Solution of Quaternion Equations with Imprecisely Defined Coefficients

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Abstract. In this paper we investigate quaternion polynomial equations with imprecisely defined coefficients, an area not previously explored. To address this uncertainty, we introduce the concept of a closed quaternion ball and develop arithmetic operations for these sets. Within this framework, we analyze specific classes of equations and identify the conditions under which solutions exist. The approach allows us to represent the solution sets within closed quaternion balls, thereby capturing the range of all possible values. In particular, we examine quadratic and cubic equations in detail and derive a quaternionic analogue of the de Moivre formula, providing explicitly the n-th roots of a closed quaternion ball.

Keywords: Closed quaternion ball · algebraic structure · quaternion polynomial equation

1 Introduction

A quaternion polynomial equation is an equation of the form $f(x) = g(x)$, where

$$f(x) = x^m + \boldsymbol{a}_{m-1}x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0 \tag{1}$$

and

$$g(x) = x^n + \boldsymbol{b}_{n-1}x^{n-1} + \cdots + \boldsymbol{b}_1 x + \boldsymbol{b}_0, \tag{2}$$

are polynomials with quaternionic coefficients. The solution set

$$S = \{x \in \mathbb{H} \ : \ f(x) = g(x)\}. \tag{3}$$

consists of the zeros of the quaternion polynomial $p(x) = f(x) - g(x)$, which can be computed using known techniques. This problem has received considerable attention in recent years; see, for example, [14] for a historical overview.

Motivated by the studies in [3–5,7,10,12,13,15–18,20], we investigate the setting in which certain coefficients in quaternion polynomial equations are not known precisely and lie within specified bounds. To the best of our knowledge,

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2026
O. Gervasi et al. (Eds.): ICCSA 2025 Workshops, LNCS 15887, pp. 249–266, 2026.
https://doi.org/10.1007/978-3-031-97589-9_18

quaternion polynomial equations with uncertain coefficients and independent terms have not been previously studied in the literature.

Representing and computing the corresponding solution sets under such uncertainty presents significant challenges. To address this, we enclose the solution set S within a closed quaternion ball and develop arithmetic operations specific to this structure. The uncertainty in each quaternion component can be modeled either by assigning intervals of the form $[a, b]$ to each component, resulting in a hypercube, or by treating the quaternion as the center of a hypersphere. In this work, we adopt the latter model.

In this work, operations on closed quaternion balls are considered in Sect. 2. First, we start with a revision on quaternion algebra, focusing on quaternion polynomials and their zeros; this is given in Subsect. 2.1. In Subsect. 2.2, we define closed quaternion balls and give two binary operations: addition and multiplication; some of their properties are established. In Sect. 3 we present some results related to some specific equations with imprecisely defined coefficients and independent term. Finally, in Sect. 4, some examples are given.

2 Preliminaries

This section presents preliminary results on quaternion algebras and closed quaternion balls.

2.1 Quaternion Algebra

Let $\mathbb{H} = \{q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$ denote the quaternion field, where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$, $\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}$, and $\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}$. For $\boldsymbol{q} = q_0+q_1\boldsymbol{i}+q_2\boldsymbol{j}+q_3\boldsymbol{k} \in \mathbb{H}$, the conjugate of $\boldsymbol{q}$ is given by $\overline{\boldsymbol{q}} = q_0-q_1\boldsymbol{i}-q_2\boldsymbol{j}-q_3\boldsymbol{k}$. The **trace** of $\boldsymbol{q}$, denoted by $t(\boldsymbol{q})$, is twice its real part: $t(\boldsymbol{q}) \equiv 2q_0 = \boldsymbol{q} + \overline{\boldsymbol{q}}$. The **norm** of $\boldsymbol{q}$ is defined by $|\boldsymbol{q}| \equiv \sqrt{\boldsymbol{q}\overline{\boldsymbol{q}}} = \sqrt{\overline{\boldsymbol{q}}\boldsymbol{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \in \mathbb{R}$. If $\boldsymbol{q} \neq 0$, then its inverse is $\boldsymbol{q}^{-1} = \overline{\boldsymbol{q}}/\,|\boldsymbol{q}|$.

We recall some basic properties and definitions.

Theorem 1. *For any* $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{H}$, $\overline{\boldsymbol{q}_1 + \boldsymbol{q}_2} = \overline{\boldsymbol{q}_1} + \overline{\boldsymbol{q}_2}$, *and* $\overline{\boldsymbol{q}_1\boldsymbol{q}_2} = \overline{\boldsymbol{q}_2}\ \overline{\boldsymbol{q}_1}$.

We now introduce the notion of similarity.

Definition 1. *Given* $\boldsymbol{q}, \boldsymbol{q}' \in \mathbb{H}, \boldsymbol{q} \sim \boldsymbol{q}'$ *if there exists* $\boldsymbol{\sigma} \in \mathbb{H}$ *such that* $\boldsymbol{q}' = \boldsymbol{\sigma}\boldsymbol{q}\boldsymbol{\sigma}^{-1}$.

