An Efficient Polynomial Framework for Optimal Control: Applications in Network Protection and Solar Energy Systems

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Abstract. This paper develops a novel polynomial-based framework for accurately and efficiently solving continuous optimal control problems, which are often challenging due to high computational cost and the limitations of existing approximation techniques. The primary objective is to construct a robust numerical scheme that avoids the need for closed-form analytical solutions while maintaining high accuracy. To this end, a new class of basis functions, namely the extended complete Vieta–Pell (ECVP) polynomials, is introduced. These polynomials are systematically constructed and employed to approximate both state and control variables over the time domain, transforming the original optimal control problem into a tractable algebraic system suitable for numerical implementation. The main contribution of this work is twofold: first, the formulation of the ECVP polynomials as a unified and efficient basis for approximation; second, the development of an effective computational framework that improves convergence behavior and numerical stability compared with existing methods. Unlike traditional approaches, the proposed method provides flexible approximation capability with enhanced accuracy as the expansion order increases. The effectiveness of the proposed framework is demonstrated through applications in solar energy systems and network protection models. Numerical results indicate rapid convergence and high precision, with the obtained solutions closely matching exact or benchmark solutions. These findings establish the proposed ECVP-based method as a reliable and powerful tool for a broad class of optimal control problems.

Keywords: Optimal control problems · New extended complete Vieta–Pell polynomials · Network protection systems · solar energy engineering · Error analysis

1 Introduction

The optimal control problem represents important problems in control theory and applied mathematics because of their significant role in modeling and analyzing a wide range of physical and engineering as well as in technological systems [1–5]. The objective of optimal control problems is to calculate a suitable control function that influence

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the dynamic system in certain way to achieve an optimal performance index criterion, subject to given dynamic and constraints. In general, the function $x(t)$ describes the state vector representing the evolution of the system based on time, while the function $u(t)$ represents the control vector that governs the behavior of the system. The control functions are usually assumed to belong to a class of admissible controls that ensures the existence and feasibility of the solution. A standard formulation of an OCP is given by

$$J = \int_{t_0}^{t_1} F(x(t), u(t), t)dt, \tag{1}$$

together with the following dynamic system and the boundary conditions

$$\dot{x}(t) = f(x(t), u(t)), \tag{2}$$

$$x(t_0) = x_a, x(t_1) = x_b. \tag{3}$$

where $F(x(t), u(t), t)$ is a positive definite performance index and f is continuously differentiable function.

From a theoretical perspective, OCPs are commonly treated using Pontryagin's minimum principle, which leads to a Hamiltonian formulation consisting of the state equations, adjoint equations, and associated optimality conditions. This formulation typically results in two-point boundary value problems (TPPVPs), which is often difficult to solve analytically or even numerically, particularly for nonlinear or constrained systems. Many numerical algorithms have been proposed to approximately solve them. Some of these numerical methods have been suggested to using the direct numerical techniques, including the both finite element and finite difference methods [6–9], the adaptive collocation method, [10], wavelet collocation method using the Chebyshev wavelets functions [11, 12], the Müntz– Legendre functions, [13] and others [14–18]. In addition, more suitable methods to solve the OCPs are applying the orthogonal basis functions such as Vieta-Pell family polynomials. As some applications of Vieta polynomial, a new type which is Vieta–Fibonacci-like polynomial was defined in [19], and then the relationship between Vieta–Fibonacci-like polynomial and the famous Vieta polynomials obtained. In addition, the generating function and the Binet formula were established with some new identities involving the sum formula of Vieta–Fibonacci-like polynomials. Also [20–22] proposed some identities containing families of Vieta functions. Other work certain family of Vieta [23], Vieta Lucas polynomials [24], Pell polynomials [25] and Lucas polynomials [26]. In addition, some recent work constructed generalized basis functions for different families of Vieta polynomials such as Generalized Vieta-Jacobsthal [27], incomplete generalized Vieta–Pell [28] and a specific modification of Vieta-Pell Lucas polynomials [29]. Such polynomials are often restricted to narrower domains unlike our suggested ECVP polynomials allows for solving problems with larger physical or mathematical boundaries without losing accuracy.

