Analyzing Zipf's Law: Reflections on Power Law or Sigmoid Distribution

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Abstract. The Urban hierarchy has often been explained with Zipf's law, a particular form of the power law where the shape parameter tends to equal 1. There is a large literature on the applicability of Zipf's law to different spatial and historical contexts, which is generally valid but not optimal. In fact, there is a wide debate about whether the mathematical form expressed by Zipf's Law is the best distribution to represent urban hierarchies since this function seems to decline in its ability to represent the "fat tail" of the distribution curve. The objective of this study is to show there is a distribution function potentially better than Zipf that is based on a sigmoid function. This particular function represents the hierarchy between urban centers, even the smallest ones, in a much more accurate way. The use of this function overcomes the problems arising when adopting a Pareto function - which is valid only for the group of major cities - either a log-normal one or a combination of both. The sigmoid function has been applied to a sample of European countries, showing constant application validity.

Keywords: Rank-size Rule · Zipf's Law · Sigmoid · City-Size · Distribution

1 Exploring Rank-Size Distribution of Cities

1.1 Limits of Zipf's Rank-Size Law

The history of Zipf's law dates back to Felix Auerbach [1] who first observed that the Pareto distribution could fit well for cities sizes. This preliminary evidence of Auerbach, received success only later, thanks to George Kingsley [2], to whom the law also owes its name. Zipf postulated the size of cities is inversely proportional to their rank. From that moment, this type of regularity, which is found in many phenomena of different nature, took the name of "rank-size rule" or "power law". This relationship implies that, in a system of cities, the largest one is about twice the size of the second, about three times the size of the third, and so on (when the exponent of the power law is 1). Deviations from Zipf's law are considered as evidence of distortions in urban systems, such as aggregations, segregations and efficiency losses [3].

According to most scholars, one of the limits of Zipf's law is its static nature. The double logarithmic scale graphs illustrating the rank-size relationship represent instant

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photographs of an urban system at a precise time. However, from a historical perspective, it could be helpful to represent the temporal dynamics of distribution. Most scholars agree that the exponent of the power law tends to remain constant. An urban system, indeed, needs a long time to change its hierarchical structure significantly: in this sense, considering "instant" photos can have a more general value [4, 5].

A second limit is that the Zipf distribution correctly estimates the rank of a city through its population only for the top cities [6]; depending on the country's demographic extension - and thus on the number of cities there, this cut-off can range from 90% to 99.5% of all the other cities. In addition, it is widely demonstrated that the chosen cut-off heavily influences the value of the estimated parameter. Another crucial point is the level of "coherence" that city data must possess to observe Zipf's law. A surprising feature that characterizes "Zipfian" sets is that in general (as observed by [7], subsets or union of Zipfanian sets are no longer Zipfanian. Examples from cities in the US and EU provide strong evidence to support this view. While Zipf's law holds approximately for the size of the cities of each European country (France, Italy, Germany, Spain, etc.), it totally fails in the European Union as a whole. On the other hand, considering all the cities of each state of the Confederation, the distribution of US cities as a whole closely follows Zipf's law (as noted by Krugman, 1996), although the data set of the cities of each state (e.g., California, New York, Illinois, Massachusetts) does not generate a Zipf distribution. These two examples suggest a general property that could be defined as "coherence" or "integration," thus recalling a characteristic of "homogeneity", an expression of a shared history. This means that Zipf's Law must be strongly connected to the evolutionary history of the entire urban system. For Europe, the geographic level consistent with the data is the nation-state when an integrated evolution is observed, whilst in the United States, the entire Confederation represents the consistency, but not each independent state. The American urban system, in fact, has evolved collectively and organically, generating an ordered distribution of cities complying with Zipf's Law.

The scholars who more recently have addressed the question of the validity (and therefore usability) of Zipf's law [1, 3, 8, 9] have highlighted empirical evidence that Zipf's law is not always a good estimation function, even by limiting the study to the most populated cities of a country/region. The contribution of this paper consists in the proposal of a new distribution function which best approximates the urban hierarchy, overcoming the limits of the Zipf's law. Furthermore, in this study we only adopt the functional definition of cities, postponing the discussion for the case of "natural" cities, to further works.

1.2 Zipf's law applied to European Countries

To test Zipf's Law, the study conducts a comparative analysis across territorially coherent European countries, focusing on continental Europe. Data is sourced from Eurostat, covering census years from 1961 to 2021. The dataset includes all data available in the Eurostat database (30 countries), excluding only Cyprus due to missing data. Therefore, we did not select a specific group of countries, but we preferred to include all for the completeness of the analysis. Most countries provide census data at ten-year intervals, starting in 1961 and ending with the most recent census in 2010–2011. Although very small nations like Malta and Luxembourg cannot reliably support Zipf's Law testing due

to their limited number of cities, their data is still included for completeness. The limited urban structure in such countries poses statistical challenges for accurate estimations. The study acknowledges that the historical political landscape of Europe-particularly the existence of former unified entities like Austria-Hungary, which included modern states such as Austria, Hungary, Slovakia, and the Czech Republic could have influenced urban development and population distribution. However, these historical factors are noted but not analyzed within the scope of this paper.