This defines an equivalence relation on $\mathbb{H}$. The **conjugacy class** of a quaternion $\boldsymbol{q}$ is denoted by $[\boldsymbol{q}] = \{x \in \mathbb{H} : x \sim \boldsymbol{q}\}$. All elements of $[\boldsymbol{q}]$ satisfy the same characteristic polynomial:

$$\Delta_{\boldsymbol{q}}(x) = x^2 - t(\boldsymbol{q})x + |\boldsymbol{q}|^2 \tag{4}$$

Let $\mathbb{H}[x]$ denote the ring of left-unilateral polynomials with quaternionic coefficients, where the variable x commutes with elements of $\mathbb{H}$. A polynomial

$p(x) = \sum_{i=0}^{n} \boldsymbol{a}_i x^i \in \mathbb{H}[x]$ is said to have degree n if $\boldsymbol{a}_n \neq 0$, and is called **monic** if $\boldsymbol{a}_n = 1$. The conjugate of $p(x)$ is given by $\overline{p}(x) = \sum_{i=0}^{n} \overline{\boldsymbol{a}}_i x^i$.

For $p(x) \in \mathbb{H}[x]$ and $\boldsymbol{q} \in \mathbb{H}$, the **evaluation** of $p(x)$ at $\boldsymbol{q}$ is defined as:

$$p(\boldsymbol{q}) = \sum_{i=0}^{n} \boldsymbol{a}_i \boldsymbol{q}^i, \tag{5}$$

where the expression must be rewritten in standard form before substitution (see p. 262 in [8]).

Although addition and multiplication in $\mathbb{H}[x]$ mimic those in the commutative case, it is important to note that in general:

$$f(x)g(x) = p(x) \Rightarrow f(\boldsymbol{q})g(\boldsymbol{q}) \neq p(\boldsymbol{q}). \tag{6}$$

We say that $\boldsymbol{q} \in \mathbb{H}$ is a *zero* of $p(x)$ if $p(\boldsymbol{q}) = 0$. The set of all such zeros is denoted by $\mathrm{Zero}(p)$.

Note that if $\boldsymbol{a}_n \neq 0$, then $p(\boldsymbol{q}) = 0$ if and only if $\boldsymbol{a}_n^{-1} p(\boldsymbol{q}) = 0$, so we may assume without loss of generality that $p(x)$ is monic.

We now present key theorems about the zeros of quaternion polynomials.

Let $p(x), q(x), r(x) \in \mathbb{H}[X]$. We define $r(x)$ as a **right divisor** of $p(x)$ if there exists a polynomial $q(x)$ such that $p(x) = q(x)r(x)$.

Theorem 2 (pp. 262 [8]). *Let $\boldsymbol{q} \in \mathbb{H}$ and $p(x) \in \mathbb{H}[x]$. Then $\boldsymbol{q} \in \mathrm{Zero}(p)$ if and only if $x - \boldsymbol{q}$ is a right divisor of $p(x)$.*

Theorem 3 (pp. 263 [8]). *Let $\boldsymbol{q} \in \mathbb{H}$ and $p(x), q(x), r(x) \in \mathbb{H}[X]$ be such that $p(x) = q(x)r(x)$. If $\boldsymbol{\gamma} = r(\boldsymbol{q}) \neq 0$, then $p(\boldsymbol{q}) = q(\boldsymbol{\gamma}\boldsymbol{q}\boldsymbol{\gamma}^{-1})r(\boldsymbol{q})$.*

Theorem 4 (Factorization Theorem, pp. 658 [6]). *All non-constant quaternion polynomial of degree m can be factorized into a product of m linear factors.*

The above theorems provide a theoretical foundation for locating the zeros of quaternion polynomials. However, due to the noncommutativity of the quaternion algebra, the computation of these zeros requires specific methods. One such approach, combining both classical and quaternion-specific techniques, is described below.

Let $p(x) = \sum_{i=0}^{n} \boldsymbol{a}_i x^i$ be a quaternion polynomial of degree n. The following algorithm, adapted from [19], outlines a systematic procedure for computing all zeros of $p(x)$:

1. Compute the polynomial $n_p(x) = p(x)\overline{p}(x)$. This is a polynomial of degree $2n$ with real coefficients.
2. Find the complex roots z_i of $n_p(x)$. These roots occur in conjugate pairs. The zeros of the original polynomial $p(x)$ will lie within the conjugacy classes $[z_i] \subset \mathbb{H}$.
3. For each complex root z_i, use Niven's algorithm [11] to extract the quaternion zeros from the class $[z_i]$. Specifically:

(a) Divide $p(x)$ on the right by the characteristic polynomial $\Delta_{z_i}(x)$, obtaining:
$$p(x) = q(x)\Delta_{z_i}(x) + \boldsymbol{\alpha}x + \boldsymbol{\beta}. \tag{7}$$
(b) if $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$, then $[z_i] \subset \text{Zero}(p)$;
(c) otherwise, the unique zero in the class is given by $\boldsymbol{q} = -\boldsymbol{\alpha}^{-1}\boldsymbol{\beta}$.

2.2 Closed Quaternion Balls

In this subsection, we introduce the notion of closed quaternion balls, adapting the definitions from [2] to the quaternionic setting. These sets serve as the foundational structure for representing uncertainty in quaternion polynomial equations.