The important problem is how can present a general expression for the original polynomials. This study addresses the numerical solution of OCPs through the use of

new extended complete Vieta–Pell polynomials. The proposed polynomial framework provides an accurate and flexible representation of the state and control variables over the time domain, allowing the approximation quality to improve systematically as the expansion order increases. To address these difficulties, an equivalent optimization-based formulation can be adopted; in which both the state and control functions are approximated using suitable basis functions. This approach enables the transformation of the Hamiltonian-based formulation into an optimization problem while preserving the essential structure of the original system and reducing computational complexity. Within this framework, the present work introduces an extension of the classical Vieta–Pell polynomials through the construction of Extended Complete Vieta–Pell Polynomials. These polynomials possess favorable mathematical properties that make them well suited as Basis functions over finite intervals. Their use facilitates accurate approximations of the variables and provides an effective foundation for the numerical treatment of OCPs arising in practical applications, including solar energy systems and network protection models.

The proposed new version of the polynomials can use on any finite interval. The proposed method converts solving optimal control problems into solving constrained optimization problem by approximating both the state and control variables using the ECVPPs. The efficiency of the extended complete approach is investigated on some test applications.

The main contributions of this work can be summarized as follows:

- Introducing a new extended complete family of Vieta–Pell polynomials (ECVP) and establishing their mathematical structure.
- Developing new operational matrices for derivatives and products based on the proposed polynomials.
- Constructing an efficient numerical framework for solving optimal control problems using direct parameterization techniques.
- Demonstrating the effectiveness of the proposed method through practical applications, including solar energy systems and network protection models.

The rest of the work is arranged in sections as follows: The new modified definitions of ECVPPs with important properties are presented in Sect. 2. Derivative and product formulation matrices of ECVPPs are given in Sects. 3 and 4 while an efficient approach is constructed in Sect. 5 for approximately treating the optimal control problems. Two applications are solved and examined in Sect. 6. The Discussion and conclusion of the study are reported in Sects. 7 and 8.

2 New Extended Complete Vieta-Pell Polynomials

In this section, a new family of polynomials, referred to as the Extended Complete Vieta–Pell Polynomials (ECVPPs), is introduced. This family represents a natural extension of the classical Vieta–Pell polynomials over a general. Finite interval while preserving the essential second–order recursive structure that characterizes the complete Vieta–Pell system.

Definition 1. Let $t \in [t_0, t_1]$, k > 0 and let $k \in N$. The extended complete Vieta–Pell polynomials, denoted by $p_{k,n}(t)$, are defined by the recurrence relation

$$p_{k,n}(t) = 2^k \xi(t)\, p_{k,n-1}(t) - p_{k,n-2}(t), n \geq 3. \tag{4}$$

$$\text{with} \quad p_{k,1}(t) = 1 \quad \text{and} \quad p_{k,2}(t) = 2^k \xi(t). \tag{5}$$

This definition introduces a scaled and shifted version of the classical complete Vieta–Pell polynomials on a general interval $[t_0, t_1]$ with $\xi(t) = \frac{(2t - t_0 - t_1)}{t_0 - t_1}$.

The general closed–form expression of the extended complete Vieta–Pell polynomials with $k = 2$ is given by

$$p_{2,n}(t) = \sum\nolimits_{r=0}^{(n-1)/2} (-1)^r \left(\frac{n-1-r}{r}\right) \left(2^2 \xi(t)\right)^{n-1-2r} n = 1,2, \ldots.$$

$$p_{2,1}(t) = 1,\ p_{2,2}(t) = 4\xi(t),\ p_{2,n}(t) = 4\xi(t) p_{2,n-1}(t) - p_{2,n-2}(t),\ n \geq 3.$$

By choosing $t_0 = -1$, $t_1 = 1$ and $k = 1$ the extended formulation in (4–5) reduces to the classical Vieta–Pell polynomials defined on the standard interval $[-1, 1]$. In particular, the basis employed in the present study corresponds to this special configuration, which is selected for its computational simplicity and numerical stability.

