

Comparison of Different Estimators for the Rayleigh Gamma Gompertz's Parameters and Reliability Function

Nadia Hashim Al-Noor[1], Rafida M. Elobaid[2], and Suzan J. Obaiys[3](✉)

[1] Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq
nadialnoor@uomustansiriyah.edu.iq

[2] Faculty of Communication, Arts and Sciences, Canadian University Dubai, Dubai, United Arab Emirates
rafida@cud.ac.ae

[3] Department of Computer System and Technology, Faculty of Computer Science and Information Technology, University of Malaya, 50603 Kuala Lumpur, Malaysia
suzan@um.edu.my

Abstract. Estimating parameters and reliability features in various situations is a key challenge for statistical researchers in many disciplines. The prime goal of this paper is to investigate the flexibility of the newly compound three-parameter Rayleigh Gamma Gompertz for estimating unknown parameters and the reliability function under different estimation methods, sample sizes, and datasets. Simulation experiments are presented to investigate how the different estimators behave under various sample sizes and parameter values. The estimators' performance is measured using mean squared error. The empirical results demonstrated the outperforming, flexibility, and consistency of maximum likelihood and maximum product of spacing estimates to estimate the parameters and reliability function, with marked superiority of maximum likelihood for estimating the reliability function, confirming that this approach is still the most often used estimating technique due to its theoretical and practical features.

Keywords: Rayleigh Gamma Gompertz · Estimation Methods · Simulation

1 Introduction

Reliability is defined as a system's or component's ability to fulfill required functions under specified conditions over time [1]. Reliability theory looks at many facets of dependability, probability, statistics, and stochastic modeling to design and scientifically explain failure mechanisms using engineering concepts. Data from various fields, including engineering, biology, medicine, ecology, economics, sociology, and the social sciences, have been examined using reliability analysis [2–4]. In some contexts, reliability data analysis is also known as lifetime, failure-time, survival, or event-time data analysis [2]. In the examination of reliability data, reliability attributes are frequently specified using the mean time to failure, reliability function, and failure rate function.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2026
O. Gervasi et al. (Eds.): ICCSA 2025 Workshops, LNCS 15887, pp. 332–346, 2026.
https://doi.org/10.1007/978-3-031-97589-9_23

Accordingly, reliability theory is adopted to study and analyze the behavior of failures and the factors behind such failures. To achieve this, the parameter estimators of failure models or probability distributions, with good performance and efficiency, are required to estimate the reliability function. Therefore, it is of keen significance to study the methods and techniques for getting the parameter estimators.

One of the most widely used distributions is the one-scale parameter Rayleigh distribution (RD) introduced by Rayleigh [5]. The origins and properties of RD are investigated by Siddiqui [6], while Sinha and Howlader [7] looked at RD inferences. Reports on RD have been increasing since 1993. It is used in applied statistics, diagnostic imaging, physical sciences, clinical subjects, survival and reliability theory, and lifetime data modeling and analysis. Furthermore, this distribution has applications in a variety of domains, including astronomy [8], medicine [9], and environmental science [10]. Much research has utilized RD to evaluate wind speeds and wind potential at different sites throughout the world because of its capacity to precisely characterize wind regimes [11–13]. As a result, this distribution has acquired prominence as a model for a variety of situations. In actuality, models with monotone hazard rates can be well modeled using the RD, while those with non-monotone hazard rates cannot be effectively modeled using this distribution. Therefore, RD has been modified by several authors for use in situations with non-monotone hazard rates. Numerous noteworthy generalized versions related to RD have been introduced in the literature (see [14–23]), and they all share the trait of having more parameters. When additional parameters are included in an existing distribution, the distribution's tail properties improve, and its goodness of fit is enhanced. Furthermore, based on different strategies, the RD is employed as a generator (family) to introduce several modified, extended, and generalized versions of different distributions to improve their flexibility to deal with the different natures of data. Among them, Rayleigh Rayleigh [24, 25], Rayleigh Generalized Gamma [26], Rayleigh Gamma Gompertz [27], Rayleigh Weibull and Rayleigh Inverted Weibull [28], Rayleigh Gompertz [29], Rayleigh Lomax [30], Rayleigh Exponential Gamma [31], Rayleigh Exponential and Rayleigh Generalized Log-Logistic [32].

Motivated by the possibility of estimating the unknown parameters and reliability function, this paper aims to perform seven estimation methods that have not been implemented for the newly flexible compound three-parameter Rayleigh Gamma Gompertz. Simulation experiments are presented to investigate how the different estimators behave under various sample sizes and parameter values. The remainder of the paper is structured as follows. Section 2 briefly describes the essential functions of the considered distribution. Seven approaches for estimating the parameters and reliability function are described in Sect. 3. Simulation results and discussions are provided in Sect. 4. Section 5 contains some concluding remarks.

2 Newly Compound Rayleigh Gamma Gompertz

The cumulative distribution function (CDF) and the probability density function (PDF) of the Rayleigh family of distributions are defined by [27]

$$F(x; a, \varphi) = e^{-\frac{a}{2}(\ln G(x;\varphi))^2}; x > 0 \tag{1}$$

and

$$f(x;a,\varphi)=a\frac{g(x;\varphi)}{G(x;\varphi)}(-\ln G(x;\varphi))e^{-\frac{a}{2}(-\ln G(x;\varphi))^2};x>0 \tag{2}$$

where $a>0$, $g(x;\varphi)$ and $G(x;\varphi)$ are the PDF and CDF of the baseline distribution with vector parameters φ.

