Funicular Analysis of a Non-symmetric Sail Vault

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Abstract. This work is part of a research project aimed at exploring the applicability of a computational funicular method to masonry structures with non-trivial geometries, with the aim of adequately describing the flow of compressive forces. As a case study, a non-symmetric vault was selected, namely a sail vault in the Fortezza Vecchia (Old Fortress) in Livorno, Italy, which covers a trapezoidal room and is characterised by an octagonal *oculus*. The study of the mechanical response of the vault is conducted by means of a funicular approach based on a mathematical programming formulation within the framework of the static theorem of limit analysis. The proposed investigation considers simplified assumptions regarding the mechanical behaviour of masonry, namely zero tensile strength, infinite compressive strength, and infinite friction. The objective function was specifically implemented to identify minimum thrust solutions. The results are discussed and compared with those obtained from a finite element analysis performed in Abaqus CAE.

Keywords: Masonry · Sail Vault · Funicular Network Analysis · Limit Analysis

1 Introduction

Modelling masonry vaults is a complex challenge, given the non-homogeneous and anisotropic nature of this material, characterized by low tensile strength and high compressive strength [1]. According to [2], the very definition of the concept of 'masonry' is a challenging task because it includes a variety of materials depending on the components used (stone, bricks, etc.), the types of blocks, the assembly method, the presence or absence of mortar and the construction technique adopted. Without claiming completeness, some contributions aim to create realistic models capable of faithfully representing the structural response of masonry structures (or masonry elements), by using homogenization techniques [3], or by proposing non-linear finite element analyses adopting different modelling strategies. Among other examples, [4] homogeneously models a masonry structure with a complex geometry, discretizing it through macro-blocks obeying a Mohr-Coulomb failure criterion; [5] proposes a discretization of the mortar in

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masonry walls subjected to in-plane loads. However, the high degree of uncertainty surrounding the material properties constitutes a disadvantage that makes it difficult to select the appropriate mechanical parameters. Observing that finite element analyses are often complex, mesh-dependent and require the application of ad hoc constitutive laws to capture the intricacy of the mechanical response of masonry structures, some studies aim to bridge the gap between sophisticated modelling and real-world implementation; see for example [6], where a simple nonlinear truss model is proposed to assess the stability of curved masonry structures. Additionally, one of the most notable characteristics of masonry is its uncertain tensile strength, which is often significantly lower than its compressive strength. In this complex context, a simplified approach based on Heyman's assumptions [7] is commonly followed, even in the most recent scientific literature [8], to evaluate the structural response of masonry structures. According to these assumptions, masonry can transmit only compressive forces; the compressive strength is assumed to be infinite, while the tensile strength is zero; furthermore, sliding between blocks is assumed to be prevented by an ideally infinite friction. Heyman's hypotheses allow the problem to be rigorously treated within the framework of the theorems of limit analysis.

In this context, Heyman's assumptions are adopted to find purely compressive load paths; see for example [9], where masonry is described as a no-tension solid continuum, or [10], where purely compressed shells are identified by exploiting the Airy stress function. Several studies have revisited the historical contributions on the thrust line method for masonry arches and vaults or the inverted catenary concept; see for example [11]. Advances in graphical statics and the development of computer-aided techniques have made it possible to design the funicular polygon corresponding to the actual loads for the safety assessment of masonry structures with complex geometry. One of the techniques arising from this background is the thrust network approach [12, 13].

The purpose of the research here presented is to investigate whether a computational funicular method can be applied to masonry structures with complex or non-symmetric geometries to accurately describe the flow of compressive forces under service conditions. The approach, in which Heyman's hypotheses are adopted regarding the mechanical behaviour of masonry, is based on the mathematical programming formulation suggested in [14]. The procedure begins with the construction of a spatial network with a specified planar geometry and considering an independent set of force densities. The load path in the vault is found by formulating an optimization problem in terms of the vertical coordinates of the constrained nodes. Depending on the goal, different objective functions can be chosen. To identify anti-funicular networks (in equilibrium with the loads) that are located within the vault's shape, restrictions are imposed on the height of the nodes and the sign of the force densities. The method, implemented in MATLAB, has been already successfully applied to the study of symmetric and non-symmetric masonry vaults subjected to their self-weight [15]. Furthermore, some applications have been provided for the study of arches and vaults subjected to seismic loads [14, 16]; recent developments of the method are able to take into account strength requirements and stereotomy [17].

This research aims to examine a significant case study of a non-symmetric vault, a sail vault in the Fortezza Vecchia in Livorno, Italy, which covers a trapezoidal room

and is distinguished by an octagonal *oculus* [18]. The geometry of the sail vault was reconstructed from the results of an accurate terrestrial laser scanner survey of the entire fortress, available as a point cloud.