The Zip's Law parameters are estimated at different truncations; in Table 1 we also show the standard deviation of residuals, and the arithmetic mean of residuals in absolute value[1]. Each column of Table 1 refers to the estimations with different truncation levels[2].

The countries in the table have been sorted into three groups: the first block concerns the top 5 nations comparable for the high number of cities (>8000); the second group of nations concerns states that are small either territorially or demographically but with a dense network of cities (between 2000 and 6000); the third group concerns small nations with few cities (from around 50 to 1000).

The analysis of Zipf's Law in urban systems reveals several recurring issues, particularly when applied to heterogeneous groups of cities. Previous studies [10] have shown that non-homogeneous distributions are common and can distort results, regardless of how strong the correlation is between observed and expected data. One key problem is the atypical behavior of the largest cities, which often do not conform to Zipf's Law. Another frequent issue arises with medium and small cities, where, beyond a certain population threshold, the distribution abruptly diverges from Zipf's expected trend. In such cases, only the largest urban agglomerations tend to align with Zipf's Law, forming a relatively homogeneous system where the distribution exponent is close to 1.

To address this, many studies apply a cutoff, including only the largest cities and excluding smaller ones, thus focusing on the homogeneous upper tail of the distribution. However, this practice raises questions about the general applicability of Zipf's Law across an entire national urban system. Another limitation lies in the assumption - common in City Size Distribution (CSD) studies - that cities within a given country represent a single coherent system. This disregards historical changes in national borders and administrative structures, which may have shaped city development in fundamentally different ways [11].

A further complication involves the concept of "primacy cities" - extremely dominant capital cities that cannot be statistically grouped with the rest of the urban system. Even in cases where Zipf's Law seems to apply, such outliers violate its core assumptions.

Despite these challenges, the analysis of data from 30 European countries reveals consistent patterns. One key finding is that the fit of Zipf's Law tends to worsen as smaller cities are included in the sample. This supports previous research [12, 13] indicating that Zipf's Law becomes less valid without data truncation. Without a cut-off, regression residuals show strong leftward asymmetry (asymmetry < 0) and high leptokurtosis (kurtosis > 0), meaning that the model systematically overestimates lower-ranked cities

[1] The lower the error standard deviation is the better the estimation is. The same holds for the residuals arithmetic mean in absolute value.

[2] All estimation has run through Gretl, an Italian software of econometric estimation.

Table 1. Zip's Law at different thresholds

Country No. Cities	tr-135	tr-500	tr-2000	No-threshold	Av. | Residuals|	Dev.St. Residuals
France (FR) - 36565	−1.4863	−1.4896	−1.2650	0.7125	0.2123	0.3051
R^2	(0.9837)	(0.9937)	(0.9909)	(0.9068)		
Italy (IT) - 8092	−1.3422	−1.4450	−1.3183	−0.6850	0.2758	0.3870
R^2	(0.9857)	(0.9930)	(0.9945)	(0.8494)		
Germany (DE) - 11255	−1.2324	−1.2831	−1.2765	−0.6113	0.2633	0.3755
R^2	(0.9918)	(0.9961)	(0.9978)	(0.8583)		
Spain (ES) - 8116	−1.3100	−1.17363	−0.9490	−0.5104	0.2232	0.3249
R^2	(0.9884)	(0.9936)	(0.9810)	(0.8938)		
U. Kingdom (GB) - 8923	−0.1609	−0.2059	−0.2347	−0.6217	0.2556	0.3244
R^2	(0.9343)	(0.9745)	(0.9755)	(0.7851)		
Austria (AT) - 2357	−1.1406	−1.3811	−1.3020	−1.0219	0.2522	0.3175
R^2	(0.9388)	(0.9684)	(0.9756)	(0.8783)		
Bulgaria (BG) - 4598	−0.9235	−0.9366	−1.0145	−0.5765	0.3168	0.4278
R^2	(0.9910)	(0.9951)	(0.9954)	(0.8153)		
Czech R. (CZ) - 6258	−1.4711	−1.4466	−0.9727	−0.7611	0.1571	0.2487
R^2	(0.9721)	(0.9904)	(0.9465)	(0.9377)		
Denmark (DK) - 2114	−4.6102	−2.2370	−0.7798	−0.7102	0.2769	0.4334
R^2	(0.9732)	(0.8714)	(0.8339)	(0.8090)		
Ireland (IE) - 3402	−2.4623	−1.9895	−1.2232	−0.938374	0.2029	0.3160
R^2	(0.9792)	(0.9775)	(0.9414)	(0.9026)		
Slovakia (SK) - 2926	−1.2101	−1.0598	−1.0599	−0.7832	0.2445	0.3316
R^2	(0.9101)	(0.9706)	(0.9877)	(0.8886)		
Swiss (CH) - 2495	−1.47113	−1.4466	−0.9467	−0.6863	0.2919	0.4088

(*continued*)

Table 1. (*continued*)