Definition 2. *Let $\boldsymbol{q} \in \mathbb{H}$ and let $r \in \mathbb{R}_0^+$. The closed ball in $\mathbb{H}$, called* ***a closed quaternion ball*** *is defined by*
$$\langle \boldsymbol{q}, r\rangle = \{\boldsymbol{x} \in \mathbb{H} : |\boldsymbol{x} - \boldsymbol{q}| \leq r\}. \tag{8}$$
If $\boldsymbol{Q} = \langle \boldsymbol{q}, r\rangle$ then the ***center*** *and the* ***radius*** *of $\boldsymbol{Q}$ are denoted by $C(\boldsymbol{Q}) = \boldsymbol{q}$ and $R(\boldsymbol{Q}) = r$, respectively.*

The set of all closed quaternion balls is denoted by $\mathcal{H}$. The zero-radius elements in $\mathcal{H}$ are called ***scalars*** *and correspond to the quaternions themselves.*

Definition 3. *Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$. The closed quaternion balls $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are* ***concentric*** *if $C(\boldsymbol{Q}_1) = C(\boldsymbol{Q}_2)$.*

If $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are concentric and $R(\boldsymbol{Q}_1) = R(\boldsymbol{Q}_2)$, then $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are equal.

We now state a useful characterization of inclusion between closed quaternion balls.

Lemma 1. *Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$ such that $\boldsymbol{Q}_1 = \langle \boldsymbol{q}_1, r_1\rangle$ and $\boldsymbol{Q}_2 = \langle \boldsymbol{q}_2, r_2\rangle$. Then $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_2$ if and only if $|\boldsymbol{q}_1 - \boldsymbol{q}_2| \leq r_2 - r_1$. In particular, if $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are concentric, then $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_2$ if and only if $r_1 \leq r_2$.*

Proof.

[⇒] Let $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_2$. The most distant $\boldsymbol{x} \in \boldsymbol{Q}_1$ from $\boldsymbol{q}_2$ is located at the boundary of $\boldsymbol{Q}_1$, along the line that extends through $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Consequently, $|\boldsymbol{x} - \boldsymbol{q}_2| = |\boldsymbol{q}_2 - \boldsymbol{q}_1| + |\boldsymbol{x} - \boldsymbol{q}_1| \leq r_2$, leading us to deduce that $|\boldsymbol{q}_2 - \boldsymbol{q}_1| \leq r_2 - r_1$.
[⇐] Suppose that $|\boldsymbol{q}_1 - \boldsymbol{q}_2| \leq r_2 - r_1$ and that $\boldsymbol{x} \in \boldsymbol{Q}_1$. Then
$$|\boldsymbol{x} - \boldsymbol{q}_2| = |\boldsymbol{x} - \boldsymbol{q}_1 + \boldsymbol{q}_1 - \boldsymbol{q}_2| \leq |\boldsymbol{x} - \boldsymbol{q}_1| + |\boldsymbol{q}_2 - \boldsymbol{q}_1| \leq r_1 + r_2 - r_1 = r_2. \tag{9}$$
Hence, $\boldsymbol{x} \in \boldsymbol{Q}_2$, and $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_2$.

In the special case where $\boldsymbol{q}_1 = \boldsymbol{q}_2$, the condition reduces to $r_1 \leq r_2$. □

Next we define algebraic operations on closed quaternion balls and study their properties.

Addition: We begin by defining a natural addition operation on closed quaternion balls and examining its algebraic properties.

Definition 4. *The binary operation* $+ : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$*, from now on referred to as* ***addition****, is defined by the equation*

$$\langle \boldsymbol{q}_1, r_1 \rangle + \langle \boldsymbol{q}_2, r_2 \rangle := \langle \boldsymbol{q}_1 + \boldsymbol{q}_2, r_1 + r_2 \rangle . \tag{10}$$

Proposition 1. *Addition in* $\mathcal{H}$ *is both commutative and associative. Furthermore,* $\langle 0, 0 \rangle$ *is the addition identity.*

Proof. The result follows immediately from the commutativity and associativity of addition in $\mathbb{H}$ and $\mathbb{R}$, and from the addition identity elements $0 \in \mathbb{H}$ and $0 \in \mathbb{R}$. □

Corollary 1. *The elements in* $\mathcal{H}$ *that have an additive inverse are the zero-radius closed quaternion balls. Moreover, the reciprocal of* $\langle \boldsymbol{q}, 0 \rangle$ *is* $\langle -\boldsymbol{q}, 0 \rangle$*.*

Proof. A direct consequence of Definition 4 and Proposition 1. □

We now provide a geometric interpretation of the addition operation.