According to (1) and (2), with Gamma Gompertz as baseline distribution, the compound three-parameter Rayleigh Gamma Gompertz (RGGo) is defined by the following CDF and PDF

$$F(x)=F(x;a,b,c)=e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx}\right)\right)^2} \tag{3}$$

$$f(x)=f(x;a,b,c)=\frac{abce^{-bcx}}{1-e^{-bcx}}\left(-\ln\left(1-e^{-bcx}\right)\right)e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx}\right)\right)^2} \tag{4}$$

where $a,b>0$ are scale parameters and $c>0$ is a shape parameter.

The reliability and failure rate functions of the failure-time random variable X can be represented as

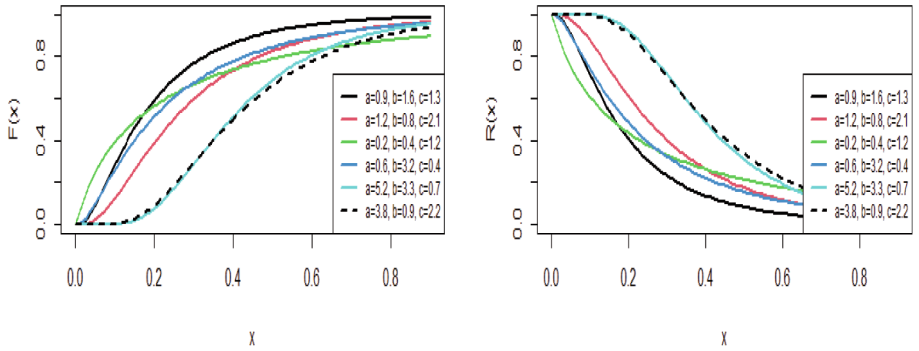
$$R(x)=1-F(x)=1-e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx}\right)\right)^2} \tag{5}$$

$$h(x)=\frac{f(x)}{R(x)}=\frac{abce^{-bcx}\left(-\ln\left(1-e^{-bcx}\right)\right)e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx}\right)\right)^2}}{\left(1-e^{-bcx}\right)\left(1-e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx}\right)\right)^2}\right)} \tag{6}$$

The quantile function can be obtained from the inverted CDF in (3) as

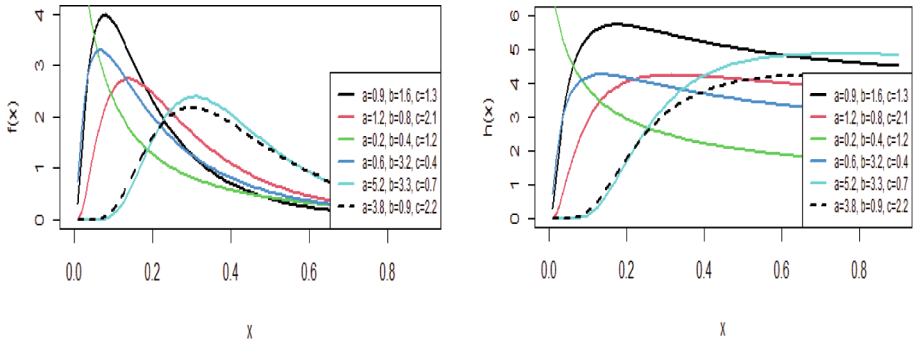
$$Q(u)=-\frac{1}{bc}\ln\left(1-e^{-\left(-\frac{2}{a}\ln(u)\right)^{\frac{1}{2}}}\right);u\in(0,1) \tag{7}$$

Some visualizations of the essential RGGo functions defined in (3)–(6) are shown in Figs. 1 and 2. The CDF and reliability function features are exhibited in Fig. 1. The PDF and failure rate function's various shapes are evident in Fig. 2, confirming that this novel distribution is highly suitable for simulating a variety of positive data.



(i) (ii)

Fig. 1. The *RGGo* (i) CDF (ii) Reliability function plots with different parameter values



(i) (ii)

Fig. 2. The *RGGo* (i) PDF (ii) failure rate function plots with different parameter values

3 Different Estimators

This section investigates the estimators of RGGo parameters $\Delta = (a, b, c)$ and the reliability function by different estimation methods. For the purposes of this paper, related to the failure-time random variable X, $(x_1, x_2, \ldots, x_n)$ denotes a random sample of size n that follows the RGGo distribution. Thus, $x_1, x_2, \ldots, x_n$ represent the data, and their ascending ordering values are denoted by $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. For more details on the considered methods, one can review [33, 34].

3.1 Maximum Likelihood (ML) Estimators

The ML estimators of the vector parameters Δ are given by $\widehat{\Delta}^{ML} = \underset{\Delta}{\text{argmax}}\, \mathrm{L}(\Delta)$ or, equivalently $\widehat{\Delta}^{ML} = \underset{\Delta}{\text{argmax}}\, \ell(\Delta)$ where $\mathrm{L}(\Delta) = \prod_{i=1}^{n} f(x_i; \Delta)$ and $\ell(\Delta) = \ln(\mathrm{L}(\Delta))$ represent the likelihood and natural log likelihood functions.