Country No. Cities	tr-135	tr-500	tr-2000	No-threshold	Av. | Residuals|	Dev.St. Residuals
R^2	(0.9721)	(0.9904)	(0.9362)	(0.8304)		
Hungary (HU) - 3176	−1.2242	−1.0172	−0.9492	−0.6730	0.2407	0.3365
R^2	(0.9345)	(0.9748)	(0.9872)	(0.8854)		
Poland (PL) - 2479	−1.2704	−1.3322	−1.3053	−1.1954	0.1152	0.1586
R^2	(0.9953)	(0.9971)	(0.9968)	(0.9744)		
Romania (RO) - 3181	−1.0200	−1.1087	−1.3658	−1.1957	0.1845	0.2520
R^2	(0.9719)	(0.9817)	(0.9741)	(0.9357)		

Note: The Average of Residuals in modulus and Standard Deviation are referred to the Residuals of OLS Zip's Law estimation run on the complete dataset of cities

and exhibits extreme variations at the distribution's tails. This challenges the power law's adequacy in modeling city rank versus population size.

Moreover, scatter plots comparing actual city ranks to those predicted by the Zipf model reveal its limitations. Only the top-ranked cities are estimated with reasonable accuracy, while lower-ranked cities are increasingly overestimated. Specifically, beyond the top 20% of cities, the model's accuracy declines sharply. Interestingly, when comparing the logarithm of estimated ranks with observed ranks, it becomes clear that the expected logarithmic trend of city ranks does not align with the mathematical behavior of the Zipf estimate. The log of the estimated rank follows a different pattern - closer to the inverse of a sigmoid function - further emphasizing the model's limitations in representing full urban systems.

2 The Sigmoid Distribution Function

2.1 Zipf and Sigmoid (Sig) Distributions

The sigmoid function is initially characterized by an increasing trend, approximating a Pareto function; after that, however, the para-exponential growth slows down until it stops at an asymptotic value. Zipf's Law is:

$$y_i = A \bullet x_i^{\beta} \tag{1}$$

A sigmoid function is of the type:

$$y_i = \frac{A}{1 + e^{\beta x}} \tag{2}$$

Table 2. Zip's Law at different thresholds

Country No. Cities	tr-135	tr-500	No-threshold	AV. |Residuals|	Dev.St. Residuals
Belgium (BE) - 589	−1.6348	−1.3727	−1.0421	0.2651	0.3780
R^2	(0.9938)	(0.9667)	(0.8508)		
Croatia (HR) - 556	−1.1649	−1.0954	−0.9736	0.1610	0.2338
R^2	(0.9883)	(0.9882)	(0.9428)		
Estonia (EE) - 226	−1.0088	−0.8398	-	0.1648	0.2784
R^2	(0.9981)	(0.9156)	-		
Finland (FI) - 336	−1.0995	−0.7566	-	0.2130	0.3162
R^2	(0.9926)	(0.8933)	-		
Greece (EL) - 1034	−1.4321	−1.1119	−0.6917	0.2808	0.4127
R^2	(0.9601)	(0.9749)	(0.8248)		
Latvia (LV) - 119	−0.9197	-	-	0.1681	0.2304
R^2	(0.9394)	-	-		
Lithuania (LT) - 60	−0.9847	-	-	0.2815	0.3621
R^2	(0.8385)	-	-		
Luxemburg (LU) - 106	−1.0378	-	-	0.1409	0.1736
R^2	(0.9652)	-	-		
Malta (MT) - 68	−0.8366	-	-	0.3327	0.4707
R^2	(0.7313)	-	-		
Netherlands (NL) - 415	−1.5578	−1.0903	-	0.2173	0.3379
R^2	(0.9911)	(0.8793)	-		
Norway (NO) - 429	−1.2497	−0.7976	-	0.2219	0.2972
R^2	(0.9899)	(0.9067)	-		
Portugal (PT) - 309	−1.2176	−0.8029	-	0.2005	0.3120
R^2	(0.9526)	(0.8957)	-		
Slovenia (SI) - 58	0.3783	-	-	0.1109	0.1692

(*continued*)

Table 2. (*continued*)

Country No. Cities	tr-135	tr-500	No-threshold	AV. |Residuals|	Dev.St. Residuals
R^2	(0.9646)	-	-		
Sweden (SE) -290	−1.3342	−0.9853	-	0.1854	0.2603
R^2	(0.9779)	(0.9272)	-		
Turkey (TR) - 957	−0.4616	−0.8532	−1.1885	0.3496	0.4208
R^2	(0.8870)	(0.9083)	(0.8677)		

Note. The Average of Residuals in modulus and Standard Deviation are referred to the Residuals of OLS Zip's Law estimation run on the complete dataset of cities