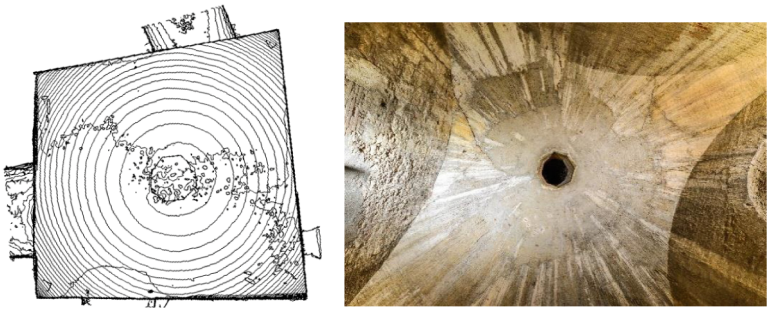
The funicular method is used to investigate networks that represent paths of compressive force flow in equilibrium with prescribed loads. In this regard, it should be remembered that this method fits into the context of limit analysis. It is based on Heyman's safe theorem [7], which states that the structural stability of masonry structures is guaranteed if at least one statically admissible compressive load path can be found within the geometry of the structure. It is therefore necessary to identify the most plausible equilibrium state among the infinite admissible ones. To this end, the "principle of least pressure", proposed by Moseley in 1833 [19] and originally conceived for masonry arches, is applied; according to this principle, the "true" thrust line is that which minimizes the horizontal component of the thrust itself. In light of the above, in the funicular method the concept of "thrust" is extended to three dimensions and the generalized horizontal "thrust" is minimized. Because only low tensile stresses are expected to develop in the vault, this choice ensures a meaningful comparison with the results of the linear elastic finite element analyses performed in Abaqus CAE, despite being based on a different modelling of the masonry material. This comparison allows one to verify the reliability of the minimum thrust funicular solution.

2 The Sail Vault of the Fortezza Vecchia in Livorno: History and Previous Structural Analyses

The history of the fortress, described in detail in [18], begins in the 16th century, when the Medici family took control of Livorno and began significantly modifying the existing buildings. Under the guidance of his brother Giuliano, Antonio da Sangallo the Elder developed the *Fortezza* plans. With the help of numerous expert engineers, its construction took place between 1519 and 1533. The structure has irregular and unique architectural lines with three imposing bastions. Inside the Canaviglia bastion, there is a characteristic trapezoidal room (Fig. 1a) covered by a sail vault with an octagonal ventilation opening (*oculus*) at the centre (Fig. 1b).

The entire fortress, including the vault, is an example of Renaissance architecture of great value, which testifies to the impact of Filippo Brunelleschi's innovations on the building panorama of the time (such as the *opus spicatum* as a construction technique for vaults) and the technical expertise of Antonio da Sangallo the Elder.

Some previous research on the structural behaviour of the vault was performed by [20–24] within a research project of the University of Pisa (programme PRA 2017, project *"Architetture toscane rinascimentali: casi studio fra indagine storica, rilievo e analisi strutturale"*). These works provided some preliminary results - within the context of the safe theorem of limit analysis - related to the behaviour of the vault subjected to vertical loads, by seeking statically admissible solutions. Assuming either Heyman's hypotheses [7, 22], or a limited compressive strength [20, 23], and simplifying the vault's geometry by adopting an analytical expression for the intrados surface, these studies were conducted both by reviewing Durand-Claye method [21–23, 25] and by considering the vault as a thin shell. For simplicity, the analyses were limited to axisymmetric stress distributions. From these studies, the vault was found to be safe under vertical loads.



(*a*) (*b*)

Fig. 1. The sail vault in the Canaviglia bastion of the Fortezza Vecchia in Livorno. (*a*) Trapezoidal shape of the plan with the contour lines describing the geometric irregularities (courtesy of Prof. Andrea Piemonte, University of Pisa). (*b*) view from the inside, with the central octagonal *oculus*.

3 From the Survey to the Structural Analysis

As part of a collaboration with the University of Pisa [18], the administrative bodies (*Autorità di Sistema Portuale del Mar Tirreno Settentrionale* - North Tyrrhenian Sea Port Authority) have made available the data collected from a terrestrial laser scanner survey (both internal and external) of the entire fortress. The *status quo* geometrical survey is described in [24].

The geometries required for the structural analysis were extracted from the survey, accessible as a point cloud provided in Autodesk ReCap format, with a density of approximately three points per square centimetre. The region that corresponded to the vault was separated from the remainder of the fortress, by operating with the CloudCompare software. The resulting model was exported in the .pts format, making it compatible with three-dimensional modelling software like AutoCAD and Rhino (Fig. 2).

