A Daily-Grid Hazards-to-Dynamics Framework for Inpatient Flow

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Abstract. Routine hospital episode data record admissions and exits at date resolution, together with exit type and a limited set of baseline covariates. This paper develops a reproducible daily-grid method for deriving inpatient occupancy and cause-specific outflow series from such data. The statistical layer estimates grouped day-of-stay exit probabilities for discharge alive and in-hospital death, and the dynamical layer propagates these probabilities through a conservative cohort update under a half-open counting convention. The formulation separates directly observed episode-based quantities from probability tables and long-stay summaries that depend on pooling, completion, or tail closure. It also links the discrete construction to a McKendrick–von Foerster transport equation as a compatible continuous-time analogue, with exact accounting established for the grouped daily scheme used in practice. The resulting outputs provide cohort-level daily flow inputs for retrospective audits, comparison of operating regimes, and scenario analysis; when capacity data are available, the same series can be combined with them to study operational bed pressure. In an application to Brazilian administrative records for femur-fracture admissions, the reference sample contained 442,646 episodes after exclusion of 5,106 same-day admission-and-exit records. As an internal reconstruction check on this retained closed-episode cohort, the model-implied occupancy closely followed the empirical census, with a mean absolute error of 53.03, or 3.91% of mean empirical occupancy. Mass-balance residuals were at numerical roundoff level, confirming conservation of the cohort update. Sensitivity analyses over support length, sparse-cell pooling, tail handling, and same-day policy yielded stable reconstruction results.

Keywords: Administrative hospital data · Age-structured transport model · Competing risks · Conservative daily cohort discretization · Daily-grid identifiability · Inpatient flow modelling

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1 Introduction

Hospitals make daily decisions about beds and patient flow under uncertainty, while routine administrative datasets record each stay as an episode defined by admission date, exit date, exit type, and a small set of admission-time covariates [1]. Without timestamps for intermediate clinical transitions, inpatient flow is reconstructed on a daily grid under explicit measurement and modelling conventions.

Admissions, length of stay, occupancy, and exits evolve jointly and have been studied through queueing, simulation, and forecasting models [2–4]. For date-recorded data, discharge alive and in-hospital death naturally define a grouped competing-risks problem [5,6], while age-of-stay-structured transport models provide a system-level description in which conservative discretizations preserve accounting links among admissions, occupancy, and outflows [7,8]. These strands provide complementary tools for capacity planning, aggregate prediction, and patient-level event modelling; this paper connects them through a daily-grid interface that maps grouped exit probabilities into an accounting-conservative inpatient population update [12].

This paper introduces a unified daily-grid framework that maps grouped competing-risks probabilities from date-recorded episodes into conservative inpatient flow dynamics and daily occupancy and outflow series. Its central identifiability distinction is that date-only episodes determine daily admissions, recorded exits, and half-open occupancy on the retained sample, whereas sub-daily exit times, intermediate clinical states, continuous-time hazard surfaces, sparse cells, and long-stay tails require modelling, pooling, completion, or tail-construction choices.

Admission-based and calendar-time formulations are handled within a common structure, clarifying how regime information enters the daily exit probabilities and their propagation through the cohort system. A McKendrick–von Foerster equation is included as a compatible continuous-time analogue of the grouped daily mechanism [7,9,10], while exact accounting results are established for the discrete formulation used in practice.

The applied analysis uses Brazilian administrative hospital records from the Hospital Information System of the Brazilian Unified Health System (SIH/SUS), accessed through the Department of Informatics of the Unified Health System (DATASUS), restricted to International Statistical Classification of Diseases and Related Health Problems, 10th Revision (ICD-10) code S72 hospitalizations for femur fractures [1,11]. The extract is used for reconstruction diagnostics and sensitivity checks on real episode data, focusing on internal consistency of the reconstructed daily series.

The remainder of the article is organized as follows. Section 2 presents the methodological framework. Section 3 describes the episode schema and computational workflow. Section 4 reports the SIH/SUS ICD-10 S72 extract, reconstruction diagnostics, and sensitivity checks. Section 5 summarizes the findings and possible extensions.

2 Methodological Framework

This section defines the daily-grid framework: retained episodes are converted into empirical daily series, grouped day-of-stay exit probabilities are estimated, and these probabilities are propagated through a conservative cohort update on the calendar grid.

2.1 Observed Episodes and Daily Series

We work with a retained analysis table containing one hospitalization episode per row. Unless otherwise stated, the development refers to the fully observed retained sample after preprocessing; same-day episodes and extraction-window censoring are discussed in Sect. 3.2. For episode $i = 1, \ldots, N$, we observe the admission day s_i, exit day e_i, exit type $c_i \in \{D, M\}$, where D denotes discharge alive and M denotes in-hospital death, and admission-time variables $\boldsymbol{x}_i$, used for stratification or optional modelling. Time is recorded on a daily grid. Under the half-open convention, episode i is present on day t if and only if $s_i \le t < e_i$, and its recorded day-resolution length of stay is $\ell_i := e_i - s_i$. This is a grouped quantity induced by recorded dates, not a directly observed continuous elapsed time. Same-day admission and exit give $\ell_i = 0$; these records are excluded from the default retained sample and handled in sensitivity analysis.

For stratified analyses, let g map admission-time variables to $\{1, \ldots, J\}$, and write $j(i) := g(\boldsymbol{x}_i)$. With $\mathbf{1}\{\cdot\}$ denoting the indicator function, the retained episode table directly determines the empirical daily series used for estimation and checking. Admissions into stratum j are

$$A_j(t) := \sum_{i=1}^{N} \mathbf{1}\{s_i = t,\ j(i) = j\},\ A(t) := \sum_{j=1}^{J} A_j(t). \tag{1}$$

Empirical cause-specific exits are

$$\widehat{D}_{\mathrm{emp},j}(t) := \sum_{i=1}^{N} \mathbf{1}\{j(i) = j,\ c_i = D,\ e_i = t\},$$

$$\widehat{M}_{\mathrm{emp},j}(t) := \sum_{i=1}^{N} \mathbf{1}\{j(i) = j,\ c_i = M,\ e_i = t\}.$$

The corresponding totals are

$$\widehat{D}_{\mathrm{emp}}(t) := \sum_{j=1}^{J} \widehat{D}_{\mathrm{emp},j}(t),\ \widehat{M}_{\mathrm{emp}}(t) := \sum_{j=1}^{J} \widehat{M}_{\mathrm{emp},j}(t).$$

Empirical occupancy is

$$\widehat{H}_{\mathrm{emp}}(t) := \sum_{i=1}^{N} \mathbf{1}\{s_i \le t < e_i\}. \tag{2}$$

These series are fully determined by the retained episode table and provide the empirical basis for probability estimation and reconstruction diagnostics.