3 Operational Differentiation Matrix for ECVPPs

Let $p_{k,n}(t)$ be the new extended complete Vieta–Pell polynomials defined on the interval $[t_0, t_1]$ and define the basis vector.

$$P(t) = \left[p_{k,1}(t), p_{k,2}(t), \ldots, p_{k,n}(t)\right], \tag{6}$$

The operational differentiation matrix $D \in R^{n \times n}$ is defined such that

$$\frac{d}{dt} p(t) = p(t) D, \tag{7}$$

To formulate the explicit form of the matrix D, let $D = \left[d_{ij}\right]$, $i, j = 1,2, \ldots, n$. The entries of D the operational differentiation matrix D are given by

$$d_{ij} = \{\frac{2^{k+1}}{t_1 - t_0} i,\ \textit{if } j > i \textit{ and } (j - i) \textit{ is odd}, 0, \quad \textit{otherwise}. \tag{8}$$

The matrix D is an upper triangular matrix. Nonzero entries appear only at positions where the index difference $(j - i)$ is odd. The magnitude of each nonzero entry is proportional to the scaling factor $\frac{2^{k+1}}{t_1 - t_0}$

Hence, the general pattern of D can be written as

$$D = \frac{2^{k+1}}{t_1 - t_0} \begin{bmatrix} 0 & 1\,0\,3\,0\,5\ldots \\ 0 & 0\,2\,0\,0\,0\ldots \\ 0 & 0\,0\,3\,0\,5\ldots \\ 0 & 0\,0\,0\,4\,0\ldots \\ 0 & 0\,0\,0\,0\,5\ldots \\ \cdots & \end{bmatrix} \tag{9}$$

The D structure illustrates that the derivative of each ECVP polynomial can be written exactly as a finite linear combination of lower-order ECVP polynomials.

4 The Product Operation ECVP Polynomials

The product of two ECVP basis functions can always be written as a linear combination of the same basis. Accordingly, the product of two ECVP basis functions can be represented in the following form

$$P_{k,n}(t)P_{k,m}(t) = p_{k,n+m-1}(t) + p_{k,n+m-3}(t) + \cdots + p_{k,|n+m|+1}(t).$$

where the number of terms in the above expansion is equal to $min(n, m)$. Such property is referred to as a linearization rule. It plays an important role in spectral and EGU-based numerical methods, as it allows the multiplication of Basis functions to be replaced by a summation of known coefficients, thereby simplifying the computational structure of the problem.

Theorem1. Let $p_{k,n}(t), n \geq 1$ be the new extended complete Vieta–Pell polynomials defined by

$$p_{k,1}(t) = 1, p_{k,2}(t) = z(t),$$

and

$$p_{k,n}(t) = z(t)p_{k,n-1}(t) - p_{k,n-2}(t), n \geq 3. \tag{10}$$

where $Z(t) = 2^k\xi(t)$. For any integers $n, m \geq 1$ we nave the following product formula

$$p_{k,n}(t)p_{k,m}(t) = \sum\nolimits_{r=0}^{(n,m)-1} p_{k,n+m-1-2r}(t). \tag{11}$$

Proof. Since the recurrence depends only on the single variable z(t), each $p_{k,n}(t)$ can be viewed as a polynomial in z. Define $Q_n(z)$, by

$$Q_1(Z) = 1, Q_1(Z) = Z, Q_n(Z) = ZQ_{n-1}(z) - Q_{n-2}(z), \text{ so that } p_{k,n}(t) = Q_n(z(t))$$

For the classical complete Vieta-Pell family $\left[Q_n(z)\right]$, the well-known linearization identity holds:

$$Q_{k,n}(z)Q_{k,m}(z) = \sum\nolimits_{r=0}^{(n,m)-1} Q_{k,n+m-1-2r}(z).$$

5 The Complete Vieta–Pell Version Algorithm for Solving the Optimal Control Problem

Consider an optimal control problem (1–3) and assume the system dynamics (2) can be written in a form that allows expressing the control as a function of the state and its first derivative:

$$u(t) = G(x(t), \dot{x}(t)). \tag{12}$$

The aim is to approximate the optimal trajectories $x^*(t)$, $u^*(t)$ and the optimal cost J^*. For a fixed truncation order n, define the basis vector as in Eq. (6).