Lemma 2. *Let* $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$*. Then* $\boldsymbol{Q}_1 + \boldsymbol{Q}_2 = \{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ *and* $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\}$*.*

Proof. Let $\boldsymbol{y}_i \in \boldsymbol{Q}_i = \langle \boldsymbol{q}_i, r_i \rangle$, for $i \in \{1, 2\}$. Then $\boldsymbol{y}_1 + \boldsymbol{y}_2 \in \{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\}$, and

$$\begin{aligned} |\boldsymbol{y}_1 + \boldsymbol{y}_2 - (\boldsymbol{q}_1 + \boldsymbol{q}_2)| &= |\boldsymbol{y}_1 - \boldsymbol{q}_1 + \boldsymbol{y}_2 - \boldsymbol{q}_2| \\ &\leq |\boldsymbol{y}_1 - \boldsymbol{q}_1| + |\boldsymbol{y}_2 - \boldsymbol{q}_2| \\ &= r_1 + r_2. \end{aligned} \tag{11}$$

Hence, $\boldsymbol{y}_1 + \boldsymbol{y}_2 \in \boldsymbol{Q}_1 + \boldsymbol{Q}_2$, and $\{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\} \subseteq \boldsymbol{Q}_1 + \boldsymbol{Q}_2$.

Conversely, for any $\boldsymbol{q} \in \boldsymbol{Q}_1 + \boldsymbol{Q}_2$, we can write $\boldsymbol{q} = \boldsymbol{y}_1 + \boldsymbol{y}_2$, where $\boldsymbol{y}_1 = \frac{r_1}{r_1 + r_2}(\boldsymbol{q} - \boldsymbol{q}_2) + \frac{r_2}{r_1 + r_2}\boldsymbol{q}_1$, and $\boldsymbol{y}_2 = \frac{r_2}{r_1 + r_2}(\boldsymbol{q} - \boldsymbol{q}_1) + \frac{r_1}{r_1 + r_2}\boldsymbol{q}_2$. Then

$$\begin{aligned} |\boldsymbol{y}_1 - \boldsymbol{q}_1| &= \left| \frac{r_1}{r_1 + r_2}(\boldsymbol{q} - \boldsymbol{q}_2) + \frac{r_2}{r_1 + r_2}\boldsymbol{q}_1 - q_1 \right| \\ &= \left| \frac{r_1}{r_1 + r_2}(\boldsymbol{q} - \boldsymbol{q}_2) - \frac{r_1}{r_1 + r_2}\boldsymbol{q}_1 \right| \\ &= \frac{r_1}{r_1 + r_2} |\boldsymbol{q} - \boldsymbol{q}_1 - \boldsymbol{q}_2| \\ &\leq \frac{r_1}{r_1 + r_2}(r_1 + r_2) = r_1. \end{aligned} \tag{12}$$

Consequently, $\boldsymbol{y}_1 \in \boldsymbol{Q}_1$. Swapping indices 1 and 2, we can also conclude that $\boldsymbol{y}_2 \in \boldsymbol{Q}_2$. Hence, $\boldsymbol{q} \in \{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\}$ and $\boldsymbol{Q}_1 + \boldsymbol{Q}_2 \subseteq \{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{Q}_1$ and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2\}$. □

We now introduce a monotonicity property under inclusion.

Definition 5. *Let $\diamond : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be a binary operation in $\mathcal{B}$. The operation $\diamond$ is* ***inclusion monotonic*** *if, for all $\boldsymbol{a}_m, \boldsymbol{b}_m \in \mathcal{B}$ such that $\boldsymbol{a}_m \subseteq \boldsymbol{b}_m$, $m \in \{1,2\}$, $\boldsymbol{a}_1 \diamond \boldsymbol{a}_2 \subseteq \boldsymbol{b}_1 \diamond \boldsymbol{b}_2$.*

Proposition 2. *The addition in $\mathcal{H}$ is inclusion monotonic.*

Proof. Let $\boldsymbol{Q}_m, \boldsymbol{R}_m \in \mathcal{H}$ such that $\boldsymbol{Q}_m \subseteq \boldsymbol{R}_m$, $m \in \{1,2\}$. By Lemma 2, $\boldsymbol{Q}_1 + \boldsymbol{Q}_2 = \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in \boldsymbol{Q}_1 \text{ and } \boldsymbol{y} \in \boldsymbol{Q}_2\} \subseteq \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in \boldsymbol{R}_1 \text{ and } \boldsymbol{y} \in \boldsymbol{R}_2\} = \boldsymbol{R}_1 + \boldsymbol{R}_2$. □

Multiplication: We now define and study multiplication of closed quaternion balls. This operation is not commutative and does not distribute over addition but retains important structural properties.

Definition 6. *The binary operation $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, referred to as* ***multiplication****, is defined by*

$$\langle \boldsymbol{q}_1, r_1 \rangle * \langle \boldsymbol{q}_2, r_2 \rangle \equiv \langle \boldsymbol{q}_1, r_1 \rangle \langle \boldsymbol{q}_2, r_2 \rangle := \langle \boldsymbol{q}_1 \boldsymbol{q}_2, r_1 |\boldsymbol{q}_2| + r_2 |\boldsymbol{q}_1| + r_1 r_2 \rangle . \tag{13}$$

Scalar multiplication by $\boldsymbol{\alpha} \in \mathbb{H}$ is defined as:

$$\boldsymbol{\alpha} \langle \boldsymbol{q}, r \rangle \equiv \langle \boldsymbol{\alpha}, 0 \rangle \langle \boldsymbol{q}, r \rangle = \langle \boldsymbol{\alpha}\boldsymbol{q}, |\boldsymbol{\alpha}| r \rangle . \tag{14}$$