Now, related to (4), the L(Δ) and $\ell(\Delta)$ are

$$L(\Delta) = (abc)^n \prod_{i=1}^{n} \frac{e^{-bcx_i}}{1-e^{-bcx_i}} \left(-\ln\left(1-e^{-bcx_i}\right)\right) e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_i}\right)\right)^2}$$

$$\ell(\Delta) = n\ln(abc) - bc\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \ln\left(1-e^{-bcx_i}\right) - \sum_{i=1}^{n} \ln\left(\ln\left(1-e^{-bcx_i}\right)\right) - \frac{a}{2}\sum_{i=1}^{n}\left(\ln\left(1-e^{-bcx_i}\right)\right)^2 \tag{8}$$

The differentiability of (8) can offer the requested estimators of vector Δ $\left(\hat{a}^{ML}, \hat{b}^{ML}, \hat{c}^{ML}\right)$ by simultaneously solving the following three non-linear equations corresponding to $\partial\ell(\Delta)/\partial\Delta = 0$ with respect to Δ, where

$$\frac{\partial \ell(\Delta)}{\partial a} = \frac{n}{a} - \frac{1}{2}\sum_{i=1}^{n}\left(\ln\left(1-e^{-bcx_i}\right)\right)^2$$

$$\frac{\partial \ell(\Delta)}{\partial b} = \frac{n}{b} - c\sum_{i=1}^{n} x_i - c\sum_{i=1}^{n} \frac{xe^{-bcx_i}}{1-e^{-bcx_i}} - c\sum_{i=1}^{n} \frac{xe^{-bcx_i}}{\left(1-e^{-bcx_i}\right)\ln\left(1-e^{-bcx_i}\right)} - ac\sum_{i=1}^{n} \frac{xe^{-bcx_i}}{\left(1-e^{-bcx_i}\right)} \ln\left(1-e^{-bcx_i}\right)$$

$$\frac{\partial \ell(\Delta)}{\partial c} = \frac{n}{c} - b\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} \frac{xe^{-bcx_i}}{1-e^{-bcx_i}} - b\sum_{i=1}^{n} \frac{xe^{-bcx_i}}{\left(1-e^{-bcx_i}\right)\ln\left(1-e^{-bcx_i}\right)} - ab\sum_{i=1}^{n} \frac{xe^{-bcx_i}}{\left(1-e^{-bcx_i}\right)} \ln\left(1-e^{-bcx_i}\right)$$

In the sequel, the ML estimator of the reliability function at each observation of failure time $x_{(i)}; i = 1,2,\ldots,n$ can be gained based on (5) by employing the invariant property as

$$\hat{R}^{ML}\left(x_{(i)}\right) = 1 - e^{-\frac{\hat{a}^{ML}}{2}\left(\ln\left(1-e^{-\hat{b}^{ML}\hat{c}^{ML}x_{(i)}}\right)\right)^2} \tag{9}$$

3.2 Ordinary Least Squares (OLS) Estimators

The OLS estimators of Δ are given by $\hat{\Delta}^{OLS} = \underset{\Delta}{\operatorname{argmin}} OLS(\Delta)$, where.

$OLS(\Delta) = \sum_{i=1}^{n}\left[F\left(x_{(i)};\Delta\right) - \frac{i}{n+1}\right]^2$.

Related to RGGo distribution, recall (3), the OLS(Δ) is

$$OLS(\Delta) = \sum_{i=1}^{n}\left[e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2} - \frac{i}{n+1}\right]^2 \tag{10}$$

The OLS estimators $(\hat{a}^{OLS}, \hat{b}^{OLS}, \hat{c}^{OLS})$ can be obtained by solving simultaneously the three non-linear equations summarized as $\partial\mathrm{OLS}(\Delta)/\partial\Delta = 0$ with respect to Δ. Also, the estimators can be found by solving the following non-linear equation.

$$\sum_{i=1}^{n}\left[e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2}-\frac{i}{n+1}\right]K_k\left(x_{(i)};\Delta\right)=0 \text{ for } k=1,2,3, \text{ where}$$

$$K_1\left(x_{(i)};\Delta\right)=\frac{\partial F\left(x_{(i)};\Delta\right)}{\partial a}=-\frac{1}{2}\sum_{i=1}^{n}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2 e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2} \tag{11}$$

$$K_2\left(x_{(i)};\Delta\right)=\frac{\partial F\left(x_{(i)};\Delta\right)}{\partial b}=-\frac{acx_{(i)}e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2}}{e^{bcx_{(i)}}-1}\ln\left(1-e^{-bcx_{(i)}}\right) \tag{12}$$

$$K_3\left(x_{(i)};\Delta\right)=\frac{\partial F\left(x_{(i)};\Delta\right)}{\partial c}=-\frac{abx_{(i)}e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2}}{e^{bcx_{(i)}}-1} \tag{13}$$

The OLS estimator of reliability function can be obtained approximately based on (5) as

$$\widehat{R}^{OLS}\left(x_{(i)}\right)\cong 1-e^{-\frac{\widehat{a}^{OLS}}{2}\left(\ln\left(1-e^{-\widehat{b}^{OLS}\widehat{c}^{OLS}x_{(i)}}\right)\right)^2} \tag{14}$$

3.3 Weighted Least Squares (WLS) Estimators

The WLS estimators of Δ are defined by $\widehat{\Delta}^{WLS}=\underset{\Delta}{\operatorname{argmin}}WLS(\Delta)$ where.

$WLS(\Delta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)};\Delta\right)-\frac{i}{n+1}\right]^2$.

Related to RGGo, recall (3), the WLS(Δ) is

$$WLS(\Delta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2}-\frac{i}{n+1}\right]^2 \tag{15}$$

Based on (15), the WLS estimators $(\widehat{a}^{WLS},\widehat{b}^{WLS},\widehat{c}^{WLS})$ can be obtained, in a similar manner to the OLS, by solving simultaneously the three non-linear equations summarized as $\partial\text{WLS}(\Delta)/\partial\Delta=0$ with respect to Δ which follow by solving

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2}-\frac{i}{n+1}\right]K_k\left(x_{(i)};\Delta\right)=0;k=1,2,3$$

with $K_k\left(x_{(i)};\Delta\right)$ for $k=1,2,3$ as given in (11), (12), (13).