In a sigmoid function $\beta < 0$; in Zipf's function $\beta > 0$. As x increases in (6), the mathematical function tends to the asymptotic value A, vice versa, in Zipf's law, when x increases, the function grows to infinity. The asymptotic point A in the sigmoid function represents the total number of cities (the city with the highest rank is the last in the ranking in order of population), x is the population while the dependent variable is the rank (for a comparison see: [14]). Rewriting (2) with the aforementioned variables:

$$R_i = \frac{R_{max}}{1 + e^{a+bln(POP_1)}} \tag{3}$$

From the empirical analysis, it seems that the population depends on the rank according to a function that is actually the inverse of the sigmoid; therefore, to correctly estimate (7), we must solve (7) for POP. Through the application of logarithms in a natural base, we obtain the following linearized function:

$$\ln(POP_i) = \alpha + \beta \ln\left(\frac{R_{max}}{R_i} - 1\right) \tag{4}$$

where $\alpha = -\frac{a}{b}$ and $\beta = \frac{1}{b}$. Now, we note that by construction[3], the arithmetic mean of $ln\left(\frac{R_{max}}{R} - 1\right) = 0$. If, in general, we use an OLS estimator to estimate the dependent variable y_i with a single regressor ($y_i = \alpha + \beta x_i$), the estimation parameter for "the constant model" is ($\alpha = \overline{y} - \beta\overline{x}$) [15]. Therefore, in this case where $y_i = \ln(POP_i)$ and $x_i = \ln\left(\frac{R_{max}}{R} - 1\right)$, the deduce that $\alpha = \overline{\ln\,(POP)} - \beta ln\left(\frac{R_{max}}{R_i} - 1\right)$, but, as just recalled, the average of $ln\left(\frac{R_{max}}{R} - 1\right) = 0$, so that:

$$\alpha = \overline{\ln\,(POP)} \tag{5}$$

[3] Firstly, by definition of logarithm we simplify the Eq. (4) as follow: $POP_i = e^{\overline{\ln\,(POP)}} \bullet \left(\frac{R_{max}-R_i}{R_i}\right)^{\beta}$, $e^{\overline{\ln\,(POP)}} = e^{\frac{1}{n}(POP_1)+\cdots+\frac{1}{n}\ln(POP_n)} = \sqrt[n]{\prod_{i=1}^{n} POP_i}$. This last finding is the geometric mean of population.

The parameter α is, therefore, depending on the magnitude of the population. More interesting, on the opposite, is the interpretation of β which does not depend on the population's size. β is totally independent of the country-size, therefore it can be used to compare different nations. The parameter β expresses the degree of population concentration among the cities of a nation. β varies in a range between $(0; +\infty)$. By construction, in fact, an increasing number (rank) is linked by a decreasing number (the population of the corresponding city); therefore, this inverse relationship forces the parameter b of Eq. (3) to be positive. By this way, a sigmoid curve decreases as the logarithm of the population increases, $b \in (0; +\infty)$. Since $\beta = \frac{1}{b}$ also $\beta \in [0; +\infty)$, varying in the opposite direction to b. If $\beta \to 0$, $\ln(POP_i) = \overline{\ln(POP)}\ \forall i$. This is the case of a perfect distribution of the population in each city; each urban agglomeration has the same number of inhabitants. Algebraically, this means that the sigmoid of Eq. (3) degenerates into a vertical line with abscissa $\overline{\ln(POP)}$. By difference, the opposite limiting case, i.e. when $\beta \to +\infty$ implies $b \to 0$ and $a \to 0$, i.e., observing Eq. (3), the sigmoid degenerates into a straight line horizontal of height $R_i = R_{max}$, i.e. the rank does not change when the cities population varies. This is the extreme case in which the entire population of a nation is concentrated in a single city, so the rank i cannot change because there is a single city. But there is also another way to interpret parameter β.

Rearranging the Eq. (4) we get:

$$POP_i = \overline{POP_g} \bullet (R_{max} - R_i)^{\beta} \bullet \left(\frac{1}{R_i}\right)^{\beta} \tag{6}$$

We can also rearrange the Zipf in this way:

$$POP_i = A \bullet \left(\frac{1}{R_i}\right)^{\beta} \tag{7}$$

Equations (6) and (7) are similar; they have a pre-multiplied constant (POPg and A); and exponent β, the rank of city i in the denominator. However, in the Sigmoid's Law there is the factor $(\mathrm{Rmax} - \mathrm{R})^{\beta}$ that varies according to R; this factor is quite constant for small values of R[4]. This means that, for the greatest cities of a country, whose rank is close to 1, the Zip's Law well approximates the Sig's Law. However, when the rank raises (i.e., medium-sized cities), the factor $(\mathrm{Rmax} - \mathrm{R})^{\beta}$ is no more close to its initial value so it reduces significantly the final outcome (POP). This factor corrects the Zip's Law that as already observed overestimates the medium-size and small cities. With this rearrangement, the meaning of β in the Sig's Law is similar to the meaning of β in the Zipf's law. If $\beta = 1$, also for the Sig's Law, the second city is approximately the half of the first one and so on. However when R increases, this rule, that remain valid for Zipf's Law, is no more true for the Sig's Law: the factor $(\mathrm{Rmax} - \mathrm{R})^{\beta}$, indeed, gains importance reducing the city-size faster than in the Zip's law. In this case the factor becomes simply $(\mathrm{Rmax} - \mathrm{R})$ since $\beta = 1$. If $\beta < 1$, it means that the city-size of the second city is less than the half of the first one, i.e. there is a less hierarchical distribution of population among cities. Moreover, the city population decreases slower by increasing the rank R