Proposition 3. *There exists a multiplicative identity in $\mathcal{H}$, namely the zero-radius ball $\langle 1, 0 \rangle$.*

Proof. For any $\langle \boldsymbol{q}, r \rangle \in \mathcal{H}$, we have $\langle \boldsymbol{q}, r \rangle \langle 1, 0 \rangle = \langle 1, 0 \rangle \langle \boldsymbol{q}, r \rangle = \langle \boldsymbol{q}, r \rangle$. □

Definition 7. *Let $\boldsymbol{Q} \in \mathcal{H}$. We define the* ***powers*** *of $\boldsymbol{Q} \neq \langle 0, 0 \rangle$ by*

$$\boldsymbol{Q}^0 = \langle 1, 0 \rangle \text{ and } \boldsymbol{Q}^k = \boldsymbol{Q}^{k-1} \boldsymbol{Q} \text{ for } k \in \mathbb{N}. \tag{15}$$

If $\boldsymbol{Q} = \langle 0, 0 \rangle$, then $\boldsymbol{Q}^k = \langle 0, 0 \rangle$, for all $k \in \mathbb{N}$.

Definition 8. *The multiplication operation is* ***power-associative*** *if, for all $\boldsymbol{Q} \in \mathcal{H}$ and for all $m, s \in \mathbb{N}$, $\boldsymbol{Q}^s \boldsymbol{Q}^m = \boldsymbol{Q}^{s+m}$.*

Proposition 4. *The multiplication in $\mathcal{H}$ is power-associative.*

Proof. Let $\boldsymbol{Q} = \langle \boldsymbol{q}, r \rangle \in \mathcal{H}$. On one hand,

$$\begin{aligned} \boldsymbol{Q}^2 \boldsymbol{Q} &= \left\langle \boldsymbol{q}^2, 2|\boldsymbol{q}| r + r^2 \right\rangle \langle \boldsymbol{q}, r \rangle \\ &= \left\langle \boldsymbol{q}^3, 3|\boldsymbol{q}|^2 r + 3|\boldsymbol{q}| r^2 + r^3 \right\rangle , \end{aligned} \tag{16}$$

and

$$\begin{aligned} \boldsymbol{Q}\boldsymbol{Q}^2 &= \langle \boldsymbol{q}, r \rangle \left\langle \boldsymbol{q}^2, 2|\boldsymbol{q}| r + r^2 \right\rangle \\ &= \left\langle \boldsymbol{q}^3, 3|\boldsymbol{q}|^2 r + 3|\boldsymbol{q}| r^2 + r^3 \right\rangle , \end{aligned} \tag{17}$$

which implies $\boldsymbol{Q}^2\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^2$.

On the other hand, we have

$$\begin{aligned}\left(\boldsymbol{Q}^2\boldsymbol{Q}\right)\boldsymbol{Q} &= \left\langle \boldsymbol{q}^3, 3\left|\boldsymbol{q}\right|^2 r + 3\left|\boldsymbol{q}\right| r^2 + r^3 \right\rangle \langle q, r\rangle \\ &= \left\langle \boldsymbol{q}^4, 4\left|\boldsymbol{q}\right|^3 r + 6\left|\boldsymbol{q}\right|^2 r^2 + \left|\boldsymbol{q}\right| r^3 + r^4 \right\rangle\end{aligned} \tag{18}$$

and

$$\begin{aligned}\boldsymbol{Q}^2\boldsymbol{Q}^2 &= \left\langle \boldsymbol{q}^2, 2\left|\boldsymbol{q}\right| r + r^2 \right\rangle \left\langle \boldsymbol{q}^2, 2\left|\boldsymbol{q}\right| r + r^2 \right\rangle \\ &= \left\langle \boldsymbol{q}^4, 4\left|\boldsymbol{q}\right|^3 r + 6\left|\boldsymbol{q}\right|^2 r^2 + \left|\boldsymbol{q}\right| r^3 + r^4 \right\rangle.\end{aligned} \tag{19}$$

As $\boldsymbol{Q}^2\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^2$ and $\left(\boldsymbol{Q}^2\boldsymbol{Q}\right)\boldsymbol{Q} = \boldsymbol{Q}^2\boldsymbol{Q}^2$, invoking [1], the result follows. □

Proposition 5. *Let $\langle \boldsymbol{q}, r\rangle \in \mathcal{H}$. Then, for all $k \in \mathbb{N}$ with $k \geq 1$,*

$$\langle \boldsymbol{q}, r\rangle^k = \left\langle \boldsymbol{q}^k, \left(\left|\boldsymbol{q}\right| + r\right)^k - \left|\boldsymbol{q}\right|^k \right\rangle. \tag{20}$$