2.2 Competing-Risks Quantities and Daily Estimands

Let S_{adm}, L, C, and $\boldsymbol{X}$ denote the random counterparts of s_i, ℓ_i, c_i, and $\boldsymbol{x}_i$. On the retained sample, $L \in \{1, 2, \dots\}$ is the grouped day-resolution length of stay. When regime stratification is used, calendar time is partitioned by a deterministic map $P(t) \in \{1, \dots, R\}$. Two conventions are distinguished. Under the admission-based convention, an episode is assigned the regime in force at admission, $P(S_{\mathrm{adm}})$. Under the calendar-time convention, an episode admitted on day s is exposed on grouped day of stay m to the regime in force on calendar day $s + m$. The primary estimands are one-day cause-specific exit probabilities on the grouped day-of-stay scale. Under the admission-based convention,

$$q_{k,j,r}(m) := \Pr\big(L = m+1,\, C = k \,\big|\, L > m,\, g(\boldsymbol{X}) = j,\, P(S_{\mathrm{adm}}) = r\big), \tag{3}$$

for $k \in \{D, M\}$ and $m \in \mathbb{N}_0$. Thus, $q_{k,j,r}(m)$ is the probability of exit by cause k at the end of grouped day m, among patients still present at the start of that grouped day. Under the calendar-time convention, the fine daily-grid probabilities are indexed by admission day:

$$q_{k,j,s}(m) := \Pr\big(L = m+1,\, C = k \,\big|\, L > m,\, g(\boldsymbol{X}) = j,\, S_{\mathrm{adm}} = s\big), \tag{4}$$

with the contemporaneous regime determined by $P(s + m)$. Regime-level calendar-time summaries may be obtained by risk-set-consistent pooling or smoothing over cells sharing the same contemporaneous regime, but their use as transportable inputs requires the assumption that, conditional on stratum and grouped day of stay, calendar time affects the exit mechanism only through that regime label. Continuous-time notation is used only as a compatible embedding of the grouped daily mechanism [5,12]. Under date-only observation, within-day exit times and continuous-time hazards are not identified. Suppressing indices, let $q(m) := \sum_k q_k(m)$. If $0 \le q(m) < 1$, one constant-within-day embedding sets

$$\Lambda(m) := -\log\{1 - q(m)\},\ \Lambda_k(m) = \begin{cases} \Lambda(m)\dfrac{q_k(m)}{q(m)}, & q(m) > 0, \\ 0, & q(m) = 0. \end{cases}$$

Under the constant-within-day embedding used here, or more generally under a within-day embedding with constant cause proportions, if cause-specific integrated hazards $\Lambda_k(m)$ are specified first and $\Lambda(m) := \sum_{k \in \{D,M\}} \Lambda_k(m)$, the induced daily probabilities are

$$q(m) = 1 - \exp\{-\Lambda(m)\},\ q_k(m) = \big(1 - \exp\{-\Lambda(m)\}\big)\frac{\Lambda_k(m)}{\Lambda(m)}, \quad k \in \{D, M\}, \tag{5}$$

with the ratio defined as 0 when $\Lambda(m) = 0$. These mappings are compatible representations of the observed daily probabilities, not additional empirical information. If $q(m) = 1$, the daily probability model remains well defined, but a finite continuous-time hazard embedding is singular.

2.3 Statistical Layer: Operational Estimands and Estimation Choices

The estimands in Sect. 2.2 enter the numerical framework through daily cause-specific exit-probability tables on the grouped day-of-stay support used by the cohort dynamics. The implementation is probability-based; integrated hazards are used only for reporting or for a compatible continuous-time embedding.

For retained episodes with $\ell_i \geq 1$, the grouped day-resolution length of stay is observed from the admission and exit dates. Under the admission-based convention, define $q_{j,r}(u) := \sum_{k\in\{D,M\}} q_{k,j,r}(u)$. For an episode i in stratum $j(i)$, admitted under regime $P(s_i)$, with length of stay ℓ_i and exit cause c_i, the grouped-data likelihood contribution is

$$\mathcal{L}_i = \left[\prod_{u=0}^{\ell_i-2} \big(1 - q_{j(i),P(s_i)}(u)\big)\right] q_{c_i,j(i),P(s_i)}\big(\ell_i - 1\big),$$

with the empty product interpreted as 1 when $\ell_i = 1$.

Under the calendar-time convention, the fine daily-grid likelihood is written in terms of admission-day-indexed probabilities.
With $q_{j,s}(u) := \sum_{k\in\{D,M\}} q_{k,j,s}(u)$, the corresponding contribution is

$$\mathcal{L}_i = \left[\prod_{u=0}^{\ell_i-2} \big(1 - q_{j(i),s_i}(u)\big)\right] q_{c_i,j(i),s_i}\big(\ell_i - 1\big).$$

A regime-pooled calendar-time implementation replaces the fine admission-day-indexed terms by pooled terms indexed by the contemporaneous regime: $P(s_i+u)$ in the survival factors and $P(s_i + \ell_i - 1)$ in the terminal factor. As noted in Sect. 2.2, this reduced representation requires a pooling or modelling assumption beyond direct cellwise identifiability.