The WLS estimator of the reliability function is obtain approximately based on (5) as

$$\widehat{R}^{WLS}\left(x_{(i)}\right)\cong 1-e^{-\frac{\widehat{a}^{WLS}}{2}\left(\ln\left(1-e^{-\widehat{b}^{WLS}\widehat{c}^{WLS}x_{(i)}}\right)\right)^2} \tag{16}$$

3.4 Cramér-Von Mises Minimum Distance (CM) Estimators

The CM estimators of Δ are given by $\widehat{\Delta}^{CM} = \underset{\Delta}{\text{argmin}} CM(\Delta)$, where.

$CM(\Delta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F\left(x_{(i)};\Delta\right) - \frac{2i-1}{2n}\right]^2$.

Related to RGGo distribution with CDF given in (3), the CM(Δ) is

$$CM(\Delta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2} - \frac{2i-1}{2n}\right]^2 \tag{17}$$

Based on (17), the CM estimators $(\hat{a}^{CM}, \hat{b}^{CM}, \hat{c}^{CM})$ can be obtained by solving simultaneously the three non-linear equations summarized as $\partial CM(\Delta)/\partial\Delta = 0$ with respect to Δ.

The CM estimator of the reliability function can be obtained approximately based on (5) as

$$\widehat{R}^{CM}\left(x_{(i)}\right) \cong 1 - e^{-\frac{\hat{a}^{CM}}{2}\left(ln\left(1-e^{-\hat{b}^{CM}\hat{c}^{CM}x_{(i)}}\right)\right)^2} \tag{18}$$

3.5 Maximum Product of Spacing (MP) Estimators

The MP estimators of Δ are given by $\widehat{\Delta}^{MP} = \underset{\Delta}{\text{argmax}} MP(\Delta)$, where.

$MP(\Delta) = \left[\prod_{i=1}^{n+1}(S_i(\Delta))\right]^{\frac{1}{n+1}}$ where $S_i(\Delta) = F\left(x_{(i)};\Delta\right) - F\left(x_{(i-1)};\Delta\right)$ with $F\left(x_{(0)};\Delta\right) = 0$ and $F\left(x_{(n+1)};\Delta\right) = 1$. Further, the MP estimators of Δ can be equivalent given by $\widehat{\Delta}^{MP} = \underset{\Delta}{\text{argmax}} LMP(\Delta)$ where $LMP(\Delta) = \ln(S_i(\Delta)) = \frac{1}{n+1}\sum_{i=1}^{n+1}\ln(S_i(\Delta))$.

Related to RGGo distribution, recall (3), the MP(Δ) and LMP(Δ) are given by

$$MP(\Delta) = \left[\prod_{i=1}^{n+1}\left(e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2} - e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i-1)}}\right)\right)^2}\right)\right]^{\frac{1}{n+1}} \tag{19}$$

$$LMP(\Delta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\ln\left(e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i)}}\right)\right)^2} - e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(i-1)}}\right)\right)^2}\right) \tag{20}$$

The MP estimators $(\hat{a}^{MP}, \hat{b}^{MP}, \hat{c}^{MP})$ can be obtained by solving simultaneously the three non-linear equations summarized as $\partial MP(\Delta)/\partial\Delta = 0$ or equivalently $\partial LMP(\Delta)/\partial\Delta = 0$ with respect to Δ.

The MP estimator of the reliability function is obtain approximately by

$$\widehat{R}^{MP}\left(x_{(i)}\right) \cong 1 - e^{-\frac{\hat{a}^{MP}}{2}\left(\ln\left(1-e^{-\hat{b}^{MP}\hat{c}^{MP}x_{(i)}}\right)\right)^2} \tag{21}$$

3.6 Percentile (PC) Estimators

The PC estimators of Δ are given by $\widehat{\Delta}^{PC} = \underset{\Delta}{argmin} PC(\Delta)$, where $PC(\Delta) = \sum_{i=1}^{n}\left[x_{(i)} - Q(p_i;\Delta)\right]^2$.

Related to RGGo distribution, recall (7) with $p_i = \frac{i}{n+1}$, the PC(Δ) is

$$PC(\Delta) = \sum_{i=1}^{n}\left[x_{(i)} + \frac{1}{bc}\ln\left(1 - e^{-\left(-\frac{2}{a}\ln(p_i)\right)^{\frac{1}{2}}}\right)\right]^2 \tag{22}$$

The PC estimators $(\hat{a}^{PC}, \hat{b}^{PC}, \hat{c}^{PC})$ can be obtained by solving simultaneously the three non-linear equations summarized as $\partial \mathrm{PC}(\Delta)/\partial\Delta = 0$ with respect to Δ.

The PC estimator of reliability function is obtain approximately by

$$\widehat{R}^{PC}\left(x_{(i)}\right) \cong 1 - e^{-\frac{\hat{a}^{PC}}{2}\left(\ln\left(1-e^{-\hat{b}^{PC}\hat{c}^{PC}x_{(i)}}\right)\right)^2} \tag{23}$$

3.7 Anderson Darling (AD) Estimators

The AD estimators of Δ are given by $\widehat{\Delta}^{AD} = \underset{\Delta}{\mathrm{argmin}} AD(\Delta)$, where.

$AD(\Delta) = -n + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\xi$, and $\xi = \ln F\left(x_{(i)};\Delta\right) + \ln\left(1 - F\left(x_{(n+1-i)};\Delta\right)\right)$.

Related to RGGo distribution, the AD(Δ) is

$$AD(\Delta) = -n + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\xi \tag{24}$$

and

$$\xi = -\frac{a}{2}\left(\ln\left(1 - e^{-bcx_{(i)}}\right)\right)^2 + \ln\left(1 - e^{-\frac{a}{2}\left(\ln\left(1-e^{-bcx_{(n+1-i)}}\right)\right)^2}\right) \tag{25}$$

The AD estimators $(\hat{a}^{AD}, \hat{b}^{AD}, \hat{c}^{AD})$ can be obtained by solving simultaneously the three non-linear equations summarized as $\partial \mathrm{AD}(\Delta)/\partial\Delta = 0$ with respect to Δ.