Step 1. Approximation of the State Variable
Approximate the state variable $x(t)$ by a finite NECVPP expansion:

$$x(t) \approx a^t P(t). \tag{13}$$

where $a = [a_1, a_2, \ldots, a_{n+1}]^T \in R^{(n+1)\times 1}$ represents the unknown parameter vector.

Step 2. State Derivative Via the Operational Differentiation Matrix
Differentiate the approximation in (1) to obtain.

$$\dot{x}(t) \approx a^T \dot{P}(t). \tag{14}$$

Define the operational differentiation matrix D associated with the NECVPP basis such that the derivative of the basis vector is represented in the same basis:
$\dot{P}(t) = P(t)D$.
Therefore, Eq. (14) can be written as

$$\dot{x}(t) \approx a^T P(t)D. \tag{15}$$

Step 3. Reconstruction of the Control Variable
Using Eq. (12) and substitute the approximations derived in Steps 1–2, yields

$$u(t) \approx G(a^T P(t), a^T P(t)D \tag{16}$$

Hence, $u(t)$ becomes an explicit function of the unknown coefficient vector a.

Step 4. Reduction of the Performance Index
Substitute the approximations in Eqns. (13) and (16) into the cost functional in (1) to obtain.

$$J(a) = \int_{t_0}^{t_1} F(a^T P(t), G(a^T P(t), a^T P(t)D), t)dt.$$

In many OCPs, the integrand $F(\cdot)$ is quadratic in x and u. In this case, $J(a)$ reduces to a quadratic form in a:

$$J(a) = a^T Ha + 2c^T a + J0,$$

where $H \in R^{\{(N+1)\times(N+1)\}}$ is symmetric, $c \in R^{\{(N+1)\times 1\}}$, and $J0$ is a constant. If no linear term is produced, then $c = 0$ and

$$J(a) = a^T Ha.$$

Note that, the entries of H (and c, if present) are computed by evaluating the required integrals involving $P(t)$ and $P(t)D$ over $[t_0, t_1]$.

Step 5. Boundary Conditions as Linear Constraints
Impose the boundary conditions in (3) on (13), yields

$$a^T P(t_0) = x_a, a^T P(t_1) = x_b$$

These constraints can be written in matrix form as $Fa = b$, where

$$[F = P(t_0)^T; P(t_1)^T] \in R^{\{2\times(N+1)\}}, b = [x_a, x_b]^T \in R^{\{2\times 1\}}$$

Step 6. Final Quadratic Programming Form
The OCP is transformed into the algebraic constrained optimization problem minimize

$$J(a) = a^T Ha + 2c^T a + J0, \text{ subject to } Fa = b.$$

This is a quadratic programming problem with linear equality constraints. It may be solved by standard QP techniques or by the Lagrange multiplier method.

Step 7. Closed-Form Coefficients
For the common case $J(a) = a^T Ha$ $(c = 0)$
with equality constraints only, the optimal coefficients admit the closed form

$$a^* = H^{-1}F^T(FH^{-1}F^T)^{-1}b \tag{17}$$

If a linear term exists $J(a) = a^T Ha + 2c^T a$,
then the constrained minimizer satisfies

$$a^* = H^{-1}(F^T\lambda - c), \text{ with } (F = H^{-1}F^T)\lambda = b + FH^{-1}c$$

Step 8. Reconstruction of the Approximate Optimal Solution After computing a^*, reconstruct

$$x^*(t) \approx (a^*)^T P(t), \dot{x}^*(t) \approx (a^*)^T P(t)D \text{ and}$$
$$u^*(t) \approx G((a^*)^T P(t), (a^*)^T P(t)D)$$

Finally, evaluate the approximate optimal cost as

$$J^* \approx J(a^*).$$

6 Applications

The effectiveness of the suggested method is illustrated throughout this section by solving some applications on OCPs. All computations in this section have been obtained using Mat lab and python. The accuracy of the adopted technique is discussed by computing the absolute error values different values of n depending on the following formulas.