The primary implementation uses the nonparametric daily estimators in Sect. 3.3, which match the date resolution of the data and the grouped update in Sect. 2.5. Alternative grouped-time, smoothed, piecewise-constant, or proportional-hazards estimators may also be used, provided that the dynamical layer receives daily probability tables on the chosen working grid.

If fitting is carried out at a finer covariate level, fitted probabilities must be aggregated before simulation. Under the admission-based convention, aggregation is over survivors sharing stratum and admission regime.

Under the calendar-time convention, aggregation must respect the surviving episode-day composition on the calendar-time by day-of-stay grid before any coarser regime-level pooling is applied. After this risk-set-consistent aggregation, the statistical layer is closed and the grouped cohort update takes the completed probability tables as fixed inputs.

2.4 Dynamical Layer: Continuous-Time Transport Analogue

The daily probability model admits a compatible continuous-time age-of-stay embedding in the McKendrick–von Foerster framework, but this embedding does not identify sub-daily inpatient dynamics from date-recorded data [7,9,10,13].

For each stratum $j \in \{1, \dots, J\}$, let $h_j(t, a) \geq 0$ denote the age-of-stay density at calendar time t and elapsed stay $a \geq 0$. Thus $h_j(t, a)\, da$ represents the number of inpatients in stratum j with elapsed stay in $[a, a + da)$.

After relabelling the daily calendar grid as consecutive integers, write $t^n = n$. Let $a_j(t) \geq 0$ be a boundary admission-rate embedding satisfying

$$\int_{t^n}^{t^{n+1}} a_j(u)\, du = A_j(t^{n+1}),\ n \in \mathbb{Z}.$$

The within-day shape of $a_j(t)$ is not identified by the data; only the daily admission totals are observed. The integral convention is chosen to match the daily boundary update used in the cohort scheme.

Let $\lambda_{k,j}(a, t) \geq 0$, $k \in \{D, M\}$, denote an aggregated cause-specific hazard surface compatible with the daily probability table. With regime labels, the calendar-time convention may be represented as $\lambda_{k,j}(a, t) = \lambda_{k,j,P(t)}(a)$, whereas the admission-based convention corresponds, along characteristics, to $\lambda_{k,j}(a, t) = \lambda_{k,j,P(t-a)}(a)$. If finer calendar-time indexing is retained, $\lambda_{k,j}(a, t)$ is interpreted as the corresponding cell-specific surface.

The continuous-time analogue is

$$\partial_t h_j(t, a) + \partial_a h_j(t, a) = -\big(\lambda_{D,j}(a, t) + \lambda_{M,j}(a, t)\big) h_j(t, a),\ t > 0,\ a > 0, \quad (6)$$

with boundary condition

$$h_j(t, 0) = a_j(t), \quad (7)$$

and initial condition $h_j(0, a) = h_{j,0}(a) \geq 0$. Exact accounting results in this paper are stated for the grouped daily scheme of Sect. 2.5; the continuous-time formulation is used only as a compatible analogue.

Define

$$H_j(t) = \int_0^\infty h_j(t, a)\, da,$$

and

$$D_j(t) = \int_0^\infty \lambda_{D,j}(a, t) h_j(t, a)\, da,\ M_j(t) = \int_0^\infty \lambda_{M,j}(a, t) h_j(t, a)\, da.$$

The corresponding totals are

$$H(t) := \sum_{j=1}^{J} H_j(t),\ D(t) := \sum_{j=1}^{J} D_j(t),\ M(t) := \sum_{j=1}^{J} M_j(t).$$

Proposition 1 (Continuous-Time Mass Balance). *Assume sufficient regularity and integrability so that $h_j(t, \cdot)$, $\lambda_{D,j}(\cdot, t) h_j(t, \cdot)$, and $\lambda_{M,j}(\cdot, t) h_j(t, \cdot)$ are integrable on $(0, \infty)$, with $h_j(t, a) \to 0$ as $a \to \infty$. Then, for each stratum j,*

$$\frac{d}{dt} H_j(t) = a_j(t) - D_j(t) - M_j(t)$$

for almost every $t \geq 0$.

Proof sketch. Integrate (6) over $a \in (0, \infty)$, use the boundary condition (7), and let the boundary term at $a \to \infty$ vanish under the stated assumptions.

2.5 Conservative Daily Cohort Discretization

The grouped daily exit mechanism induces a conservative cohort update on the same daily grid as the data. After relabelling the grid as consecutive integers, let $\Delta = 1$ day, $t^n := n\Delta$, $a^m := m\Delta$. For each stratum j, let $N_j^{n,m}$ denote the number, or expected number, of patients present on day t^n in grouped day-of-stay class m. This is a start-of-day presence state under the half-open convention: ordinary cells $m = 0, \ldots, m_{\max} - 1$ correspond to episodes admitted on day t^{n-m} and still present on day t^n. Admissions dated t^{n+1} enter through the boundary state $N_j^{n+1,0}$.

Let $\mathcal{M}_{\text{ret}}$ be the retained set of state indices. With a single retained tail cell, $\mathcal{M}_{\text{ret}} = \{0, \ldots, m_{\max}\}$, where $m = m_{\max}$ groups patients with day of stay at least $m_{\max}$. With truncation, $\mathcal{M}_{\text{ret}} = \{0, \ldots, m_{\max} - 1\}$, and survivor mass leaving the retained support is recorded as truncation loss. Grid-respecting retained tails are obtained by applying the same survivor-shift logic across additional tail cells. For grid-respecting retained tail cells that keep a unique grouped day of stay, the same probability-assignment principle applies to those retained cells. Occupancies are

$$H_j^n := \sum_{m \in \mathcal{M}_{\text{ret}}} N_j^{n,m}, \; H^n := \sum_{j=1}^{J} H_j^n.$$

When the retained state represents the intended grid exactly, H^n is directly comparable with $\widehat{H}_{\text{emp}}(t^n)$; under a reduced-tail closure, the comparison is a reduced-state reconstruction.