[4] Image a country like Italy with more than 8000 cities; with small R, 8000-R is however very close to 8000.

since the factor $(\text{Rmax} - \text{R})^{\beta}$ is smaller when β pass from 1 to < 1. Finally, if $\beta > 1$, the opposite is true: there is a distribution more hierarchic than in a power law, for the biggest cities. Meanwhile, the city population decreases faster by increasing the rank R since the factor $(\text{Rmax} - \text{R})^{\beta}$ is higher when β pass from 1 to > 1. To sum up, β provides information not only about the distribution of a country population among its cities, but also the rate at which cities decline as their rank increases.

2.2 The Rank-Size City Distribution: A Logistic Theoretical Interpretation

The variation in the natural landscape may usefully be regarded as random, and that this random variation could produce a sigmoid law. The logistic regression model of estimation for the relationship rank-urban population, has been initially thought looking to the data available (data- driven approach). However, there are significant theoretical reasons to support the hypothesis of a sigmoid distribution for the cities population. We assume two relevant hypotheses widely confirmed by the empirical literature:

Hp 1) the population growth of a city follows a logistic curve[5];

Hp 2) irregularity in resources distribution and in territory (plains, mountains, rivers, lakes, row materials, fertile lands, etc.) follows a Gaussian spatial-distribution.

As for Hp 1), Northam [16] demonstrated the increase in the urbanization level over time exhibits an S-shaped curve that can be formulated as a sigmoid function (also called known as model). For several years, the UN assumed that the urban-rural growth difference of countries follows a logistic path and estimated it based on the experience of numerous countries [17, 18]. According to this hypothesis the population growth for a generic city i would be:

$$POP_{it} = \frac{K_i}{1 + A \bullet e^{-rt}} \tag{8}$$

where POP_{it}, is the population of city i at time t; r is the population growth rate, assumed fixed for simplicity, A is a constant parameter, t is the number of year passed from the starting point and K_i is the carrying capacity of the environment to sustain humankind in a specific area. In any given country, there are a limited number of areas, denoted as K_i, that are potentially suitable for urbanization. Each area, based on its geographic features and natural resources, can support a city or a system of cities with up to K_i inhabitants. Initially, not all of these areas are populated; early settlers tend to choose the most favorable locations, leading to the concentration of populations in select areas. As rural populations grow - driven by technological advancement, social stratification, and wealth accumulation - some of the other potentially urbanized areas may eventually become inhabited. The population growth of cities is generally assumed to follow a logistic pattern under normal conditions. However, external disruptions such as wars, climate shifts, or plagues can halt or reverse growth, sometimes destroying cities entirely. Despite these short-term shocks, the analysis focuses on long-term urban development and distribution, as these exogenous events are ultimately neutralized over time.

[5] Near the upper asymptote threshold of the logistic curve, the population of a city reaches a state of equilibrium which remains stable until technological shocks occur. Also war or environmental (famines and plagues) shocks may shift the population city out of its logistic equilibrium.

As for Hp 2) Geostatistics literature widely employ Gaussian function to model the spatial distribution of natural resources. Geostatistics was born in the 1960s from the need to estimate reserves in mineral deposits [19]. If the continuous spatial phenomenon $\{z(u), u \in A\}$ is generated by the sum of a (not too large) number of independent sources $\{yk(u), u \in A\}, k = 1,\ldots, K$, with similar spatial distributions, then its spatial distribution can be modeled by a multivariate Gaussian RF model. Multivariate Gaussian models are highly effective and well-established in spatial analysis, with a strong record of successful applications [20]. In the classical Kriging model, which is widely used for spatial estimation, a normal distribution is assumed. Kriging is a regression technique that interpolates quantities in space by minimizing the mean square error, based on the assumption that similar values are geographically close to each other, as per Tobler's law. Several studies have applied Gaussian and plurigaussian models in resource estimation, including Betzhold and Roth [21], who used plurigaussian simulations for copper deposits in Chile, and Veliz [22], who studied geological uncertainty in copper deposits. Other applications include simulating uranium mineralization [23] and lithological resource evaluation [24]. Sadeghi [25] combined fractal theory and Gaussian simulation for mineral resource classification. Gaussian copulas, used in water resource estimation [26], also account for spatial variability in unsampled locations.

According to Hp 1) and 2), our thesis is that the final population distribution among cities is close to a Sigmoid distribution. The brief demonstration (logic) is the following.