Proof. Let $\langle \boldsymbol{q}, r\rangle \in \mathcal{H}$. We will prove by induction. For $k = 1$, the equation states

$$\langle \boldsymbol{q}, r\rangle = \langle \boldsymbol{q}, \left|\boldsymbol{q}\right| + r - \left|\boldsymbol{q}\right|\rangle = \langle \boldsymbol{q}, r\rangle, \tag{21}$$

which is clearly true. Suppose that the proposition is true for k. Then, for $k+1$ we have

$$\begin{aligned}\langle \boldsymbol{q}, r\rangle^{k+1} &= \langle \boldsymbol{q}, r\rangle^k \langle \boldsymbol{q}, r\rangle \\ &= \left\langle \boldsymbol{q}^k, \left(\left|\boldsymbol{q}\right| + r\right)^k - \left|\boldsymbol{q}\right|^k \right\rangle \langle \boldsymbol{q}, r\rangle \\ &= \left\langle \boldsymbol{q}^{k+1}, \left|\boldsymbol{q}\right| \left(\left(\left|\boldsymbol{q}\right| + r\right)^k - \left|\boldsymbol{q}\right|^k\right) + \left|\boldsymbol{q}^k\right| r + \left(\left(\left|\boldsymbol{q}\right| + r\right)^k - \left|\boldsymbol{q}\right|^k\right) r \right\rangle \\ &= \left\langle \boldsymbol{q}^k, \left|\boldsymbol{q}\right| \left(\left|\boldsymbol{q}\right| + r\right)^k - \left|\boldsymbol{q}\right|^{k+1} + \left|\boldsymbol{q}\right|^k r + \left(\left|\boldsymbol{q}\right| + r\right)^k r - \left|\boldsymbol{q}\right|^k r \right\rangle \\ &= \left\langle \boldsymbol{q}^{k+1}, \left(\left|\boldsymbol{q}\right| + r\right)^{k+1} - \left|\boldsymbol{q}\right|^{k+1} \right\rangle.\end{aligned} \tag{22}$$

By mathematical induction, it is proved that for all $k \geq 1$ the statement is true. □

Proposition 6. *The multiplication in $\mathcal{H}$ is not distributive with respect to the addition.*

Proof. Let $\boldsymbol{Q}_1 = \langle \boldsymbol{i}, 1\rangle$, $\boldsymbol{Q}_2 = \langle \boldsymbol{j}, 1\rangle$, and $\boldsymbol{Q}_3 = \langle \boldsymbol{k}, 1\rangle$. Then

$$\boldsymbol{Q}_1\left(\boldsymbol{Q}_2 + \boldsymbol{Q}_3\right) = \langle \boldsymbol{i}, 1\rangle \langle \boldsymbol{j} + \boldsymbol{k}, 2\rangle = \left\langle \boldsymbol{k} - \boldsymbol{j}, 4 + \sqrt{2} \right\rangle \tag{23}$$

and

$$\boldsymbol{Q}_1\boldsymbol{Q}_2 + \boldsymbol{Q}_1\boldsymbol{Q}_3 = \langle \boldsymbol{i}, 1\rangle \langle \boldsymbol{j}, 1\rangle + \langle \boldsymbol{i}, 1\rangle \langle \boldsymbol{k}, 1\rangle = \langle \boldsymbol{k}, 3\rangle + \langle -\boldsymbol{j}, 3\rangle = \langle \boldsymbol{k} - \boldsymbol{j}, 6\rangle. \tag{24}$$

Hence,

$$\boldsymbol{Q}_1\left(\boldsymbol{Q}_2 + \boldsymbol{Q}_3\right) \neq \boldsymbol{Q}_1\boldsymbol{Q}_2 + \boldsymbol{Q}_1\boldsymbol{Q}_3. \tag{25}$$

□

We have seen that the set $\{\boldsymbol{x}_1+\boldsymbol{x}_2 : \boldsymbol{x}_i \in \boldsymbol{Q}_i,\ i=1,2\}$ remains a closed quaternion ball. However, the set $\{\boldsymbol{x}_1\boldsymbol{x}_2 : \boldsymbol{x}_i \in \boldsymbol{Q}_i,\ i=1,2\}$ is generally not a closed quaternion ball. Nonetheless, it is bounded. Among all closed quaternion balls centered at $C(\boldsymbol{Q}_1)C(\boldsymbol{Q}_2)$, containing the set $\{\boldsymbol{x}_1\boldsymbol{x}_2 : \boldsymbol{x}_i \in \boldsymbol{Q}_i,\ i=1,2\}$, we will demonstrate that the one with the minimal radius is the closed quaternion ball $\boldsymbol{Q}_1\boldsymbol{Q}_2$.

Proposition 7. *The set of all products of two quaternions from distinct closed quaternion balls is not necessarily a closed quaternion ball.*

Proof. It is sufficient to consider a counterexample. Let $\boldsymbol{Q}_1 = \langle -5-2\boldsymbol{i}, 2\rangle$ and $\boldsymbol{Q}_2 = \langle 5+2j, 4\rangle$. Then $\boldsymbol{Q}_1\boldsymbol{Q}_2 = \left\langle -25-10\boldsymbol{i}-10\boldsymbol{j}-4\boldsymbol{k}, 8+6\sqrt{29}\right\rangle$. Given that $|-25-10\boldsymbol{i}-10\boldsymbol{j}-4\boldsymbol{k}| \leq 8+6\sqrt{29}$, we deduce that $0 \in \boldsymbol{Q}_1\boldsymbol{Q}_2$, while $0 \notin \boldsymbol{Q}_1$ and $0 \notin \boldsymbol{Q}_2$. □