For each cause $k \in \{D, M\}$, let $q_{k,j}^{n,m} \in [0,1]$ be the one-day exit probability applied to cell (j,m) over $(t^n, t^{n+1}]$, and define $q_j^{n,m} := q_{D,j}^{n,m} + q_{M,j}^{n,m}$. On ordinary cells $m = 0, \ldots, m_{\max} - 1$, the admission day is uniquely determined by (n,m), namely t^{n-m}. Hence, the probabilities are assigned according to the working representation

$$q_{k,j}^{n,m} = \begin{cases} q_{k,j,t^{n-m}}(m), & \text{calendar-time,} \\ & \text{admission-day indexed,} \\ q_{k,j,P(t^n)}(m), & \text{calendar-time, regime-pooled,} \\ q_{k,j,P(t^{n-m})}(m), & \text{admission-based,} \end{cases} \quad m = 0, \ldots, m_{\max} - 1.$$

In the single-tail implementation, $N_j^{n,m_{\max}}$ aggregates all patients whose grouped day of stay is at least $m_{\max}$. Therefore $(n, m_{\max})$ does not determine a unique admission day, admission regime, or admission-day index. The tail probability is therefore a closure probability for the aggregated long-stay state. We write $q_{k,j}^{n,m_{\max}} := q_{k,j}^{n,\text{tail}}$, $q_j^{n,m_{\max}} := q_{D,j}^{n,\text{tail}} + q_{M,j}^{n,\text{tail}}$, $q_j^{n,m_{\max}} \leq 1$.

For a calendar-time regime-pooled tail closure, $q_{k,j}^{n,\text{tail}} = q_{k,j,P(t^n)}^{\text{tail}}$, because the contemporaneous regime is common to all patients in the tail at calendar day t^n. For admission-based or admission-day-indexed calendar-time representations, an unindexed single-tail state is compatible only with a marginal or composition-averaged tail closure.

Exact retention of admission-regime-specific or admission-day-specific long-stay effects requires either a grid-respecting tail representation or an augmented tail state carrying the relevant index.

For ordinary interior cells, $N_j^{n+1,m+1} = N_j^{n,m}\big(1-q_j^{n,m}\big),\ m = 0,\dots,m_{\max}-2$, and the boundary update is $N_j^{n+1,0} = A_j(t^{n+1})$, where $A_j(t)$ is defined in (1). In the single-tail case,

$$N_j^{n+1,m_{\max}} = N_j^{n,m_{\max}-1}\big(1-q_j^{n,m_{\max}-1}\big) + N_j^{n,m_{\max}}\big(1-q_j^{n,m_{\max}}\big).$$

This update is conservative for the implemented reduced state; exact recovery of all day-specific long-stay behaviour requires a grid-respecting tail representation or an equivalent exact tail construction.

The cause-specific removals over $(t^n, t^{n+1}]$ are

$$\widetilde{D}_j^n := \sum_{m\in\mathcal{M}_{\mathrm{ret}}} N_j^{n,m} q_{D,j}^{n,m},\ \widetilde{M}_j^n := \sum_{m\in\mathcal{M}_{\mathrm{ret}}} N_j^{n,m} q_{M,j}^{n,m}.$$

Thus, H^n is occupancy on day t^n, while $\widetilde{D}_j^n$ and $\widetilde{M}_j^n$ have exit date t^{n+1}.

At the first analysis day t^0, the state may be initialized directly from the episode table:

$$N_j^{0,m} = \sum_{i=1}^{N} \mathbf{1}\{j(i)=j,\ s_i \le t^0 < e_i,\ t^0 - s_i = m\},\ m = 0,\dots,m_{\max}-1.$$

If a single tail cell is retained, then

$$N_j^{0,m_{\max}} = \sum_{i=1}^{N} \mathbf{1}\{j(i)=j,\ s_i \le t^0 < e_i,\ t^0 - s_i \ge m_{\max}\}.$$

Grid-respecting tails are initialized analogously. If a pre-window warm start is used instead, the initialization choice and affected days should be reported.

Proposition 2 (Discrete Positivity and Mass Balance). *Assume that the initial state is nonnegative, $A_j(t^n) \ge 0$, and*

$$q_{D,j}^{n,m} \ge 0,\ q_{M,j}^{n,m} \ge 0,\ q_{D,j}^{n,m} + q_{M,j}^{n,m} \le 1$$

for all retained state components. Then all retained state components remain nonnegative.

If the retained state is closed, so that every non-exiting survivor is reassigned to a retained component, then, for each stratum j and step $n \ge 0$,

$$H_j^{n+1} - H_j^n = A_j(t^{n+1}) - \widetilde{D}_j^n - \widetilde{M}_j^n. \tag{8}$$

If the scheme is truncated at the last retained ordinary cell, then

$$H_j^{n+1} - H_j^n = A_j(t^{n+1}) - \widetilde{D}_j^n - \widetilde{M}_j^n - U_j^n, \quad \textit{where} \tag{9}$$

$$U_j^n := N_j^{n,m_{\max}-1}\big(1-q_j^{n,m_{\max}-1}\big)$$

is the truncation loss over $(t^n, t^{n+1}]$.

Proof. Nonnegativity follows because each update is a nonnegative combination of nonnegative state components and the nonnegative boundary input $A_j(t^{n+1})$. Summing the update equations over retained components gives the mass balance: in the closed case, survivor transfers cancel internally and only admissions and cause-specific removals remain. Under truncation, the uncancelled survivor flow is U_j^n, yielding (9).

Equation (8) is the accounting identity used in diagnostics for closed retained-tail implementations, including grid-respecting tails and conservative reduced closures. For truncated implementations, the adjusted balance (9) must be used. Because reduced-tail rows are closure quantities, applications should report sensitivity to $m_{\max}$ and retained-tail treatment.

2.6 Identifiability with Minimal Episode Variables

The framework distinguishes quantities determined directly by the retained episode table from quantities requiring pooling, completion, tail closure, or modelling assumptions. Given $(s_i, e_i, c_i, \boldsymbol{x}_i)$, or equivalently $(s_i, \ell_i, c_i, \boldsymbol{x}_i)$ after preprocessing, and a specified regime map $P(\cdot)$ when used, the empirical series in (1)–(2) are reconstructed directly. Under the half-open convention, the retained data determine daily admissions, cause-specific exits, and occupancy on the retained sample.