The AD estimator of the reliability function is obtain approximately by

$$\widehat{R}^{AD}\left(x_{(i)}\right) \cong 1 - e^{-\frac{\hat{a}^{AD}}{2}\left(\ln\left(1-e^{-\hat{b}^{AD}\hat{c}^{AD}x_{(i)}}\right)\right)^2} \tag{26}$$

4 Simulation and Results

This section includes simulation experiments to evaluate the behavior of the ML, OLS, WLS, CM, MP, PC, AD estimators for the parameters and reliability function presented above under various sample sizes and configurations. The following procedure is considered with the R software.

1. Generate 1000 Random Samples of the Type $(x_1, x_2, \ldots, x_n)$ from the RGGo Distribution via the So-Called Inverse Transform Sampling Based on the Quantile Function Given by (7) as

$$x_i = -\frac{1}{bc} ln\left(1 - e^{-\left(-\frac{2}{a}\ln(u_i)\right)^{\frac{1}{2}}}\right); i = 1,2, \ldots, n \tag{27}$$

   where $u_i; i = 1,2, \ldots, n$ represents a sample of values drawn from the standard uniform distribution.
2. Investigate different sample sizes as, $n = 25$, 50, 100, and 200.
3. Select three distinct configurations for the vector of parameters (a, b, c) as, I.(0.9, 1.6, 1.3); II.(0.2, 0.4, 1.2); and III.(3.8, 0.9, 2.2).
4. For the configurations under consideration, the three parameters' average estimates (AEP) related to ML, OLS, WLS, CM, MP, PC, AD are computed, as well as their mean squared errors (MSEP) and mean squared errors of reliability (MSER).

Numerical results are listed in Tables 1, 2, and 3. Further, the best estimation method of the parameters and reliability function associated with the lowest values of MSEP and MSER are listed in Tables 4 and 5.

The most important empirical results can be summarized as follows:

- As the sample size increases, the AEP values often become closer to the default values.
- For all sample sizes, the parameter's lowest MSEP value corresponds to its lowest default value.
- As the sample size increases, the performance of the estimation methods tends to align more closely.
- The MSEP and MSER values seem to decline with increasing sample size, indicating the estimators' consistency.
- The MP method performed the best when estimating the first parameter across a range of sample sizes and datasets.
- The ML method always performed the best when estimating the second parameter across a range of sample sizes and the III dataset. It also performed the best for the other datasets besides the MP method.
- The ML method always performed the best when estimating the third parameter across a range of sample sizes and datasets, except for a single appearance of the AD method when $n = 200$ with the I dataset and the MP method when $n = 25$ with the III dataset.
- The ML method always performed the best for estimating the reliability function across a range of sample sizes and datasets, except for the MP method's two appearances when $n = 25$ related to I and II datasets.

Table 1. The values of *AEP* corresponding to each configuration

n	$\widehat{\Delta}$	ML	OLS	WLS	CM	MP	PC	AD
I. (0.9, 1.6, 1.3)								
25	$\hat{a}$	1.13148	1.02615	1.02882	1.23547	0.81919	0.91255	1.03489
	$\hat{b}$	1.54534	1.59526	1.59060	1.52914	1.65478	1.69776	1.58040
	$\hat{c}$	1.41754	1.42973	1.40753	1.39957	1.48267	1.44558	1.42797
50	$\hat{a}$	0.97968	0.93226	0.93966	1.01579	0.80987	0.82263	0.94294
	$\hat{b}$	1.58803	1.61085	1.60657	1.57809	1.65344	1.73300	1.60380
	$\hat{c}$	1.44307	1.43581	1.44014	1.41129	1.47936	1.47153	1.43310
100	$\hat{a}$	0.94749	0.92769	0.93306	0.96738	0.84718	0.83074	0.93117
	$\hat{b}$	1.58693	1.59664	1.59334	1.57990	1.62698	1.69595	1.59375
	$\hat{c}$	1.44161	1.43469	1.43744	1.42567	1.45972	1.44482	1.43282
200	$\hat{a}$	0.92007	0.91108	0.91469	0.92999	0.86234	0.83848	0.91262
	$\hat{b}$	1.60184	1.60495	1.60371	1.59658	1.62520	1.68097	1.60437
	$\hat{c}$	1.43739	1.43318	1.42983	1.42644	1.45146	1.44895	1.42731
II. (0.2, 0.4, 1.2)								
25	$\hat{a}$	0.24661	0.22319	0.22661	0.25813	0.18966	0.22046	0.22644
	$\hat{b}$	0.41720	0.39134	0.39666	0.40801	0.38123	0.37220	0.40333
	$\hat{c}$	1.28665	1.32432	1.31451	1.29164	1.33231	1.31024	1.31526
50	$\hat{a}$	0.21516	0.20380	0.20643	0.21835	0.18412	0.19123	0.20647
	$\hat{b}$	0.40970	0.39769	0.40181	0.40716	0.38573	0.39266	0.40294
	$\hat{c}$	1.29882	1.32814	1.31914	1.31153	1.32682	1.31192	1.31444
100	$\hat{a}$	0.20795	0.20162	0.20393	0.20853	0.18967	0.18332	0.20374
	$\hat{b}$	0.40534	0.39922	0.40173	0.40403	0.39099	0.40841	0.40195
	$\hat{c}$	1.30211	1.31977	1.31539	1.31199	1.32130	1.30536	1.31178
200	$\hat{a}$	0.20343	0.20015	0.20153	0.20353	0.19288	0.18574	0.20121
	$\hat{b}$	0.40057	0.39713	0.39869	0.39955	0.39221	0.40675	0.39847
	$\hat{c}$	1.30292	1.31809	1.31545	1.30987	1.31697	1.30278	1.30952
III. (3.8, 0.9, 2.2)								
25	$\hat{a}$	5.08310	4.50637	4.55605	5.71466	3.38162	3.78575	4.55088
	$\hat{b}$	0.92759	0.88353	0.89214	0.92692	0.84393	0.83944	0.90093
	$\hat{c}$	2.47578	2.66155	2.61975	2.44447	3.10244	2.71506	2.40753
50	$\hat{a}$	4.40682	4.23950	4.23352	4.73492	3.47259	3.57327	4.20369