$$\epsilon_x = |x(t_i) - x_n(t_i)|, \epsilon_u = |u(t_i) - u_n(t_i)|, E_{n,state} = |x(t_i) - x_n(t_i)|,$$
$$E_{n,control} = |u(t_i) - u_n(t_i)|, E_J = |J - J_n|, E_{L1,x} = \sum_{i=1}^{n} |x(t_i) - x_n(t)|\Delta t,$$
$$E_{L1,u} = \sum_{i=1}^{n} |u(t_i) - u_n(t_i)|\Delta t, E_{L2,x} = (\sum_{i=1}^{n} |f(x_i) - f_n(x_i)|^2 \Delta t)^{\frac{1}{2}},$$
$$E_{L2,u} = (\sum_{i=1}^{n} |f(u_i) - f_n(u_i)|^2 \Delta t)^{\frac{1}{2}}.$$

where $x(t_i)$, $u(t_i)$ and J stand for the exact solutions $x(t)$, $u(t)$ and the cost function value while and $x_n(t_i)$, $u_n(t_i)$ and J_n indicate the approximate solutions to $x(t)$, $u(t)$,and the cost function value respectively.

6.1 Application 1. Optimal Control System for Smart Grid Network Security

With the increasing prevalence of cyber threats, there is a growing need to design intelligent protection systems that balance target security levels with operational costs. This research presents an optimal control model for distributing security investments in smart grid networks, aiming to reduce total costs while maintaining an acceptable security level. We assume that the security state of the network $x(t)$ is described by the following dynamic equation:

$$\dot{x}(t) = -0.4x(t) + u(t), \tag{18}$$

where $x(t)$ represents the security level of the network, taking the value 1 when the network is completely secure, and 0 when it is completely insecure. Here, $u(t)$ represents the security investment (cost of protection system and monitoring). We seek to minimize the total cost J, which combines both security risk resulting from reduced protection levels and the cost of protection systems and monitoring

$$J = \int_0^1 \Big(x^2(t) + u^2(t)\Big)dt, 0 \le t \le 1 \tag{19}$$

with initial condition $x(0) = 1$ (the network is completely secure at the beginning) and the final condition $x(1) = 0.5$ (we accept a lower security level at the end to reduce costs).

The OCP in Application 1 explores the critical trade-off between operational security integrity and expenditure management, where the calculated value is $J = 0.5696851572$. Tables 1, 2, 3 and 4 give the impact of the performance for Application 1. The optimal performance value is obtained at $n = 7$, where the error metrics values tends to minimum values.

Table 1. The $E_{n,\ state}$ values with n = 5, 6 and 7 for Application 1.

t	$E_{5,state} \times 10^{-4}$	$E_{6,statel} \times 10^{-5}$	$E_{7,staete} \times 10^{-7}$
0	0.00	0.00	0.00
0.2	2.42	0.521	9.16
0.4	3.56	1.20	5.06
0.6	3.31	1.30	4.54
0.8	2.50	0.341	8.75
1	0.00	0.00	0.00

Table 2. The $E_{n,\ control}$ values with n = 5, 6 and 7 for Application 1.

t	$E_{5,control} \times 10^{-3}$	$E_{6,control} \times 10^{-4}$	$E_{7,control} \times 10^{-5}$
0	7.92	4.24	3.54
0.2	2.71	1.65	0,539
0.4	2.12	0,945	1.05
0.6	2.02	0.837	1.03
0.8	2.59	1.51	0,506
1	7.14	3.55	3.31

Table 3. The max and mean values for Application 1.

Metric	$N = 4 \times 10^{-3}$	$N = 5 \times 10^{-4}$	$N = 6 \times 10^{-5}$
Mean error $M\epsilon_x$	0.196	0.05.61	0.00458
Max error ϵ_x	0.356	0.130	0.00916
Mean error $M\epsilon_u$	4.08	2.12	1.66
Max error ϵ_u	7.92	4.24	3.54

Table 4. The Approximate J and E_J values for Application 1.

n	J	$E_J \times 10^{-6}$
4	0.5696931846	8.0274
5	0.5696851740	0.00168
6	0.5696851573	0.00001

Figures 1, 2 and 3 graph the approximate solutions for state and control using various orders of ECVP, the errors values and the evolution of error metrics with increasing basis function order, respectively. They clearly illustrate a consistent improvement as the order of the basis function increases in the mean error and related indicators. Further, the data highlights the convergence behavior of the system over given interval.

