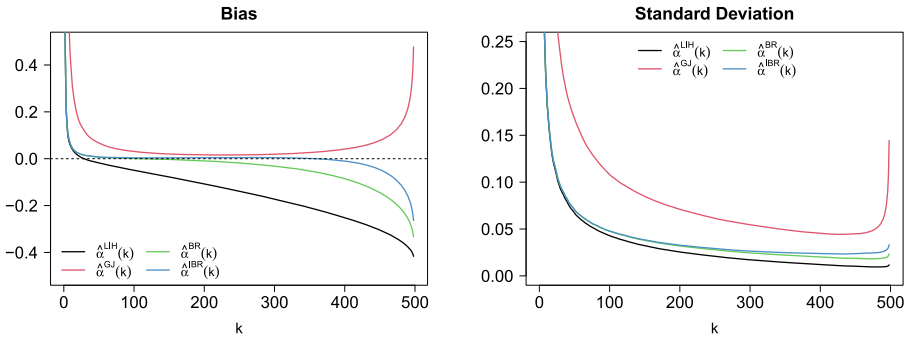

Fig. 2. Simulated bias and standard deviation of the estimators $\hat{\alpha}^{LIH}(k)$, $\hat{\alpha}^{GJ}(k)$, $\hat{\alpha}^{BR}(k)$, and $\hat{\alpha}^{IBR}(k)$, as a function of k. Data generated from a log-logistic model with parameters $(\alpha, \mu, \sigma) = (0.5, 1.5, 1)$ and sample size $n = 500$.

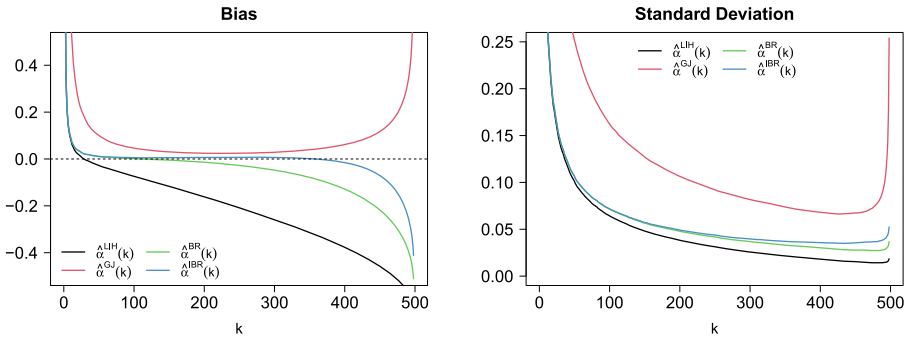

Fig. 3. Simulated bias and standard deviation of the estimators $\hat{\alpha}^{LIH}(k)$, $\hat{\alpha}^{GJ}(k)$, $\hat{\alpha}^{BR}(k)$, and $\hat{\alpha}^{IBR}(k)$, as a function of k. Data generated from a log-logistic model with parameters $(\alpha, \mu, \sigma) = (0.75, 1.5, 1)$ and sample size $n = 500$.

Using the maximum likelihood (ML) method, Ahmad et al. [1] computed a shape parameter estimate of $\hat{c} = 0.531$. By the invariance property of ML estimators, the corresponding estimate for $\alpha = c^{-1}$ is $\hat{\alpha}^{ML} = 1.883$. The estimates for the shape parameter α provided by the alternative estimators studied in this paper are

$$\hat{\alpha}_0^{LIH} = 1.389,$$
$$\hat{\alpha}_E^{GJ} = 2.090,$$
$$\hat{\alpha}_E^{RB} = 1.599,$$
$$\hat{\alpha}_E^{IRB} = 1.902.$$

The results indicate that the reduced bias estimates, $\hat{\alpha}_E^{GJ}$, $\hat{\alpha}_E^{RB}$ and $\hat{\alpha}_E^{IRB}$ align more closely with the ML estimate. Moreover, the $\hat{\alpha}_E^{IRB}$ estimate of 1.902 is similar to the ML estimate.