The Rhino graphics software was used to create a solid surface from the point cloud of the vault through the *Patch* command. This procedure enabled reproduction of the intrados surface with sufficient precision, including its irregularities. Furthermore, the vault's intrados was chosen as the reference surface for building the model of the extrados surface (which is inaccessible for direct surveying due to the fill layers and flooring). To this end, the *OffsetSrf* command in Rhino was used, by assuming a constant average vault thickness of 0.5 m, based on previous research and results of an experimental campaign, as described in [23].

The 3D survey and the previous contributions on the structural response of the sail vault cited in Sect. 2 constitute the starting point for further investigation, developed in [26]. As will be described in Sects. 4 and 5, the research here proposed – using both the funicular method and finite element analyses – aims to fully understand the three-dimensional behaviour of the vault under service conditions, exploring the effects of non-symmetry on the equilibrium solutions.

Once the 3D model definition was completed, it was exported in .sat format and integrated into the Abaqus CAE structural calculation software, with the aim to perform

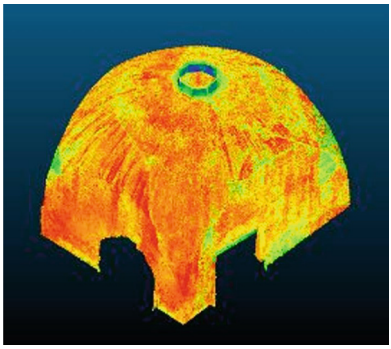
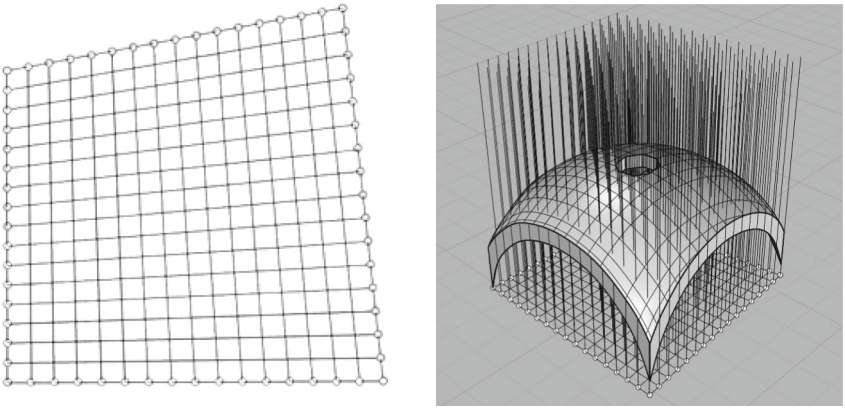


Fig. 2. Point cloud of the sail vault, with the side walls, extracted using the CloudCompare software.

FEM analyses to benchmark the results obtained via the funicular approach, as will be illustrated in Sect. 5.

To perform the funicular analysis of the vault under gravitational loads, it was necessary to define the network nodes through geometric processing operating in Rhino. In order to derive a network with a fixed projection on the horizontal plane, a planar grid was obtained by subdividing the opposite sides of the trapezoidal base perimeter into a number of equal segments (through the command *Divide*) and connecting the points belonging to opposite sides, as shown in Fig. 3a. In the present study, a 16×16 subdivision was adopted. After extracting the intersection points of the planar grid (using the commands *Explode* and *ExtractPoints*), vertical lines were conducted from each base point. By means of the *Trim* command, using the vault surfaces as the cutting object, and, again, the *ExtractPoint* command, the intersection points between the vertical lines and the intrados and extrados vault's surfaces were obtained.

The *EvaluatePt* command, an internal Rhino function, is used to extract the x, y, z^{min} and z^{max} coordinates, with z^{min} (respectively, z^{max}) derived from the intrados (resp., extrados) surface.



(*a*) (*b*)

Fig. 3. Procedure for obtaining a network with fixed projection on the horizontal plane. (*a*) Planar grid (16 × 16 subdivision). (*b*) The intrados and extrados points are derived by intersecting the vertical lines drawn starting from the planar grid with the intrados and extrados surfaces.

4 Funicular Analysis of the Sail Vault

4.1 The Force Density Method

The method adopted here [14] builds upon the concept of force density, as presented by H. J. Schek [27]. The Force Density Method is used to determine the geometric static equilibrium configurations of spatial structures subjected to external loads relying on linear equations. Through this approach, it is possible to manage the equilibrium of anti-funicular networks consisting of $n_s = n + n_f$ nodes (where n and n_f are the number of unrestrained and restrained nodes, respectively) which connect m branches.