On the grouped day-of-stay scale induced by recorded dates, the main nonparametrically estimable quantities are one-day cause-specific exit probabilities for cells with nonempty risk sets. Under the admission-based convention, these are the stratum- and regime-specific probabilities in (3). Under the calendar-time convention, the directly indexed daily-grid quantities are the admission-day-specific probabilities in (4), or the equivalent calendar-time by day-of-stay cells.

Regime-pooled calendar-time probabilities can be computed as descriptive pooled ratios, but their use as transportable inputs assumes that, conditional on stratum and grouped day of stay, calendar time affects exits only through the contemporaneous regime label. The same distinction applies to long-stay cells: ordinary cells with positive exposure support day-specific estimation, whereas retained tail cells are pooled closure representations of long-stay behaviour. Once daily probabilities are defined on a chosen support, grouped survival, cumulative incidence, and length-of-stay quantities follow algebraically. Let α denote the relevant index, with $\alpha = r$ for the admission-based convention and $\alpha = s$ for admission-day-indexed calendar time, and define

$$q_{j,\alpha}(u) := \sum_{k \in \{D,M\}} q_{k,j,\alpha}(u). \text{ Then}$$

$$S_{j,\alpha}(m) = \prod_{u=0}^{m-1} \big(1 - q_{j,\alpha}(u)\big), \; S_{j,\alpha}(0) = 1, \text{ and}$$

$$F_{k,j,\alpha}(m) = \sum_{u=0}^{m-1} S_{j,\alpha}(u) q_{k,j,\alpha}(u), \ k \in \{D, M\}.$$

If probabilities are available on the full support of L, then

$$\Pr(L = m+1 \mid j, \alpha) = S_{j,\alpha}(m) q_{j,\alpha}(m), \ \mathbb{E}[L \mid j, \alpha] = \sum_{m=0}^{\infty} S_{j,\alpha}(m),$$

whenever the series converges.

For the regime-pooled calendar-time representation, the regime index is generally path-dependent along the stay. Writing

$$q_{j,r}(u) := \sum_{k \in \{D,M\}} q_{k,j,r}(u),$$

the grouped survival function for an episode admitted on day s is

$$S_{j,s}^{\text{pool}}(m) = \prod_{u=0}^{m-1} \left\{1 - q_{j,P(s+u)}(u)\right\}, \ S_{j,s}^{\text{pool}}(0) = 1.$$

The corresponding cause-specific cumulative incidence is

$$F_{k,j,s}^{\text{pool}}(m) = \sum_{u=0}^{m-1} S_{j,s}^{\text{pool}}(u) q_{k,j,P(s+u)}(u), \ k \in \{D, M\}.$$

Thus, under regime-pooled calendar-time indexing, there is no single fixed regime for the whole stay unless $P(s+u)$ remains constant over the relevant grouped days. The corresponding admission-day-specific length-of-stay quantities are

$$\overset{\text{pool}}{\Pr}(L = m+1 \mid j, s) = S_{j,s}^{\text{pool}}(m) q_{j,P(s+m)}(m), \ \mathbb{E}^{\text{pool}}[L \mid j, s] = \sum_{m=0}^{\infty} S_{j,s}^{\text{pool}}(m),$$

whenever the series converges. With finite support, the tail of the distribution and the mean beyond that support are determined by the chosen completion or tail closure.

Date-recorded episodes identify grouped daily quantities only; the mapping in (5) and the boundary-rate embedding in Sect. 2.4 are compatible representations, not additional empirical information. Once the analysis cells are fixed, the framework delivers exit probabilities and flow quantities at that aggregation level; richer covariate or state detail requires enlarging the state space.

Because admissions, exits, and occupancy are reconstructed directly from the retained episode table, the model-implied series can be compared with their empirical counterparts under the same state definition. These comparisons are internal reconstruction checks, with differences interpreted in light of numerical roundoff, sparse-cell pooling, finite-support completion, and the chosen tail treatment.

3 Data and Computational Workflow

This section describes the applied-data workflow: episode construction, preprocessing, daily probability estimation, cohort simulation, diagnostics, and reproducible outputs.

3.1 Data Source and Episode Schema

The framework uses one preprocessed row per hospitalization episode, with admission date, exit date, exit type, and admission-time variables for stratification or optional modelling. The applied extract was built from Brazilian SIH/SUS records restricted to ICD-10 S72 femur-fracture hospitalizations [1,11]. Source-level preprocessing removed duplicate hospitalization authorization identifiers, standardized dates and variables, and retained additional fields for traceability and descriptive quality control. The default stratified analysis used admission date, exit date, exit type, age, and sex.

For reproducibility, each computational run stores the analysis configuration, software session information, cleaned episode table, empirical series, probability tables, simulation outputs, reconstruction metrics, support diagnostics, figures, and article-ready summary tables.

3.2 Preprocessing, Quality Control, and Empirical Series

Preprocessing defines the retained sample and aligns the episode table with the half-open daily-grid convention of Sect. 2. Recorded length of stay is treated as $\ell_i := e_i - s_i$. Quality control checks missing or invalid dates, invalid exit types, missing age or sex, impossible ages, and inconsistent chronology such as $e_i < s_i$. All restrictions are applied consistently to the empirical series and to the estimated probability tables.

The default implementation uses fully observed episodes with recorded exit date and exit type. If an extraction window induces right-censoring, the likelihood in Sect. 2.3 and the estimators in Sect. 3.3 require censoring-aware extensions. In the applied SIH/S72 table, no missing admission dates, exit dates, exit types, age, or sex values remained after source-level preprocessing. However, because the analysis conditions on observed closed episodes, late-window estimates should be interpreted as conditional on the observed closed-episode extract rather than as censoring-adjusted estimates if later exits were unavailable in the source files.