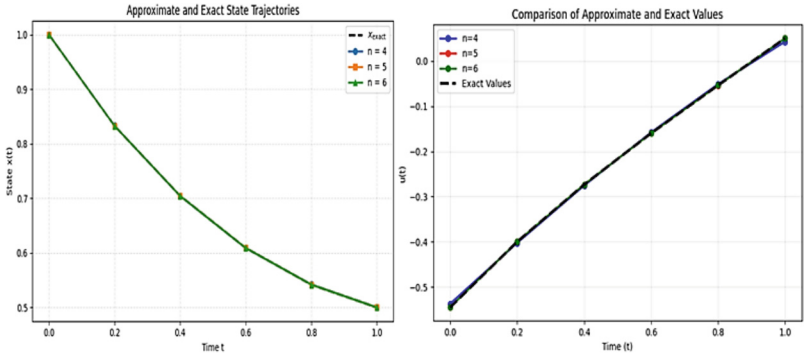

Fig. 1. Approximate solutions for state and control using various orders of ECVP for Application 1**.**

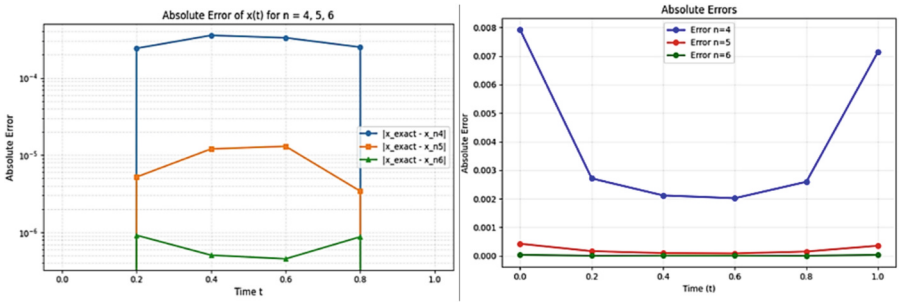

Fig. 2. Plot the errors values for Application 1 using different basis function orders

Note that the exact state trajectory is given by $x(t) = Ae^{st} + Be^{-st}$, $u(t) = (s + 0.4)Ae^{st} + (-s + 0.4)Be^{-st}$, where $A = 0.06141573559053107$, $B = 0.938584264409469$ and $s = 1.077032961426901$.

Overall, these results confirm the effectiveness of the proposed approach in producing accurate and stable approximate solutions for the optimal control problem under consideration.

6.2 Application. Solar Energy Systems

An office building equipped with a solar power system aims to optimally manage energy storage and consumption over a 24-h period. The system variables are the energy stored

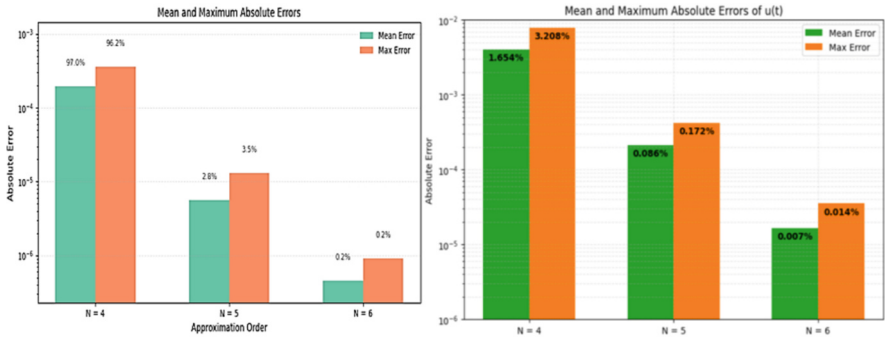

Fig. 3. Evolution of error metrics for Application 1 with increasing basis function order

in the batteries $x(t)$ and the applied charging/discharging power $u(t)$ while the solar cost function can be defined as below

$$Min\ j = \frac{1}{2}\int_{-1}^{1}\left(u^2(t) + 3x^2(t)\right)dt, \tag{20}$$

where $u^2(t)$ and $3x^2(t)$ represent the energy loss during conversion processes and the deviation from maximum power point tracking (MPPT) respectively. In addition, the solar system dynamics and solar boundary conditions are defined respectively as follow