(*continued*)

Table 1. (*continued*)

n	$\widehat{\Delta}$	ML	OLS	WLS	CM	MP	PC	AD
	$\hat{b}$	0.91668	0.89941	0.90413	0.92110	0.86634	0.85695	0.90474
	$\hat{c}$	2.51190	2.59480	2.57113	2.32585	3.02081	2.75859	2.61021
100	$\hat{a}$	4.06552	3.89513	3.95695	4.10228	3.54143	3.60048	3.94173
	$\hat{b}$	0.90714	0.89419	0.89963	0.90478	0.87762	0.87311	0.89929
	$\hat{c}$	2.61617	2.74884	2.75445	2.68774	2.91313	2.75791	2.71057
200	$\hat{a}$	3.91868	3.86716	3.88936	3.96850	3.61716	3.57208	3.87402
	$\hat{b}$	0.90279	0.89785	0.90026	0.90321	0.88552	0.87948	0.89962
	$\hat{c}$	2.59134	2.74460	2.63975	2.65203	2.79387	2.85113	2.69336

Table 2. The values of *MSEP* corresponding to each configuration

n	$\widehat{\Delta}$	ML	OLS	WLS	CM	MP	PC	AD
I. (0.9, 1.6, 1.3)								
25	$\hat{a}$	0.35418	0.91416	0.62239	1.72802	0.14521	0.44576	0.35146
	$\hat{b}$	0.11449	0.13165	0.12304	0.14345	0.10858	0.26457	0.11579
	$\hat{c}$	0.06553	0.37434	0.99632	1.12120	0.07121	0.32875	0.10520
50	$\hat{a}$	0.08874	0.12258	0.09856	0.16515	0.06144	0.19016	0.08957
	$\hat{b}$	0.05339	0.06504	0.05870	0.06756	0.05406	0.18790	0.05732
	$\hat{c}$	0.03391	0.06085	0.04447	0.06880	0.04348	0.14164	0.04509
100	$\hat{a}$	0.03545	0.05552	0.04474	0.06550	0.02839	0.10245	0.04000
	$\hat{b}$	0.02477	0.02977	0.02678	0.03059	0.02481	0.09791	0.02627
	$\hat{c}$	0.02711	0.03719	0.03075	0.03586	0.03242	0.06031	0.02921
200	$\hat{a}$	0.01673	0.02430	0.01961	0.02634	0.01550	0.06037	0.01861
	$\hat{b}$	0.01469	0.01668	0.01546	0.01674	0.01515	0.05936	0.01532
	$\hat{c}$	0.02362	0.02704	0.02380	0.02454	0.02721	0.06769	0.02282
II. (0.2, 0.4, 1.2)								
25	$\hat{a}$	0.00935	0.03516	0.03338	0.05500	0.00405	0.02748	0.00770
	$\hat{b}$	0.01542	0.01737	0.01603	0.01826	0.01377	0.01560	0.01563
	$\hat{c}$	0.01353	0.09403	0.07423	0.09161	0.02419	0.05382	0.02552
50	$\hat{a}$	0.00254	0.00408	0.00326	0.00520	0.00188	0.01350	0.00258
	$\hat{b}$	0.00719	0.00836	0.00781	0.00865	0.00677	0.00894	0.00756
	$\hat{c}$	0.01322	0.03446	0.02566	0.02486	0.01966	0.02413	0.01990
100	$\hat{a}$	0.00108	0.00167	0.00132	0.00189	0.00093	0.00772	0.00122
	$\hat{b}$	0.00364	0.00437	0.00397	0.00446	0.00350	0.00661	0.00393

(*continued*)

Table 2. (*continued*)

n	$\widehat{\Delta}$	ML	OLS	WLS	CM	MP	PC	AD
	$\hat{c}$	0.01263	0.02262	0.01955	0.02031	0.01764	0.01852	0.01709
200	$\hat{a}$	0.00055	0.00086	0.00067	0.00090	0.00053	0.00460	0.00065
	$\hat{b}$	0.00166	0.00207	0.00185	0.00208	0.00168	0.00376	0.00182
	$\hat{c}$	0.01188	0.01929	0.01816	0.01672	0.01532	0.01683	0.01523
III. (3.8, 0.9, 2.2)								
25	$\hat{a}$	12.5427	21.5825	17.9790	44.76931	4.33821	12.1620	10.3980
	$\hat{b}$	0.01437	0.01958	0.01728	0.02155	0.01560	0.02536	0.01494
	$\hat{c}$	5.22534	14.0249	9.74865	35.7509	3.34283	7.21264	7.19477
50	$\hat{a}$	3.56113	5.63766	4.18173	8.16952	1.93357	4.46351	3.52512
	$\hat{b}$	0.00727	0.01025	0.00852	0.01110	0.00780	0.01486	0.00778
	$\hat{c}$	1.86306	7.03200	5.13464	8.48300	2.39344	3.84600	4.31666
100	$\hat{a}$	1.14066	1.76167	1.41664	2.09099	0.84262	2.04212	1.26210
	$\hat{b}$	0.00332	0.00494	0.00398	0.00501	0.00367	0.00690	0.00379
	$\hat{c}$	0.89823	2.86871	1.81010	2.08994	1.99767	1.82369	1.65567
200	$\hat{a}$	0.45762	0.75369	0.59100	0.82507	0.40178	1.05355	0.54802
	$\hat{b}$	0.00155	0.00235	0.00189	0.00237	0.00172	0.00357	0.00181
	$\hat{c}$	0.55086	2.62489	1.88195	1.77239	0.93217	1.89191	1.28415