In the applications, masonry vaults are modelled as statically indeterminate networks subjected only to compressive stresses and in equilibrium with the loads applied at the nodes. It is recalled that, according to Heyman's safe theorem [7], a masonry arch is in equilibrium if there exists a thrust line contained within the arch. This theorem implicitly refers to the concept of 'line of resistance' [28], related to the centre of pressure of internal reaction inside any joint, whereas anti-funicular polygons correspond to the concept of 'line of pressure' [28], see also the discussion in [29]. In this contribution joints are disregarded. An appropriate set of enforcements is formulated to prescribe lower and upper bounds for the vertical coordinates z of the nodes, to ensure that the retrieved anti-funicular networks remain within the vault's thickness, i.e., $z^{min} \leq z \leq z^{max}$, for the assigned loads. These constraints provide a good approximation of the conditions arising from the safe theorem in order to guarantee the equilibrium of the vault.

The introduction of the force densities vector, $\mathbf{q}$, collecting the force-to-length ratio in each branch of the network, allows formulating the equilibrium of the n unrestrained nodes as a set of linear equations. Furthermore, the condition $\mathbf{q}_j \leq 0, For\, j = 1, ..., m$, allows for enforcing that the branches are subjected to compression, ensuring a no-tension behaviour.

For the details on the procedure here applied, the interested reader is referred to [14], where the optimization problem is conveniently formulated in terms of the reduced set of independent density forces, $\overline{\mathbf{q}}$, calculated via a preliminary Gauss-Jordan elimination, and solved by using sequential convex programming techniques, such as the Method of Moving Asymptotes [30].

For the purposes of this contribution, it is sufficient to recall that the required input data are the planimetric coordinates of the nodes (x, y), the vertical limits for the height of the nodes, z^{min} and z^{max}, the nodal loads, and the connectivity between the elements.

The optimization algorithm searches for the anti-funicular network that, respecting the limits described above, corresponds to the minimum thrust based on the objective functions defined below:

$$f(\mathbf{q}) = \sum_{h}^{n_f} \sqrt{R_{xh}^2 + R_{yh}^2}, \tag{1}$$

where R_{xh} and R_{yh} are the reactions along the x and y direction. Equation (1) is intended to be an extension of thrust concept, commonly used for the equilibrium of masonry arches, to three-dimensional vaults. As regards the minimization of the horizontal thrust, this choice is based on the suggestions provided in [19] and on the discussion proposed in [31].

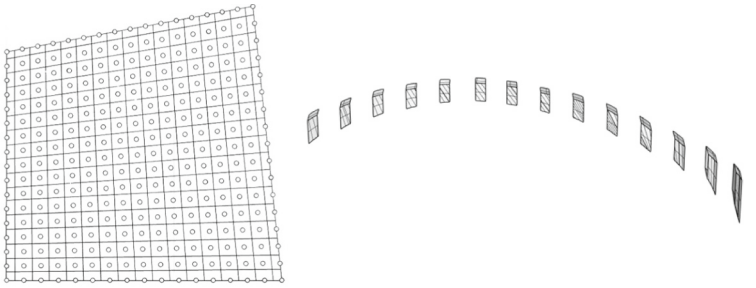
4.2 Definition of the Nodal Loads

To assign loads to each node in the network, we proceed by constructing a secondary planar grid, as shown in Fig. 4a, dividing each segment of the original grid in Fig. 3a in half. The three-dimensional model of the vault (in Rhino) is then sectioned using vertical planes drawn from the lines belonging to this new planar grid; in this way, irregular blocks with a quadrangular base are obtained, see Fig. 4b. It is thus possible to compute, after some processing in Rhino, the volume of each block. The centre of gravity of each block lies on a straight vertical line that passes approximately through each of the primary grid nodes, see the white dots in Fig. 4a.

The procedure just described allows us to evaluate the nodal load corresponding to the self-weight, by multiplying the volume of each block by the unit weight of masonry, assumed to be equal to 18 kN/m^3. For brevity, the procedure for evaluating the nodal load corresponding to the infill between the vault's extrados and the upper floor level is omitted. It is sufficient to remember that a specific weight of 17 kN/m^3 is assumed for the infill, consistent with both previous studies on the vault's structural behaviour and the technical documentation available [18, 20–24]. The nodal loads are then obtained by adding the weights of masonry and infill.