$$u(t) = 2\dot{x}(t) - x(t) \tag{21}$$

$$x(-1) = 1, x(1) = 0.51314538 \tag{22}$$

Note that, at sunrise ($t = -1$) and at solar peak ($t = 1$), Eq. (20) represent the minimum energy stored at dawn and the optimal utilization of solar energy at peak hours respectively. To evaluate the precision of the approximation, the absolute errors for different values of n are presented in Tables 5 and 6, demonstrating the convergence behavior of the model. In addition, the performance evaluation of Application 2 is summarized in Tables 7 and 8, presenting the key metric values calculated over the interval $t \in [0,1]$.

The approximate solutions of the state and control variables are demonstrated in Fig. 4 which provides a comparison of the numerical approximation as the order of ECVP increases, leading to the convergence of the trajectories. Figures 5 and 6, respectively illustrate the approximate solutions for state and control using various orders of ECVP and the errors values and the evolution of error metrics with increasing basis function order**.**

The analytic solution for Application 1 is given as below

$x(t) = c_1e^t + c_2e^{-t}$, where $c_1 =$ 0.1415817270521744 and $c_2 =$ 0.3487184380397075 and the performance index value is equal to $J = 2.791658875$.

Table 5. The $E_{n,\ state}$ values with n = 5, 6 and 7 for Application 2.

t	$E_{5,state} \times 10^{-4}$	$E_{6,statel} \times 10^{-5}$	$E_{7,staete} \times 10^{-7}$
−1	0.0000000000	0.0000000000	0.0000000000
−0.6	0.7063790000	2.8826600000	7.6439999996
−0.2	1.1606690000	1.5799500000	9.3020000003
0.2	1.4578580000	1.3919300000	9.5000000000
0.6	0.1449210000	2.7319200000	7.4300000003
1	0.0000000000	0.0000000000	0.0000000000

Table 6. The $E_{n,\ control}$ values with n = 5, 6 and 7 for Application 2.

t	$E_{5,control} \times 10^{-3}$	$E_{6,control} \times 10^{-4}$	$E_{7,control} \times 10^{-5}$
−1	5.3259409000	1.1381079000	4.9236800000
−0.6	1.7699820000	1.2995950000	1.0188300000
−0.2	1.3706939000	3.0745330000	0.6369699999
0.2	0.4558060000	3.4370160000	0.0140000000
0.6	1.5234988000	2.0014630000	0.3874100000
1	3.1331186000	1.0547144000	3.4156700000

Table 7. Different Error values over t ∈ [0,1] for Application 2.

Metric	$n = 5(\times 10^{-3})$	$n = 6(\times 10^{-4})$	$n = 7(\times 10^{-5})$
$L_{1,x}$	0.184600820	0.34975760000	0.0982599999
$L_{2,x}$	0.153331079	0..8379040898	0.0836086246
$L_{\infty,x}$	0.177312600	0.32950600000	0.0950000000
$L_{1,u}$	3.5738867800	9.3117706000	3.4485520000
$L_{2,u}$	3.1976624223	8.0238865428	3.0702165957
$L_{\infty,u}$	5.3259409000	1.1381079000	4.9236800000

Table 8. The Approximate J and E_J values for Application 2**.**

n	J	E_J $(\times 10^{-6})$
5	2.7916620247	3.1497
6	2.7916600829	1.2079
7	2.7916599754	1.1004

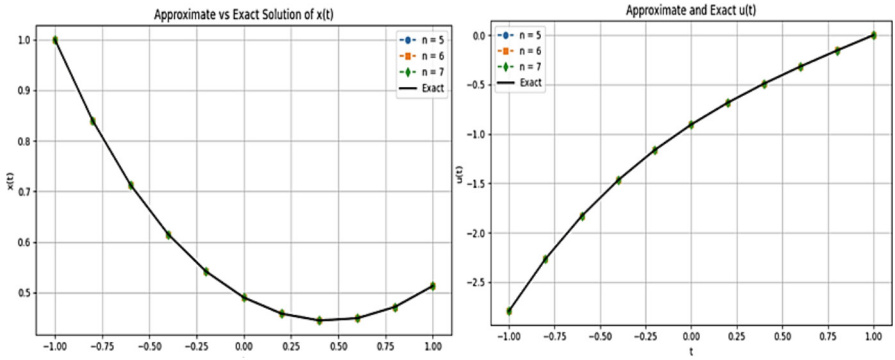

Fig. 4. Approximate solutions for state and control using various orders of ECVPPs for Application 2**.**