Lemma 3. *Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$. Then $\{\boldsymbol{x}_1\boldsymbol{x}_2 : \boldsymbol{x}_i \in \boldsymbol{Q}_i,\ i=1,2\} \subseteq \boldsymbol{Q}_1\boldsymbol{Q}_2$.*

Proof. Let $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{H}$, $\boldsymbol{x}_1 \in \boldsymbol{Q}_1 = \langle \boldsymbol{q}_1, r_1\rangle$, and $\boldsymbol{x}_2 \in \boldsymbol{Q}_2 = \langle \boldsymbol{q}_2, r_2\rangle$. Then

$$\begin{aligned}
|\boldsymbol{x}_1\boldsymbol{x}_2 - \boldsymbol{q}_1\boldsymbol{q}_2| &= |\boldsymbol{x}_1\boldsymbol{x}_2 - \boldsymbol{x}_1\boldsymbol{q}_2 + \boldsymbol{x}_1\boldsymbol{q}_2 - \boldsymbol{q}_1\boldsymbol{q}_2| \\
&= |\boldsymbol{x}_1(\boldsymbol{x}_2-\boldsymbol{q}_2) + (\boldsymbol{x}_1-\boldsymbol{q}_1)\boldsymbol{q}_2| \\
&= |\boldsymbol{x}_1(\boldsymbol{x}_2-\boldsymbol{q}_2) - \boldsymbol{q}_1(\boldsymbol{x}_2-\boldsymbol{q}_2) + \boldsymbol{q}_1(\boldsymbol{x}_2-\boldsymbol{q}_2) + (\boldsymbol{x}_1-\boldsymbol{q}_1)\boldsymbol{q}_2| \\
&= |(\boldsymbol{x}_1-\boldsymbol{q}_1)(\boldsymbol{x}_2-\boldsymbol{q}_2) + \boldsymbol{q}_1(\boldsymbol{x}_2-\boldsymbol{q}_2) + (\boldsymbol{x}_1-\boldsymbol{q}_1)\boldsymbol{q}_2| \qquad (26) \\
&\leq |\boldsymbol{x}_1-\boldsymbol{q}_1|\,|\boldsymbol{x}_2-\boldsymbol{q}_2| + |\boldsymbol{q}_1|\,|\boldsymbol{x}_2-\boldsymbol{q}_2| + |\boldsymbol{x}_1-\boldsymbol{q}_1|\,|\boldsymbol{q}_2| \\
&= r_1r_2 + |\boldsymbol{q}_1|\,r_2 + |\boldsymbol{q}_2|\,r_1.
\end{aligned}$$

Hence, $\boldsymbol{x}_1\boldsymbol{x}_2 \in \boldsymbol{Q}_1\boldsymbol{Q}_2$. □

Proposition 8. *Given the closed quaternion balls $\boldsymbol{Q}_i = \langle \boldsymbol{q}_i, r_i\rangle$, for $i=1,2$, the closed quaternion ball centered at $\boldsymbol{q}_1\boldsymbol{q}_2$ with the smallest radius that contains $\{\boldsymbol{x}_1\boldsymbol{x}_2 : \boldsymbol{x}_i \in \boldsymbol{Q}_i,\ i=1,2\}$ is $\boldsymbol{Q}_1\boldsymbol{Q}_2$.*

Proof. Let $\boldsymbol{Q}_1 = \langle \boldsymbol{q}_1, r_1\rangle$, $\boldsymbol{Q}_2 = \langle \boldsymbol{q}_2, r_2\rangle$, and $\boldsymbol{y} = \boldsymbol{x}_1\boldsymbol{x}_2$. If $\boldsymbol{q}_1 = \boldsymbol{q}_2 = 0$, then $\boldsymbol{x}_i = \frac{\boldsymbol{\sigma}_i}{|\boldsymbol{\sigma}_i|}r_i \in \boldsymbol{Q}_i$, for any $0 \neq \boldsymbol{\sigma}_i \in \mathbb{H}$, $i=1,2$. Furthermore, $|\boldsymbol{x}_1\boldsymbol{x}_2| = r_1r_2$, from where we conclude that $\boldsymbol{y}$ belongs to the border of $\boldsymbol{Q}_1\boldsymbol{Q}_2$.

If only one of the balls is centered at 0, and without loss of generality, suppose that $\boldsymbol{q}_1 = 0$, then $\boldsymbol{x}_1 = \frac{\boldsymbol{\sigma}}{|\boldsymbol{\sigma}|}r_1 \in \boldsymbol{Q}_1$, for any $0 \neq \boldsymbol{\sigma} \in \mathbb{H}$, and $\boldsymbol{x}_2 = \left(1+\frac{r_2}{|\boldsymbol{q}_2|}\right)\boldsymbol{q}_2 \in \boldsymbol{Q}_2$. Since $\boldsymbol{x}_1\boldsymbol{x}_2 = \frac{\boldsymbol{\sigma}}{|\boldsymbol{\sigma}|}\left(|\boldsymbol{q}_2|\,r_1 + r_1r_2\right)\frac{\boldsymbol{q}_2}{|\boldsymbol{q}_2|}$ and

$$\boldsymbol{Q}_1\boldsymbol{Q}_2 = \{\boldsymbol{x} \in \mathbb{H} : |\boldsymbol{x}| \leq |\boldsymbol{q}_2|\,r_1 + r_1r_2\}, \qquad (27)$$

we also conclude that $\boldsymbol{y}$ belongs to the border of $\boldsymbol{Q}_1\boldsymbol{Q}_2$.