Same-day admission-and-exit records have $\ell_i = 0$ under date arithmetic. The default analysis excluded them. A sensitivity analysis recoded them as one-day stays by shifting the recorded exit date to the next calendar day, so that they contribute to day-0 occupancy and day-0 exit probabilities. Under this recoding, empirical admissions, occupancy, and outflows are recomputed; hence the diagnostic target differs from the default retained-sample target.

The applied analysis used 10 age-sex strata. The reference run used a deterministic regime map with cut points 2020-03-01 and 2022-01-01, defining pre,

COVID, and post periods, and adopted the calendar-time regime-pooled representation. Admission-based and admission-day-indexed calendar-time representations remain available as alternative working choices.

After preprocessing, empirical daily admissions, cause-specific exits, and occupancy are reconstructed directly from admission dates, exit dates, exit types, and stratum assignments under the same half-open convention. These series provide the observed inflow for the dynamical layer, the risk sets and event counts for probability estimation, and the comparison series for reconstruction diagnostics.

3.3 Daily Exit Probabilities and Hazard-Scale Reporting

Daily exit probabilities are estimated on the grouped day-of-stay support used by the cohort dynamics. Ordinary cells are $m = 0, \ldots, m_{\max} - 1$. If a single retained tail is used, the additional cell $m = m_{\max}$ aggregates episode-days with grouped day of stay at least $m_{\max}$; this row is a pooled closure for the retained long-stay state, not a day-specific probability for each $m \geq m_{\max}$.

Let $W_i(m)$ be the row index assigned to episode i on grouped day of stay m:

$$W_i(m) = \begin{cases} P(s_i), & \text{admission-based convention,} \\ s_i, & \text{calendar-time, admission-day indexed,} \\ P(s_i + m), & \text{calendar-time, regime-pooled.} \end{cases}$$

For stratum j, row w, and ordinary grouped day $m = 0, \ldots, m_{\max} - 1$, define

$$Y_{j,w}(m) := \sum_{i=1}^{N} \mathbf{1}\{j(i) = j,\ \ell_i > m,\ W_i(m) = w\}, \tag{10}$$

$$Z_{k,j,w}(m) := \sum_{i=1}^{N} \mathbf{1}\{j(i) = j,\ c_i = k,\ \ell_i = m + 1,\ W_i(m) = w\},\ k \in \{D, M\}. \tag{11}$$

When $Y_{j,w}(m) > 0$,

$$\widehat{q}_{k,j,w}(m) = \frac{Z_{k,j,w}(m)}{Y_{j,w}(m)},\ \widehat{q}_{j,w}(m) = \sum_{k \in \{D,M\}} \widehat{q}_{k,j,w}(m).$$

For a single retained tail, probabilities must be estimated at the same resolution as the tail state propagated by the cohort update. Let v denote the retained tail-closure index. For an unindexed marginal tail, $v = *$ and long-stay exposure is collapsed over admission regimes or admission days. For a calendar-time regime-pooled tail, v may be the contemporaneous regime, since all patients in the tail at calendar day t^n are exposed to $P(t^n)$. Define

$$V_i(u) = \begin{cases} P(s_i + u), & \text{calendar-time regime-pooled tail closure,} \\ *, & \text{unindexed marginal single-tail closure.} \end{cases}$$

Then, for each retained tail index v,

$$Y^{\text{tail}}_{j,v} := \sum_{i=1}^{N} \sum_{u=m_{\max}}^{\ell_i - 1} \mathbf{1}\{j(i) = j,\ V_i(u) = v\}, \text{ and}$$

$$Z^{\text{tail}}_{k,j,v} := \sum_{i=1}^{N} \mathbf{1}\{j(i) = j,\ c_i = k,\ \ell_i - 1 \geq m_{\max},\ V_i(\ell_i - 1) = v\},\ k \in \{D, M\}.$$

If $Y^{\text{tail}}_{j,v} > 0$, the tail-closure row is

$$\widehat{q}^{\text{tail}}_{k,j,v} = \frac{Z^{\text{tail}}_{k,j,v}}{Y^{\text{tail}}_{j,v}},\ \widehat{q}^{\text{tail}}_{j,v} = \sum_{k \in \{D,M\}} \widehat{q}^{\text{tail}}_{k,j,v}.$$

Tail-closure rows with zero exposure are completed by the same fallback rule used for no-exposure ordinary rows, and this choice is recorded with the other completion settings.

In the cohort update,

$$q^{n,\text{tail}}_{k,j} = \begin{cases} \widehat{q}^{\text{tail}}_{k,j,P(t^n)}, & \text{calendar-time regime-pooled tail closure,} \\ \widehat{q}^{\text{tail}}_{k,j,*}, & \text{unindexed marginal single-tail closure.} \end{cases}$$

Tail counts indexed by admission regime or admission day may be used as diagnostic summaries, but they are valid simulation inputs only if the tail state carries the same index. Otherwise they must be collapsed into the marginal tail closure above. If grid-respecting tail states are retained, the ordinary day-specific definitions in (10)–(11) extend to the chosen tail support.

Sparse rows are completed by deterministic forward adjacent-bin pooling. Given $Y_{\min} \geq 1$, the algorithm starts at the first uncompleted day m and selects the shortest block $B = [m, m']$ satisfying

$$\sum_{u=m}^{m'} Y_{j,w}(u) \geq Y_{\min}.$$

The pooled numerator and denominator over B define a common probability for that block. A remaining positive-exposure suffix below threshold is pooled and reported; no-exposure cells carry forward the last completed value when available, otherwise use the prespecified fallback rule; and rows with no positive exposure use the same fallback rule. The applied choices for $m_{\max}$, $Y_{\min}$, tail treatment, and same-day handling are reported in Sect. 4.