1) At the starting point, in a generic country, there are K area that can be potentially urbanized.
2) Each area has its specific demographic carrying capacity K_i.
3) the carrying capacity Ki depends on natural resources distribution and on territories distribution.
4) Therefore, for Hp 2, the distribution of Ki follows a Gaussian distribution.
5) We start simulation with only 2 inhabitants per each area K.
6) We let cities population grow according to a logistic function, following the Hp 1).
7) The more time passes, the more the population grows unevenly between cities; those with small K tend to stabilize at low values while those with large K grow faster; however, cities stabilize at the same speed.
8) For t tends to infinity the distribution of population among cities reach the carrying capacity, therefore its distribution coincide to the distribution of K_i, i.e. it is Gaussian.
9) By ordering the observations of a Normal random variable from smallest to largest, we obtain the cumulative distribution function.
10) The cumulative distribution function of a Gaussian density function is very close to a logistic function.

In conclusion, the theoretical interpretation of the sigmoid distribution for population among cities is the following: Single cities grow according to a logistic curve that reflects natural demographic dynamics, but each city has its own carrying capacity of inhabitants which depends on external conditions (natural resources and type of territory on which city lies). The logistical distribution of cities therefore reflects the random (Normal) distribution of resources and the random type of environments and territories that exist within the same country.

Human activity can influence the "natural" pathways of territorial development in two key ways.

Technological progress can increase a territory's carrying capacity (K_i) by enhancing productivity, allowing populations to exceed previous limits. However, this doesn't alter the distribution of Ki, as innovations spread evenly across a country or region, benefiting all cities proportionally.

Human activity also affects population concentration within cities. While the fundamental distribution remains unchanged, the degree of urban concentration can shift. Institutions play a role in this by influencing how resources are distributed. For example, unilateral transfers, such as those from natural resource rents, can concentrate wealth in specific regions. Additionally, governmental transfers, like taxes, are often directed to capital cities from border areas, where public services are more concentrated. These persistent transfers lead to ongoing population concentration in a few cities. The spatial allocation of natural resource rents depends on property rights, not just resource distribution [27].

2.3 Comparison Between Zipf's Law and the Sigmoid Function

This section shows the application of the Sigmoid function to the same data previously used to study Zipf's Law. The Eq. (4) is used to estimate the city population, getting the parameters α and β. Together with the parameters in Tables 3 and 4, we report some fundamental statistics that measure the goodness of the estimates: the R^2 index, the arithmetic mean of the absolute value of the residuals and the standard deviation of residuals. By comparing these three variability indices with those found in Tables 1 and 2, it is possible to appreciate the improvement to which the estimation model proposed in this paper leads.

3 A Statistical Comparison Between Zipf's Law and Sig's Law

The equations estimated without truncation, have much lower residuals standard deviations, for all countries, compared to the Zip's law. The comparison between the Zipf's model and the Sigmoid model (from now, for convenience, we call it Sig's Law) can be better appreciated through a quick graphical inspection of the Fig. 1 proposed below. The graphs are a scatter plot between the statistical index R2 referred to as both Zip's Law and Sig's Law estimates. The bisector of the 1st and 3th quadrant is drawn to help the visual inspection of the outcomes; on the abscissas, there is the R2 referred to the Zip's Law; on the ordinate there are the R2 values for the estimates run with Sig's Law. In this way, when a point of the scatter plot (representing a country) lies above the bisector line, it means that the R2 is greater when we use the sigmoid estimate; vice versa, if a point of the scatter plot lies below the bisector Zipf's Law estimates fit better.

Figure 1.A compares the goodness of fit index (R2) for the Zips' Law and the Sig's Law regression without setting any threshold. What graphical inspection suggests is a noticeable overall improvement in estimates when applying Sig's Law. This new estimation method obtains precise estimates ($0.95 < R^2 < 1$) even when no threshold is set. On the opposite the R^2 indices of the Zip's Law are between 0.8–0.9. Almost every

Table 3. Sig's Law at different thresholds

Country & No. Cities	tr-135 β	tr-500 β	tr-2000 β	No-treshold β	Av. |Residuals|	St.Dev. Res
France - 36565	0.6612	0.6646	0.7728	0.7318	0.1255	0.1769
R^2	(0.9837)	(0.9938)	(0.9922)	(0.9825)		
Italy - 8092	0.7309	0.6759	0.7052	0.7389	0.0670	0.1057
R^2	(0.9853)	(0.9927)	(0.9979)	(0.9938)		
Germany - 11255	0.8022	0.7672	0.7452	0.8476	0.1028	0.1257
R^2	(0.9920)	(0.9957)	(0.9984)	(0.9931)		
Spain - 8116	0.7512	0.8332	0.9694	1.0045	0.2395	0.3243
R^2	(0.9886)	(0.9944)	(0.9901)	(0.9692)		
UK - 8923	0.1603	0.2030	0.2216	0.3865	0.0812	0.1202
R^2	(0.9352)	(0.9745)	(0.9837)	(0.9704)		
Austria - 2357	0.8085	0.6582	0.5256	0.5000	0.0500	0.1102
R^2	(0.9356)	(0.9616)	(0.9798)	(0.9853)		
Bulgaria - 4598	1.0647	1.0312	0.8559	0.8836	0.0784	0.1110
R^2	(0.9914)	(0.9938)	(0.9871)	(0.9946)		
Czech R. - 6258	0.6566	0.6704	0.8970	0.6916	0.2177	0.2710
R^2	(0.9714)	(0.9912)	(0.9675)	(0.9543)		
Denmark - 2114	0.2076	0.3692	0.7236	0.6886	0.1862	0.2438
R^2	(0.9756)	(0.8954)	(0.9627)	(0.9623)		
Ireland - 3402	0.4419	0.5048	0.6636	0.5562	0.1502	0.1953
R^2	(0.9871)	(0.9907)	(0.9846)	(0.9636)		
Slovakia - 2926	0.7439	0.8744	0.8395	0.7278	0.0777	0.1414
R^2	(0.9137)	(0.9728)	(0.9843)	(0.9894)		
Swiss - 2495	0.6503	0.6478	0.7414	0.7379	0.0527	0.0744
R^2	(0.9703)	(0.9918)	(0.9954)	(0.9968)		
Hungary - 3176	0.7554	0.9198	0.8395	0.7655	0.0866	0.1138
R^2	(0.9371)	(0.9795)	(0.9928)	(0.9894)		
Poland - 2479	0.7716	0.7061	0.5465	0.4372	0.1489	0,2230
R^2	(0.9952)	(0.9939)	(0.9746)	(0.9259)		
Romania - 3181	0.9423	0.8459	0.5622	0.4286	0.1253	0.2183
R^2	(0.9732)	(0.9772)	(0.9334)	(0.9263)		