4.3 Different Configurations of the Network

The vault analyses were conducted considering different configurations of the funicular network of increasing complexity. The starting point was a network with relatively low computational complexity, according to a quadrilateral grid scheme like the one shown in Fig. 5a ($n_s = 256$; $m = 472$). A second configuration is obtained by adding connectivity

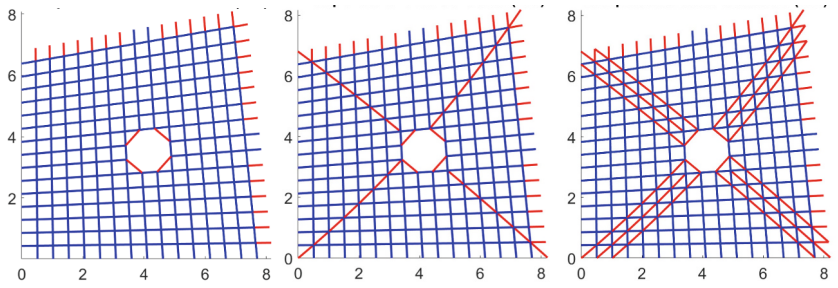


(*a*) (*b*)

Fig. 4. Procedure for assigning loads to each node. (*a*) Secondary planar grid (straight lines) and intersection points of the primary grid (white dots). (*b*) Blocks are obtained by sectioning the 3D model of the vault with vertical planes passing through the secondary grid lines (discretization of an arch belonging to the vault).

restrictions between nodes located along the diagonals of the trapezoidal plan; see Fig. 5b ($n_s = 256$; $m = 499$).

Increasing the degree of redundancy, additional branches are added along lines parallel to the main diagonals of the trapezoidal plan and starting from the vertices of the *oculus* (Fig. 5c): the asymmetry of the ensuing grid is clearly visible ($n_s = 256$; $m = 549$).



(*a*) (*b*) (*c*)

Fig. 5. Different configurations of the anti-funicular networks for the analysis of the sail vault. Branches where independent force densities act are marked in red. (*a*) Base configuration. (*b*) Addition of branches corresponding to the diagonals of the trapezoidal plan. (*c*) Addition of further branches starting from the vertices of the *oculus*.

To identify any significant compressive flows inside the dome, a fourth configuration adds branches concentric to the octagonal *oculus*, so as to simulate circumferential stresses; see Fig. 6a ($n_s = 256$; $m = 693$).

By adding branches corresponding to the diagonals of each quadrangular module (Fig. 6b), a network of greater complexity is obtained, characterized by an increase in the

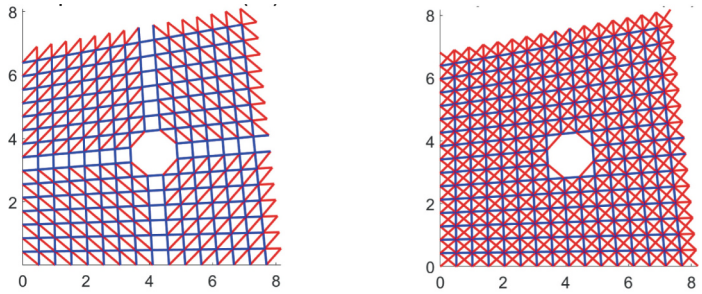


Fig. 6. Different configurations of the anti-funicular networks for the analysis of the sail vault. Branches where independent force densities act are marked in red. (*a*) Addition of branches concentric to the octagonal *oculus*. (*b*) Addition of branches corresponding to the diagonals of each quadrangular module.

degree of static indeterminacy ($n_s = 256$; $m = 996$). This configuration was designed to capture more general compression flows.

4.4 Anti-Funicular Networks of Minimum Thrust

The results of the equilibrium analyses conducted using the network configurations defined in Sect. 4.3 will be now illustrated.

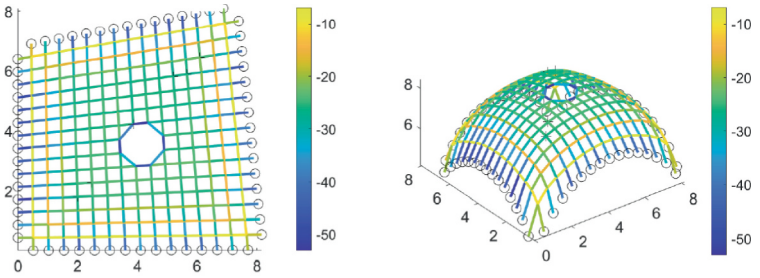
For the simplest network, corresponding to Fig. 5a, the anti-funicular network of minimum thrust is shown in Fig. 7a (plan) and Fig. 7b (axonometric view). In this case, the computation time required for the complete analysis was less than 1 min; CPU times referred to a processor Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz (1.80 GHz).

Longer computation times are required as the degree of static indeterminacy (see the number of network elements) increases. For the second configuration, where additional branches corresponding to the diagonals of the trapezoidal plan were added (Figs. 5b and 8), the time required for the complete analysis was 1 h 45 min 27 s; for the third one, with additional branches starting from the vertices of the octagonal *oculus* (Fig. 5c and Fig. 9), it was 3 h 47 min 24 s.

Given the simplified nature of the networks, the first three scenarios just described constitute a first validation phase of the method, with particular attention to the reliability of the results (i.e., the attainment of convergence to a solution that is fully admissible with respect to the prescribed constraints) rather than their capability of reproducing the actual behaviour of the vault.