The completed probability tables are passed directly to the cohort dynamics. Hazard-scale quantities are used only for reporting or for compatible continuous-time embeddings. For this purpose, let ω denote a generic completed row index: $\omega = w$ for ordinary cells and $\omega = v$ for retained tail-closure rows. For tail rows, set

$$\widehat{q}_{k,j,\omega}(m_{\max}) := \widehat{q}^{\text{tail}}_{k,j,\omega},\ \widehat{q}_{j,\omega}(m_{\max}) := \widehat{q}^{\text{tail}}_{j,\omega}.$$

Given completed probabilities $\widehat{q}_{j,\omega}(m)$ and $\widehat{q}_{k,j,\omega}(m)$, and $\epsilon > 0$, define

$$\widetilde{q}_{j,\omega}(m) := \min\{\max(\widehat{q}_{j,\omega}(m), 0),\, 1-\epsilon\},\ \ \widetilde{\Lambda}_{j,\omega}(m) := -\log\big(1-\widetilde{q}_{j,\omega}(m)\big),$$

and

$$\widetilde{\Lambda}_{k,j,\omega}(m) := \begin{cases} \widetilde{\Lambda}_{j,\omega}(m)\dfrac{\widehat{q}_{k,j,\omega}(m)}{\widehat{q}_{j,\omega}(m)}, & \widehat{q}_{j,\omega}(m) > 0, \\ 0, & \widehat{q}_{j,\omega}(m) = 0, \end{cases} \quad k \in \{D, M\}.$$

This preserves the cause proportions while replacing any unit total probability by a finite-hazard approximation. All completion and tail-closure choices are recorded with $m_{\max}$, $Y_{\min}$, ϵ, same-day handling, and retained-tail treatment.

3.4 Hazards-To-Dynamics Simulation and Internal Consistency Checks

After completion on the chosen working grid, the daily exit-probability tables are combined with the reconstructed admission series $A_j(t)$ and propagated through the cohort update of Sect. 2.5. The model-implied calendar-time series are

$$\widehat{H}_{\mathrm{sim}}(t^n) := \sum_{j=1}^{J} H_j^n,\ \ \widehat{D}_{\mathrm{sim}}(t^n) := \sum_{j=1}^{J} \widetilde{D}_j^{n-1},\ \ \widehat{M}_{\mathrm{sim}}(t^n) := \sum_{j=1}^{J} \widetilde{M}_j^{n-1},\ \ n \geq 1, \tag{12}$$

with $\widehat{H}_{\mathrm{sim}}(t^0) = \sum_{j=1}^{J} H_j^0$. Thus, $\widehat{H}_{\mathrm{sim}}(t^n)$ is occupancy on day t^n, whereas $\widehat{D}_{\mathrm{sim}}(t^n)$ and $\widehat{M}_{\mathrm{sim}}(t^n)$ are outflows with exit date t^n.

These series are compared with empirical occupancy and outflows reconstructed from episode dates under the same half-open convention. If the working grid is represented exactly, the comparison is direct. Under a conservative reduced-tail closure, it is a reduced-state reconstruction. Under truncation, survivor mass beyond the last retained cell must be reported.

For closed retained-tail implementations, the accounting residual is

$$\mathrm{Res}(t^n) := \widehat{H}_{\mathrm{sim}}(t^n) - \widehat{H}_{\mathrm{sim}}(t^{n-1}) - A(t^n) + \widehat{D}_{\mathrm{sim}}(t^n) + \widehat{M}_{\mathrm{sim}}(t^n),\ \ n \geq 1. \tag{13}$$

By Proposition 2, this residual is zero in exact arithmetic once the initialization, timing convention, and retained-tail update have been fixed. With a reduced tail closure, a zero residual verifies conservation of the implemented reduced scheme, not exact recovery of all day-specific long-stay behaviour.

For truncated implementations, the modified balance in (9) applies. With $U(t^n) := \sum_{j=1}^{J} U_j^{n-1}$, the adjusted residual is

$$\mathrm{Res}^*(t^n) := \mathrm{Res}(t^n) + U(t^n),\ \ n \geq 1,$$

and is expected to vanish in exact arithmetic.

The residuals are implementation diagnostics, not empirical fit measures. Reconstruction quality is assessed separately by comparing (12) with the empirical series from Sect. 3.2 under the same state definition. Because both the estimated probability tables and the comparison series are derived from the retained

sample, this is an internal reconstruction check rather than external predictive validation. For summary reporting, the workflow computes occupancy mean absolute error and root mean squared error, together with mean absolute errors for discharge and death outflows, over all calendar days after the initial analysis day. It also stores probability-table support diagnostics, completion summaries, model-state support by grouped day of stay, accounting residuals, sensitivity summaries, and figures.

4 Results and Diagnostics

This section reports the applied episode extract, the support of the estimated probability tables, the reconstruction diagnostics, and the sensitivity checks. All comparisons are internal reconstruction diagnostics on the retained episode data.

4.1 Applied Administrative Hospitalization Extract

The model-ready table contained 447,752 hospitalization episodes, with admission dates from 2017-01-01 to 2023-12-31 and recorded exit dates from 2017-01-02 to 2024-03-11. It included 428,968 discharges alive and 18,784 in-hospital deaths. The table was treated as an observed closed-episode extract, and the reconstruction diagnostics reported below are internal to the corresponding analysis cohort.

If later exits for admissions close to 2023-12-31 were unavailable in the source files, late-window estimates, especially for long-stay cells, should be interpreted as conditional on the observed closed-episode extract rather than as censoring-adjusted estimates.

The reference analysis used the default preprocessing rule, excluding 5,106 same-day admission-and-exit episodes. This left 442,646 retained episodes, with 424,164 discharges alive and 18,482 in-hospital deaths. The analysis used 10 age-sex strata, a regime-pooled calendar-time representation, $m_{\max} = 60$, $Y_{\min} = 25$, and a single retained tail cell. Here $m_{\max} = 60$ means ordinary day-of-stay cells $m = 0, \ldots, 59$, plus a retained calendar-time regime-pooled tail row for $m \geq 60$. The regime cut points were 2020-03-01 and 2022-01-01, defining pre-COVID, COVID, and post-COVID periods. The empirical occupancy series reconstructed from episode dates covered 2,627 calendar days, with mean empirical occupancy 1,356.01. Outflow summaries were computed over the 2,626 post-initial dates, giving mean daily discharge and death outflows of 161.52 and 7.04, respectively. The maximum recorded length of stay in the retained sample was 305 days.