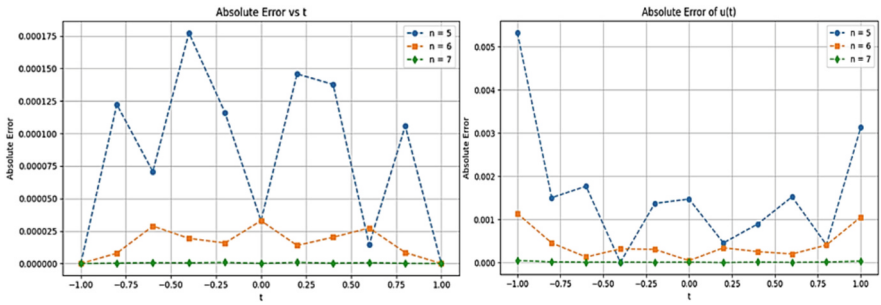

Fig. 5. Plot the error values for Application 2 using different basis function orders**.**

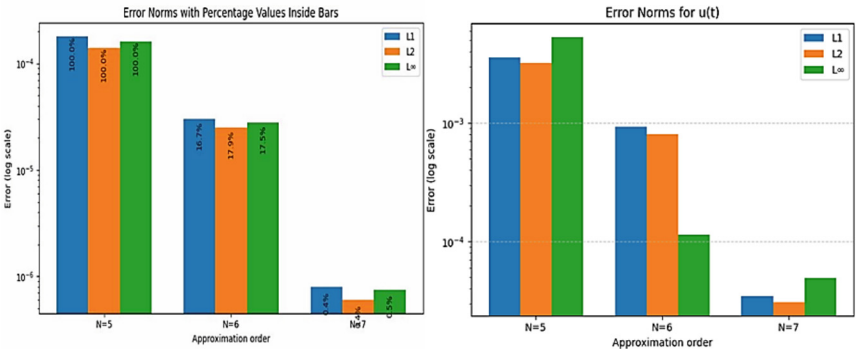

Fig. 6. Evolution of error metrics for Application 2 with increasing basis function order.

7 Discussion the Results

By analyzing the obtained results for the two applications including solar energy systems and network protection, many observations can be conclude concerning both the convergence and stability of the framework. The numerical approximation of the suggested algorithm illustrates a consistent correlation between the approximation order and the precision of the obtained solutions. One can see a clear development in the accuracy of the solution as the approximation order increases from $n = 4$ to $n = 7$ for the variables $x(t)$ and $u(t)$. In addition, the absolute error analysis demonstrates that the smaller errors near the boundary points of the time domain and within the interior region as the error values decrease by certain orders of magnitude when moving from $n = 4$ to $n = 7$. Furthermore, both the mean absolute error and the maximum absolute error exhibit a systematic reduction with increasing n. The convergence trend is also supported by the different error norms. For both the state and control variables, all norms decrease monotonically as the approximation order increases. The computed values of the performance index J show a steady convergence toward the exact value as n increases.

8 Conclusion

The definition of ECVP polynomials is constructed in this work and hen proposed an efficient numerical approximation technique for solving problems in optimal control. The extended polynomials enabled higher-order accuracy and better flexibility in representing the state and control variables over any time interval. This behavior was observed in the numerical results, where the error norms decreased as n increased. A quadratic programming approach was modified to solve the obtained optimization problem. By formulating the performance index as a quadratic functional and transforming the system dynamics together with the boundary conditions as linear equality constraints, the original OCP was transformed into a standard quadratic programming problem. The advantage of the presenting direct formulation avoids the complexity of indirect methods and eliminates the need to explicitly solve adjoint equations. The findings demonstrate that the proposed technique with ECVP polynomials is efficient, accurate and stable.

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