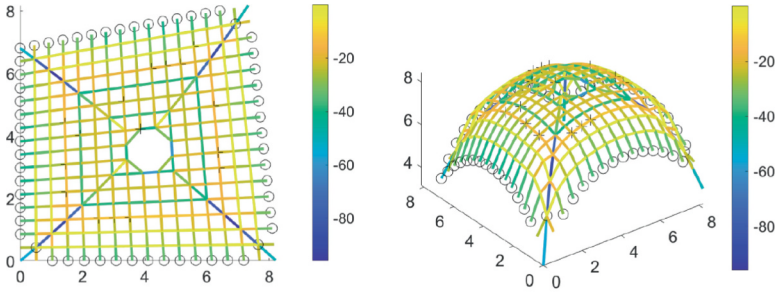
Despite a considerable increase in computation time, the more complex networks, corresponding to the configurations depicted in Figs. 6a and 6b, are the most suitable for capturing the three-dimensional behaviour of the dome, as will be clarified comparing the results deriving from the funicular method with the analyses conducted with Abaqus CAE in Sect. 5.

For the configuration with additional concentric branches (Fig. 6a), the minimum-thrust anti-funicular network is shown in Fig. 10. The computation time is 5 h 42 min



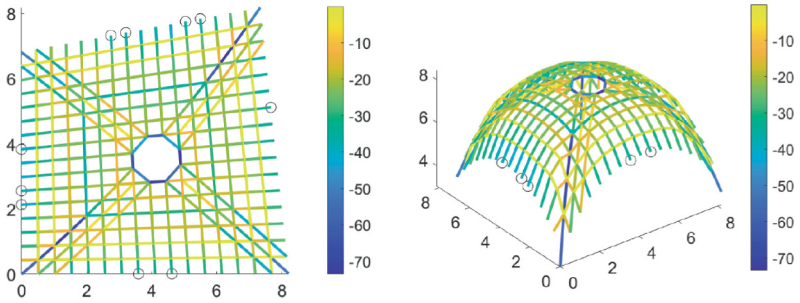
(*a*) (*b*)

Fig. 7. Anti-funicular network of minimum thrust corresponding to the simplest configuration (Fig. 5a): (*a*) plan view; (*b*) axonometric view. Forces are in kN.



(*a*) (*b*)

Fig. 8. Anti-funicular network of minimum thrust corresponding to the configuration with additional branches along the diagonals of the trapezoidal plan (Fig. 5b): (*a*) plan view; (*b*) axonometric view. Forces are in kN.



(*a*) (*b*)

Fig. 9. Anti-funicular network of minimum thrust corresponding to the configuration with additional diagonal branches starting from the vertices of the octagonal *oculus* (Fig. 5c): (*a*) plan view; (*b*) axonometric view. Forces are in kN.

39 s. For the most complex case, i.e., the network with branches along the diagonals of each quadrangular module (Fig. 6b), it increases further and reaches 21 h 12 min 16 s. The related anti-funicular network is shown in Fig. 11.

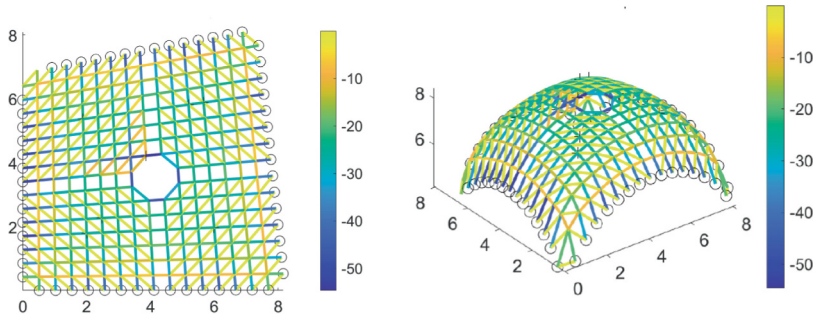


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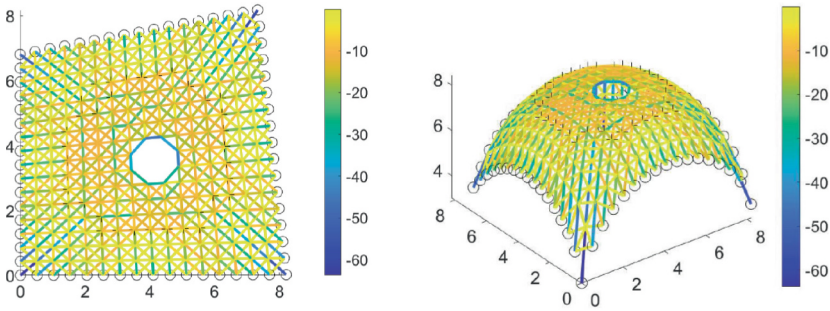
Fig. 10. Anti-funicular network of minimum thrust corresponding to the configuration with the addition of concentric branches (Fig. 6a): (*a*) plan view; (*b*) axonometric view. Forces are in kN.