4.2 Probability-Table Support

The reference probability table had 1,830 cells, corresponding to 10 strata, 3 regimes, and 61 day-of-stay states, including the retained tail cell. Positive exposure was present in 1,791 cells, or 97.87% of the table, and no cell had total estimated exit probability equal to one. The large majority of exposure was supported by raw cell estimates. Raw cells accounted for 99.887% of total episode-day exposure, while adjacent pooled cells and suffix pooled cells below $Y_{\min}$

accounted for 0.111% and 0.001%, respectively. The remaining 39 no-exposure cells were completed by carrying forward the last available value within the corresponding row. The exposure-weighted total one-day exit probability was 0.1243, decomposed into 0.1191 for discharge and 0.00519 for in-hospital death. Exposure was concentrated in the early and intermediate days of stay: the exposure-weighted median grouped day of stay was $m = 4$, and 90%, 95%, and 99% of exposure were accumulated by $m = 17$, $m = 23$, and $m = 43$. By $m = 30$, 97.42% of total exposure had already been observed. The retained tail cell $m \geq 60$ represented 0.40% of empirical episode-day exposure.

4.3 Reconstruction and Accounting Diagnostics

The model-implied occupancy and cause-specific outflow series were compared with the empirical series reconstructed directly from admission and exit dates under the same half-open convention. For occupancy, the reference implementation achieved $\mathrm{MAE}_{\mathrm{occ}} = 53.03$, $\mathrm{RMSE}_{\mathrm{occ}} = 67.38$. Relative to the mean empirical occupancy, these errors correspond to 3.91% and 4.97%. For daily outflows, the mean absolute errors were $\mathrm{MAE}_{\mathrm{dis}} = 26.69$, $\mathrm{MAE}_{\mathrm{death}} = 2.10$. These correspond to 16.52% of the mean daily discharge outflow and 29.80% of the mean daily in-hospital death outflow. The higher relative error for deaths is consistent with the lower volume and greater discreteness of the death outflow series. Accounting residuals computed from (13) were negligible, with mean and maximum absolute values 7.63×10^{-14} and 2.45×10^{-13}, respectively, consistent with exact discrete conservation up to numerical roundoff. The zero-residual diagnostic verifies conservation of the reduced closed-tail scheme, not exact recovery of all day-specific long-stay exit probabilities beyond $m_{\max}$.

4.4 Sensitivity Checks

Sensitivity was assessed with respect to finite support, sparse-cell pooling, retained-tail treatment, and same-day handling. The grid used $m_{\max} \in \{36, 60, 90\}$, $Y_{\min} \in \{12, 25, 50\}$, single-cell and grid-respecting retained-tail treatments, and two same-day policies: exclusion and recoding as one-day stays. This produced 36 configurations. The reconstruction metrics changed little across these settings. Occupancy MAE ranged from 52.99 to 53.29, and occupancy RMSE from 67.34 to 67.68. Daily discharge MAE ranged from 26.69 to 26.76, while daily death MAE ranged from 2.097 to 2.110. In relative terms, occupancy MAE remained between 3.91% and 3.92% of mean empirical occupancy, and occupancy RMSE between 4.97% and 4.98%. The corresponding ranges were 16.38%–16.52% for discharge outflow MAE and 29.49%–29.80% for death outflow MAE.

Recoding same-day episodes increased the retained sample from 442,646 to 447,752 records, but did not materially affect the reconstruction metrics. Similarly, varying $m_{\max}$, $Y_{\min}$, or the retained-tail treatment produced only small changes in the main errors. All closed-tail configurations preserved mass balance to numerical precision.

5 Conclusions

This paper proposed a reproducible daily-grid framework for converting administrative hospital episode data recorded at date resolution into daily occupancy and cause-specific outflow series. The framework links grouped-time competing-risks estimation with conservative daily cohort dynamics under a half-open timing convention, making the accounting links among admissions, occupancy, discharges, and in-hospital deaths explicit. Its main methodological contribution is to provide a transparent hazards-to-dynamics route, operationalised through grouped daily exit probabilities, while separating quantities directly supported by retained episode data from those requiring pooling, completion, tail closure, or modelling assumptions. Admission and exit dates determine the empirical daily series and support one-day exit probabilities on grouped day-of-stay cells with nonempty risk sets. Continuous-time hazards are used only as compatible embeddings, and long-stay behaviour is represented through an explicit retained-tail construction.

In the Brazilian femur-fracture hospitalization application, the reference reconstruction produced small relative occupancy error and preserved mass balance to numerical precision. Sensitivity checks over finite support, sparse-cell pooling, same-day handling, and retained-tail treatment showed stable reconstruction metrics. Because the empirical application uses one diagnosis group from one administrative data system, these results establish internal reconstruction consistency for this retained cohort; generalisability across hospitals, diagnoses, or health systems remains a question for external validation.

In practice, the workflow is most useful for analysts, planners, and health-services researchers who need to translate closed episode records into calendar-time summaries of inpatient flow. It can be run on routine administrative episode extracts to produce daily occupancy reconstructions, cause-specific outflow profiles, and day-of-stay exit probabilities by age, sex, diagnosis group, ward, hospital, or policy period, depending on the fields available in the administrative extract. These outputs can support retrospective flow audits, comparison of operating regimes, and scenario analyses in which alternative admission streams or exit-probability tables are propagated through the same conservative cohort update. They are intended to complement, rather than replace, real-time operational bed-management systems: the framework provides daily cohort occupancy and outflow inputs that can be combined with capacity information when operational bed pressure is the quantity of interest.

Natural extensions include uncertainty quantification, smoother or hierarchical estimation of daily exit probabilities, richer state definitions, multi-state models when additional timestamps are available, and external validation against independent operational data or additional clinical cohorts.

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