(*continued*)

Table 3. (*continued*)

Country & No. Cities	tr-135 β	tr-500 β	tr-2000 β	No-treshold β	Av. |Residuals|	St.Dev. Res
Belgium - 589	0.5666	0.4967	0.4817	0.0404	0.1064	
R^2	(0.9902)	(0.9947)	(0.9849)			
Croatia - 556	0.7872	0.5916	0.5357	0.1342	0.1953	
R^2	(0.9847)	(0.9692)	(0.9599)			
Estonia - 226	0.7821	0.6054	-	0.1631	0.2493	
R^2	(0.9760)	(0.9479)	-			
Finland - 336	0.7946	0.6797	-	0.0985	0.1240	
R^2	(0.9968)	(0.9894)	-			
Greece - 1034	0.6473	0.7558	0.7193	0.0745	0.0936	
R^2	(0.9649)	(0.9885)	(0.9945)			
Latvia - 119	0.5555	-	-	0.1530	0.2378	
R^2	(0.9418)	-	-			
Lithuania - 60	0.4844	-	-	0.1572	0.2162	
R^2	(0.9334)	-	-			
Luxemburg - 106	0.4976	-	-	0.1367	0.2181	
R^2	(0.9386)	-	-			
Malta - 68	0.5417	-	-	0.1277	0.1789	
R^2	(0.9629)	-	-			
Netherlands - 415	0.5747	0.4663	-	0.0753	0.1170	
R^2	(0.9903)	(0.9805)	-			
Norway - 429	0.7191	0.6477	-	0.1137	0.1587	
R^2	(0.9918)	(0.9814)	-			
Portugal - 309	0.6858	0.6353	-	0.1521	0.1910	
R^2	(0.9769)	(0.9719)	-			
Slovenia - 58	0.3783	-	-	0.1525	0.2175	
R^2	(0.8938)	-	-			
Sweden - 290	0.6306	0.5261	-	0.1170	0.1632	
R^2	(0.9878)	(0.9700)	-			
Turkey - 957	0.4463	0.7356	0.6902	0.1602	0.2168	
R^2	(0.8999)	(0.9515)	(0.9705)			

Note. The Average of Residuals in modulus and Standard Deviation are referred to the Residuals of OLS Sig's Law estimation run on the complete dataset of cities

country lies above the bisector, and almost all the dots are aligned to the horizontal line close to 1, i.e. Sig's Law improves the estimates, reaching a high level of accuracy. The concrete advantage is that these outcomes without the necessity to insert any threshold.

In Fig. 1.B the R^2 estimates using Zipf's and Sig's Law were compared, applying the threshold to the first 135 cities, as a large part of the literature suggested. In this case, two facts stand out. The first is that both models estimate rank with high accuracy: $0.94 < R^2 < 1$. The second empirical evidence is that the points of the scatter plot lie with great precision on the bisector line, that is, for very narrow thresholds, the Zipf's Law and the Sig's Law are equivalent. This is also consistent with the model algebra used in this work. In fact, mathematically, the sigmoid function assumes a para-exponential trend before slowing down its growth stabilizing close to its asymptote.

In Fig. 1.C the goodness of fit of both models was compared with a broader threshold, grouping the top 500 cities. Almost all the points of the scatter plot lie above the bisector. Therefore, the situation is similar to Fig. 1.A. Sig's Law replicates the observed trend better than Zip's Law, already slightly expanding the range of cities considered (from 135 to 500). In addition, it was observed that Sig's Law increases the goodness of fit of estimations for all countries up to a value of $0.95 < R2 < 0.99$ except for one country. The improvement is even more evident from Fig. 1.D, when the threshold bar is raised to the top 2000 cities6. Also in this case the Sigs' Law is associated with goodness of fit indices that are significantly better than the Zip's Law, passing from $R^2 = 0.85$ to $R^2 = 0.98$.