(*a*) (*b*)

Fig. 11. Anti-funicular network of minimum thrust corresponding to the configuration with additional branches along the diagonals of each quadrangular module (Fig. 6b): (*a*) plan view; (*b*) axonometric view. Forces are in kN.

From Figs. 10 and 11, the dome's portion near the octagonal *oculus* is visibly subjected to compressive stresses; the shape of the compressed cap is consistent with quasi-circumferential forces. In Fig. 10, the compressive forces increase near the side walls, as indicated by the blue branches; the corners appear to be less stressed. In Fig. 11 a different response is observed, as an increase in compressive forces can be seen along the main diagonals from top to bottom (corresponding to meridional forces). According to the comparison with the finite element results presented in Sect. 5, the most suitable configuration is found to be that shown in Fig. 10 (see also Fig. 6a).

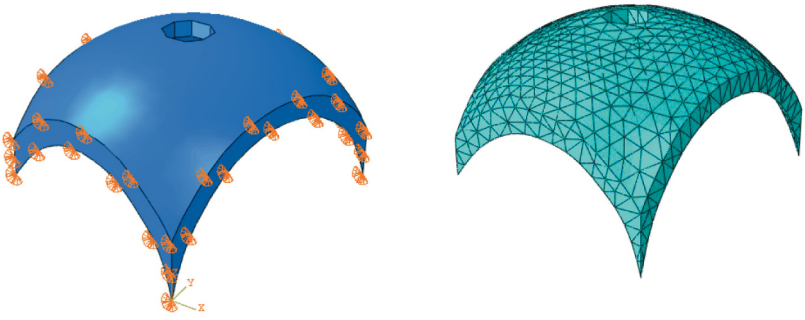
5 A Comparison Between Minimum Thrust Funicular Solutions and Finite Element Analyses with Abaqus CAE

To compare the results obtained with the funicular method (Sect. 4) with those of a more standardized approach, a simplified analysis of the structural response of the vault under gravitational loads was carried out using the Finite Element Abaqus CAE software, modelling masonry as an isotropic linear elastic material.

To represent the confinement exerted by the perimeter walls, kinematic constraints were imposed (pinned boundary condition), in which the displacements in all three directions were blocked (Fig. 12a).

The self-weight of the vault was applied through the *Gravity* command. The load from the infill was modelled as a vertical *Surface traction* acting on the extrados of the vault. The FE mesh consists of solid tetrahedral elements, considering that the dome is not thin enough to use shell elements (Fig. 12b). A quadratic interpolation was chosen to achieve a clearer representation of the stress variation along the vault's thickness. The vault was discretized into 3,730 quadratic tetrahedral finite elements, resulting in 111,900 degrees of freedom.

To facilitate the comparison of the stresses obtained with the two methods, a spherical reference system was adopted in the FE analysis.



(*a*) (*b*)

Fig. 12. Finite element analyses with Abaqus CAE: (*a*) pinned boundary conditions; (*b*) FE mesh of the sail vault.

The results obtained from the linear elastic analysis of the vault are presented below. Specifically, for subsequent comparison with the funicular model, the values, expressed in Pa, of the minimum principal stresses (Fig. 13), of the circumferential stresses (S22, Fig. 14), and of the meridional stresses (S33, Fig. 15) are reported.

From the analyses in Abaqus CAE it is observed that the upper portion of the dome is subjected to circumferential compression (Figs. 13a, 14a), while asymmetries in the stress distribution are evident.

The internal forces acting along the branches of the funicular network and their orientation with respect to the spherical coordinate system adopted in the FE analyses are also evaluated. The procedure, omitted here for brevity, calculates the ratio of

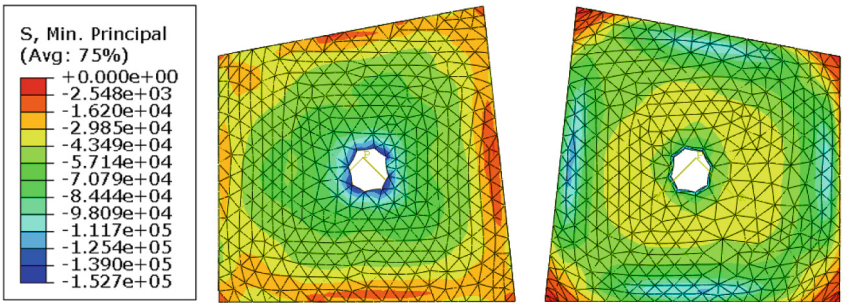


Fig. 13. Finite element analyses with Abaqus CAE: minimum principal stresses (*a*) extrados; (*b*) intrados. Stresses are measured in Pa.