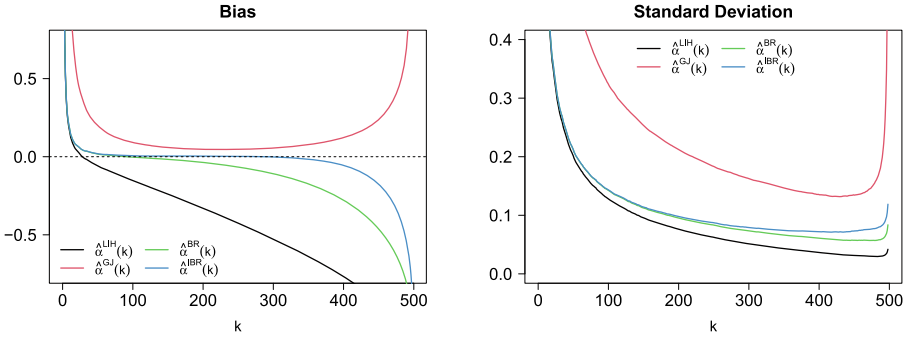

Fig. 4. Simulated bias and standard deviation of the estimators $\hat{\alpha}^{LIH}(k)$, $\hat{\alpha}^{GJ}(k)$, $\hat{\alpha}^{BR}(k)$, and $\hat{\alpha}^{IBR}(k)$, as a function of k. Data generated from a log-logistic model with parameters $(\alpha, \mu, \sigma) = (1.5, 1.5, 1)$ and sample size $n = 500$.

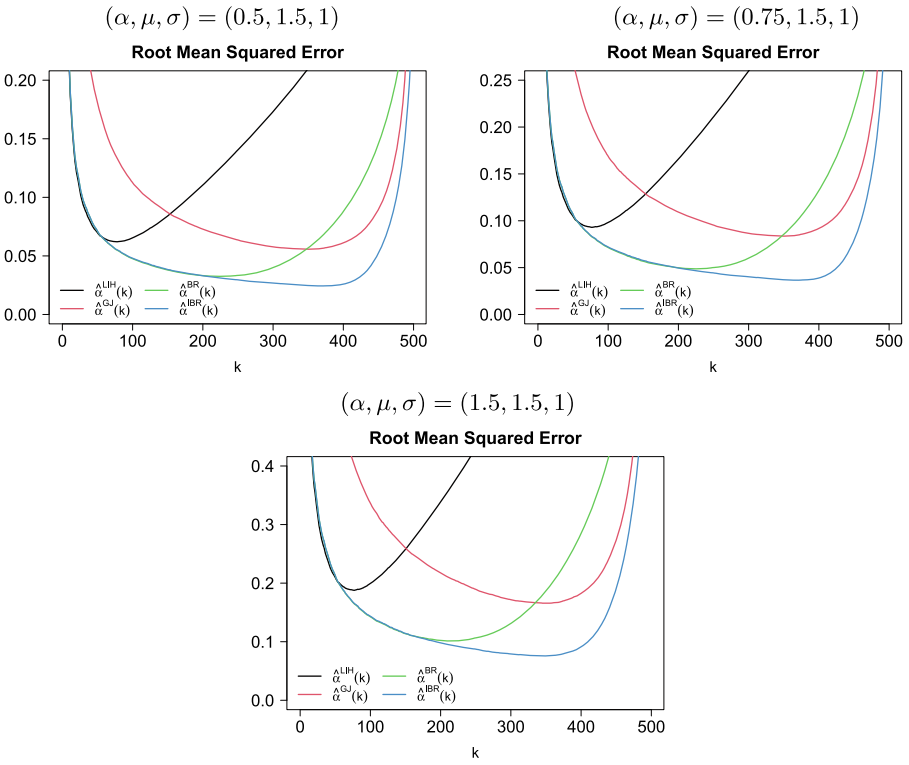

Fig. 5. Simulated RMSE of the estimators $\hat{\alpha}^{LIH}(k)$, $\hat{\alpha}^{GJ}(k)$, $\hat{\alpha}^{BR}(k)$, and $\hat{\alpha}^{IBR}(k)$, as a function of k.

5 Concluding Remarks

In this research, we introduced a modified bias reduced location-invariant estimator for the shape parameter of the three-parameter log-logistic model, which provides an improved bias correction. Our simulation results confirm that this new approach offers superior performance, consistently outperforming existing estimators, including other bias-reduced estimators, in terms of both bias reduction and RMSE. Furthermore, the conceptual framework presented here may be extended to correct higher order bias terms.

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