Table 3. The values of *MSER* corresponding to each configuration

n	ML	OLS	WLS	CM	MP	PC	AD
I. (0.9, 1.6, 1.3)							
25	0.004300	0.004912	0.004561	0.005034	0.004261	0.008547	0.004432
50	0.002039	0.002446	0.002221	0.002463	0.002086	0.004969	0.002185
100	0.000994	0.001185	0.001083	0.001195	0.001005	0.002388	0.001066
200	0.000561	0.000651	0.000601	0.000653	0.000566	0.001346	0.000596
II. (0.2, 0.4, 1.2)							
25	0.003775	0.004419	0.004044	0.004563	0.003631	0.014120	0.003904
50	0.001888	0.002331	0.002112	0.002349	0.001915	0.012360	0.002052
100	0.000993	0.001215	0.001093	0.001220	0.001008	0.007622	0.001079
200	0.000486	0.000601	0.000536	0.000601	0.000496	0.004050	0.000533
III. (3.8, 0.9, 2.2)							
25	0.004275	0.004931	0.004607	0.005000	0.004325	0.005657	0.004445
50	0.002111	0.002409	0.002236	0.002440	0.002137	0.003014	0.002180
100	0.001100	0.001280	0.001172	0.001282	0.001117	0.001574	0.001163
200	0.000515	0.000617	0.000558	0.000618	0.000525	0.000762	0.000553

Table 4. The best estimation method for the parameters

n	$\widehat{\Delta}$	I. (0.9, 1.6, 1.3)	II. (0.2, 0.4, 1.2)	III. (3.8, 0.9, 2.2)
25	$\hat{a}$	MP	MP	MP
	$\hat{b}$	MP	MP	ML
	$\hat{c}$	ML	ML	MP
50	$\hat{a}$	MP	MP	MP
	$\hat{b}$	ML	MP	ML
	$\hat{c}$	ML	ML	ML
100	$\hat{a}$	MP	MP	MP
	$\hat{b}$	ML	MP	ML
	$\hat{c}$	ML	ML	ML
200	$\hat{a}$	MP	MP	MP
	$\hat{b}$	ML	ML	ML
	$\hat{c}$	AD	ML	ML

Table 5. The best estimation method for the reliability function

n	I. (0.9, 1.6, 1.3)	II. (0.2, 0.4, 1.2)	III. (3.8, 0.9, 2.2)
25	MP	MP	ML
50	ML	ML	ML
100	ML	ML	ML
200	ML	ML	ML

5 Conclusion

The one-scale parameter Rayleigh distribution is one of the most extensively utilized. It has gained popularity as a model for various situations involving monotone hazard rates. It is also employed as a generator family to introduce many notable generalized variants of different probability distributions. This paper focused on Rayleigh Gamma Gompertz (RGGo) as one of these distributions. The RGGo's capabilities and flexibility in estimating unknown parameters and the reliability function are investigated under various sample sizes and datasets by employing seven estimation methods: ML, OLS, WLS, CM, MP, PC, and AD. The estimators' performance is measured using mean squared error. The simulation results generally demonstrated the outperforming, flexibility, and consistency of ML and MP estimates, which deal with maximizing specific objective functions for estimating the parameters for various datasets, while PC, CM, and OLS, which deal with minimizing specific objective functions, have recorded the lowest performance depending on certain configurations. Regarding

estimating the reliability function, the ML and MP are the most performed methods, while the least performed is always associated with PC. Consequently, for estimating the parameters and reliability function of RGGo, it is recommended to use ML and MP rather than utilizing PC. In the same context, as an open topic for future research, other generalized and extended distributions may be examined.