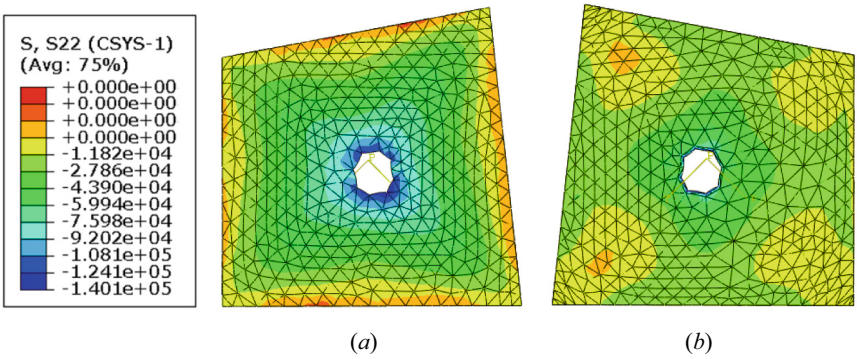


Fig. 14. Finite element analyses with Abaqus CAE: circumferential stresses, S22 (*a*) extrados; (*b*) intrados. Stresses are measured in Pa.

the internal forces to the corresponding cross-section of the vault, with some geometric simplifications. The network of Fig. 10 is found to better capture the mechanical response of the vault: the presence of a compressed cap is confirmed, and the circumferential stresses obtained from the funicular model substantially match those derived from the finite element model. Furthermore, both Fig. 15b, which reports the intrados meridional stresses obtained with the finite element analyses, and Fig. 10, where the most compressed branches are highlighted in blue, are consistent with an increase in the meridional compressive stresses towards the central portion of the sides of the trapezoidal plan, while the corners are less loaded.

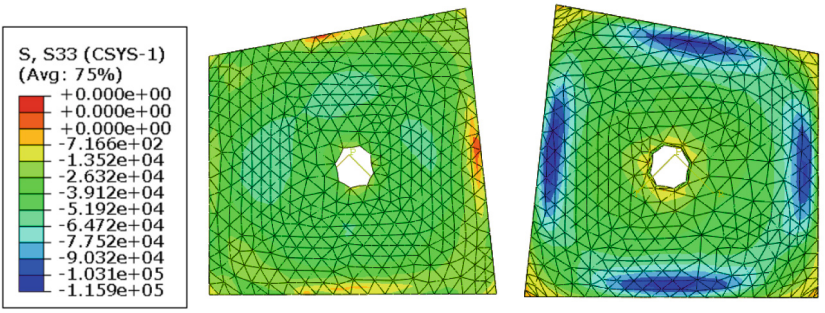


Fig. 15. Finite element analyses with Abaqus CAE: meridional stresses, S33 (*a*) extrados; (*b*) intrados. Stresses are measured in Pa.

6 Concluding Remarks

In this research, the analysis of the sail vault of the *Fortezza Vecchia* in Livorno is conducted by adopting a funicular method exploiting both the concept of force density and a mathematical programming formulation [14].

The objective function was specifically implemented to identify minimum thrust solutions. As a first step, the vault was modelled using a network with relatively low computational complexity, connecting each node to its adjacent ones so as to obtain a network composed of quadrilaterals. Subsequently, increasingly complex network configurations were considered, introducing diagonal branches, which allow some features of the structural response to be captured more accurately.

To benchmark the results thus obtained, simplified finite element analyses were performed in Abaqus CAE by modelling masonry as a linear elastic isotropic material.

Although the two methods - funicular analysis and finite element analysis - belong to different theoretical frameworks (limit analysis and elastic analysis), the minimum thrust solutions derived from the funicular network were shown to be compatible with the results obtained with Abaqus CAE, in particular with reference to circumferential stresses.

Further investigations are needed to fully interpret the real crack pattern observed in the vault: in this regard, the simplified results just obtained seem to highlight the importance of considering a limited compressive strength to describe masonry (observing both meridional and circumferential cracks). Future developments will consider the presence of ground settlements, and seismic loading conditions. Furthermore, in finite element analyses with Abaqus CAE, both the consideration of masonry plasticity and the adoption of a concrete damaged plasticity model can be crucial to capture cracking, crushing and stiffness degradation. In this context, a further development of this research could be the determination of a geometric safety factor, i.e. the ratio between the current thickness and the limit thickness (obtained by progressively reducing the thickness of the vault until collapse) considering different modelling strategies and material properties.

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