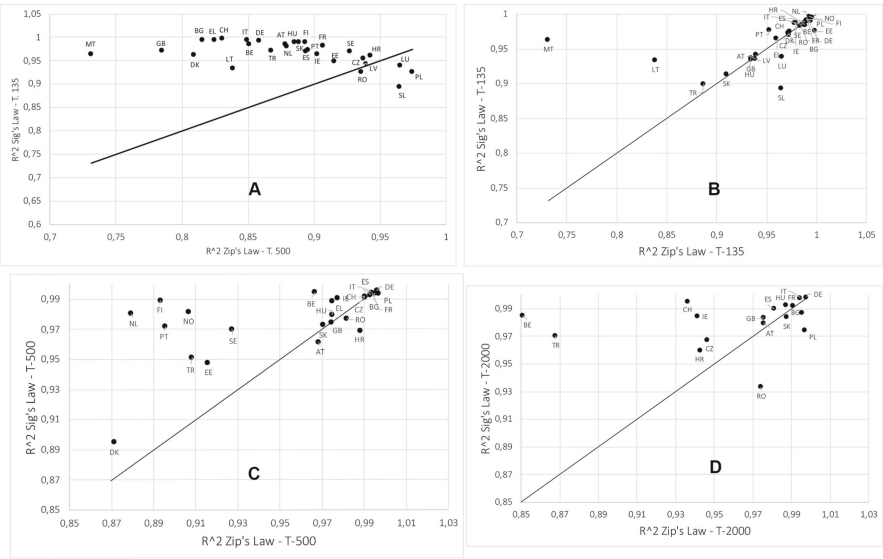

Fig. 1. Scatter Plot Dev.St. Residuals, Zip's Law vs Sig's Law (No Threshold).

Lastly, the researchers show that Sig's Law can also be employed to study how the population concentration among cities has changed over time, in the same country. To this aim, the 4 countries with the most cities according to available Eurostat data,

namely Italy, France, Germany and Spain, were considered. The data available refer to the decennial censuses conducted since 1961. The table shows only the value of the parameter β, always estimated through an OLS estimator, using always the Eq. (4).

Table 4. The population distribution among cities over Time 1961–2011 (β)[6]

	Italy	France	Germany	Spain
1961	0.5974 (0.0024)***	0.6225 (0.0013)***	0.8111 (0.0012)***	0.6859 (0.0034)***
1971	0.6411 (0.0025)***	0.6697 (0.0014)***	0.8353 (0.0011)***	0.7810 (0.0033)***
1981	0.6733 (0.0023)***	0.7057 (0.0014)***	0.8545 (0.0013)***	0.8831 (0.0033)***
1991	0.6933 (0.0022)***	0.7243 (0.0013)***	0.8588 (0.0016)***	0.9348 (0.0036)***
2001	0.7196 (0.0020)***	0.7295 (0.0013)***	0.8560 (0.0013)***	0.9536 (0.0042)***
2011	0.7389 (0.0019)***	0.7318 (0.0018)***	0.8672 (0.0014)***	1.0045 (0.0046)***
Observations	8092	36565	11755	8116

Table 4 highlights a steady rise in population inequality among cities, reflecting broader socio-economic changes since the 1960s. Industrialization and increased capital and labor productivity have led to population concentrating in fewer urban centers, where economies of scale are more effectively utilized. France and Italy show similar trends in both the value of the inequality parameter (β) and its evolution over time. In contrast, Germany and Spain demonstrate divergent patterns. Germany had a high level of urban population concentration as early as the 1960s, surpassing the levels that France and Italy would only reach decades later, though its inequality grew slowly over time. Spain, by comparison, began with a more balanced population distribution but experienced a sharp rise in inequality. By 1981, Spain had already surpassed Germany, and by 2011, it had reached one of the highest β values, indicating a significant shift toward urban population concentration.

[6] Notes (Table 4):

1. Significance Level: * 10%, ** 5% and ***1%
2. Standard errors in parentheses. Full tables are available from the authors upon request.
3. As for α coefficients, they are not reported for ease of exposition.

4 Conclusion

While Zipf's Law is commonly used to describe the hierarchical structure of urban systems, it fails to accurately represent the full distribution of cities, from the largest to the smallest. Instead, the sigmoid function provides a more precise model. Tested across 30 European countries with diverse urban histories and varying city sizes, the sigmoid function consistently outperformed Zipf's Law in statistical accuracy. This suggests the existence of a deeper, underlying regularity in the development of urban systems. Early city formation likely followed a hierarchical structure based on environmental factors such as climate, terrain, and resource availability. As urban areas expanded over time, they maintained this inherent hierarchy. Beyond this structural consistency, the organization of cities within each country has also been shaped by its unique economic and civil development. Thus, both historical foundations and modern socio-economic trajectories influence how urban systems evolve and distribute populations.

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