References

1. Safari, M.A.M., Masseran, N., Abdul Majid, M.H.: Robust reliability estimation for Lindley distribution-a probability integral transform statistical approach. Mathematics **8**, 1634 (2020)
2. Meeker, W.Q., Escobar, L.A.: Statistical Methods for Reliability Data. John Wiley and Sons, New York (2014)
3. Blischke, W.R., Murthy, D.N.P.: Reliability: Modeling, Prediction, and Optimization. John Wiley and Sons, New York (2000)
4. Lee, E.T., Wang, J.: Statistical Methods for Survival Data Analysis, 3rd edn., p. 476. John Wiley and Sons, New York (2003)
5. Rayleigh, J.W.S.: On the resultant of a large number of vibrations of the same pitch and of arbitrary phase. London Edinburgh and Dublin Philos. Mag. J. Sci. - 5th Ser. **10**, 73–78 (1880)
6. Siddiqui, M.M.: Some problems connected with Rayleigh distributions. J. Res. Natl. Bureau Standard **66D**, 167–174 (1962)
7. Sinha, S.K., Howlader, H.A.: Credible and HPD intervals of the parameter and reliability of Rayleigh distribution. IEEE Trans. Reliab. **32**, 217–220 (1993)
8. Bovaird, T., Lineweaver, C.H.: A flat inner disc model as an alternative to the Kepler dichotomy in the Q1–Q16 planet population. Mon. Not. R. Astron. Soc. **468**, 1493–1504 (2017)
9. Belaid, A., Boukerroui, D.: Local maximum likelihood segmentation of echocardiographic images with Rayleigh distribution. SIViP **12**, 1087–1096 (2018)
10. Casas-Prat, M., Holthuijsen, L.H.: Short-term statistics of waves observed in deep water. J. Geophys. Res.: Oceans **115**, 1–20 (2010)
11. Morgan, E.C., Lackner, M., Vogel, R.M., Baise, L.G.: Probability distributions for offshore wind speeds. Energy Convers. Manage. **52**, 15–26 (2011)
12. Sohoni, V., Gupta, S., Nema, R.: A comparative analysis of wind speed probability distributions for wind power assessment of four sites. Turk. J. Electr. Eng. Comput. Sci. **24**, 4724–4735 (2016)
13. Bidaoui, H., El Abbassi, I., El Bouardi, A., Darcherif, A.: Wind speed data analysis using Weibull and Rayleigh distribution functions, case study: five cities northern Morocco. Procedia Manuf. **32**, 786–793 (2019)
14. Bekker, A., Roux, J.J.J., Mosteit, P.J.: A generalization of the compound Rayleigh distribution: using a Bayesian method on cancer survival times. Commun. Stat.-Theory Methods **29**(7), 1419–1433 (2000)
15. Kundu, D., Raqab, M.Z.: Generalized Rayleigh distribution: different methods of estimations. Comput. Stat. Data Anal. **49**(1), 187–200 (2005)
16. Dey, S., Dey, T., Kundu, D.: Two-parameter Rayleigh distribution: different methods of estimation. Am. J. Math. Manag. Sci. **33**(1), 55–74 (2014)
17. Merovci, F., Elbatal, I.: Weibull Rayleigh distribution: theory and applications. Appl. Math. Inf. Sci. **9**(4), 2127–2137 (2015)
18. Almongy, H.M., Almetwally, E.M., Aljohani, H.M., Alghamdi, A.S., Hafez, E.H.: A new extended Rayleigh distribution with applications of COVID-19 data. Results Phys. **23**, 104012 (2021)

19. Bhat, A.A., Ahmad, S.P., Almetwally, E.M., Yehia, N., Alsadat, N., Tolba, A.H.: The odd Lindley power Rayleigh distribution: properties, classical and Bayesian estimation with applications. Sci. Afr. **20**, e01736 (2023)
20. Hilal, O.A., Al-Noor, N.H.: Theory and applications of truncated exponential Topp Leone Rayleigh distribution. AIP Conf. Proc. **2414**, 040055 (2023)
21. Abd El-Raouf, M.M., AbaOud, M.: A novel extension of generalized Rayleigh model with engineering applications. Alexandria Eng. J. **73**, 269–283 (2023)
22. Kalaf, B.A., Jabar, N.A.A., Madaki, U.Y.: Truncated inverse generalized Rayleigh distribution and some properties. Ibn AL-Haitham J. Pure Appl. Sci. **36**(4), 414–428 (2023)
23. Haj Ahmad, H., Ramadan, D.A., Almetwally, E.M.: Evaluating the discrete generalized Rayleigh distribution: statistical inferences and applications to real data analysis. Mathematics **12**(2), 183 (2024)
24. Ateeq, K., Qasim, T.B., Alvi, A.R.: An extension of Rayleigh distribution and applications. Cogent Math. Stat. **6**(1), 1–16 (2019)
25. Al-Noor, N.H., Assi, N.K.: Rayleigh-Rayleigh distribution: properties and applications. J. Phys.: Conf. Ser. IOP Publ. **1591**, 1–15 (2020)
26. Bentoumi, A., Mezache, A., Oudira, H.: Parameter estimation of Rayleigh-generalized gamma mixture model. Instrum. Mesure Métrologie **19**(1), 59–64 (2020)
27. Al-Noor, N.H., Assi, N.K.: Rayleigh gamma Gompertz distribution: properties and applications. AIP Conf. Proc. **2334**(1), 090003 (2021)
28. Smadi, M.M., Alrefaei, M.H.: New extensions of Rayleigh distribution based on inverted-Weibull and Weibull distributions. Int. J. Electr. Comput. Eng. **11**(6), 5107–5118 (2021)
29. Al-Noor, N.H., Khaleel, M.A., Assi, N.K.: The Rayleigh Gompertz distribution: theory and real applications. Int. J. Nonlinear Anal. Appl. **13**(1), 3505–3516 (2022)
30. Fatima, K., Jan, U., Ahmad, S.P.: Statistical properties of Rayleigh Lomax distribution with applications in survival analysis. J. Data Sci. **16**(3), 531–548 (2022)
31. Michael, A.T., Ayobami, A.I., Ayooluwa, O.E.: Rayleigh-exponential-gamma distribution: theory and properties. Int. J. Res. Publ. Rev. **4**(8), 33–39 (2023)
32. Farhin, S., Khan, A.A.: Bayesian survival analysis of Rayleigh-X family with time varying covariate. Appl. Math. E-Notes **23**, 124–145 (2023)
33. Hassan, E.A.A., Elgarhy, M., Eldessouky, E.A., Hassan, O.H.M., Amin, E.A., Almetwally, E.M.: Different estimation methods for new probability distribution approach based on environmental and medical data. Axioms **12**, 220 (2023)
34. Afify, A.Z., Hussein, E.A., Al-Mofeh, H., Alsultan, R., Aljohani, H.M.: The flexible Frechet distribution: properties, inference, and medical applications. J. Math. **2600560**, 1–27 (2024)