If both balls are not centered at 0, then $\boldsymbol{x}_i = \left(1+\frac{r_i}{|\boldsymbol{q}_i|}\right)\boldsymbol{q}_i \in \boldsymbol{Q}_i$, for $i=1,2$.

But

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{x}_1 \boldsymbol{x}_2 \\ &= \left(1 + \frac{r_1}{|\boldsymbol{q}_1|}\right) \boldsymbol{q}_1 \left(1 + \frac{r_2}{|\boldsymbol{q}_2|}\right) \boldsymbol{q}_2 \\ &= \boldsymbol{q}_1 \boldsymbol{q}_2 + \frac{|\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2}{|\boldsymbol{q}_1|\, |\boldsymbol{q}_2|} \boldsymbol{q}_1 \boldsymbol{q}_2. \end{aligned} \tag{28}$$

Since

$$\boldsymbol{Q}_1 \boldsymbol{Q}_2 = \{\boldsymbol{x} \in \mathbb{H} \ : \ |\boldsymbol{x} - \boldsymbol{q}_1 \boldsymbol{q}_2| \leq |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2\}, \tag{29}$$

one can verify that $\boldsymbol{y}$ belongs again to the border of $\boldsymbol{Q}_1 \boldsymbol{Q}_2$. Since $\boldsymbol{y}$ belongs to the border in all cases, and taking into account Lemma 3, we conclude the proposition. □

Proposition 9. *The multiplication in $\mathcal{H}$ is inclusion monotonic.*

Proof. Let $\boldsymbol{Q}_1 = \langle \boldsymbol{q}_1, r_1 \rangle$, $\boldsymbol{Q}_2 = \langle \boldsymbol{q}_2, r_2 \rangle$, $\boldsymbol{Q}_3 = \langle \boldsymbol{q}_3, r_3 \rangle$, and $\boldsymbol{Q}_4 = \langle \boldsymbol{q}_4, r_4 \rangle$, such that $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_3$ and $\boldsymbol{Q}_2 \subseteq \boldsymbol{Q}_4$. Then

$$\boldsymbol{Q}_1 \boldsymbol{Q}_2 = \langle \boldsymbol{q}_1 \boldsymbol{q}_2, |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2 \rangle\,. \tag{30}$$

Let $\boldsymbol{x} \in \boldsymbol{Q}_1 \boldsymbol{Q}_2$. Then

$$\begin{aligned} |\boldsymbol{x} - \boldsymbol{q}_3 \boldsymbol{q}_4| &= |\boldsymbol{x} - \boldsymbol{q}_1 \boldsymbol{q}_2 + \boldsymbol{q}_1 \boldsymbol{q}_2 - \boldsymbol{q}_3 \boldsymbol{q}_4| \\ &\leq |\boldsymbol{x} - \boldsymbol{q}_1 \boldsymbol{q}_2| + |\boldsymbol{q}_1 \boldsymbol{q}_2 - \boldsymbol{q}_3 \boldsymbol{q}_4| \\ &\leq |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2 + |\boldsymbol{q}_1 \boldsymbol{q}_2 - \boldsymbol{q}_1 \boldsymbol{q}_4 + \boldsymbol{q}_1 \boldsymbol{q}_4 - \boldsymbol{q}_3 \boldsymbol{q}_4| \\ &\leq |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2 + |\boldsymbol{q}_1|\, |\boldsymbol{q}_4 - \boldsymbol{q}_2| + |\boldsymbol{q}_4|\, |\boldsymbol{q}_3 - \boldsymbol{q}_1|\,. \end{aligned} \tag{31}$$

Since $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_3$ and $\boldsymbol{Q}_2 \subseteq \boldsymbol{Q}_4$, by Lemma 1,

$$\begin{aligned} |\boldsymbol{x} - \boldsymbol{q}_3 \boldsymbol{q}_4| &\leq |\boldsymbol{q}_1|\, r_2 + |\boldsymbol{q}_2|\, r_1 + r_1 r_2 + |\boldsymbol{q}_1|\, (r_4 - r_2) + |\boldsymbol{q}_4|\, (r_3 - r_1) \\ &= (|\boldsymbol{q}_2| - |\boldsymbol{q}_4|) r_1 + r_1 r_2 + |\boldsymbol{q}_1|\, r_4 + |\boldsymbol{q}_4|\, r_3 \\ &\leq |\boldsymbol{q}_4 - \boldsymbol{q}_2|\, r_1 + r_1 r_2 + |\boldsymbol{q}_1|\, r_4 + |\boldsymbol{q}_4|\, r_3. \end{aligned} \tag{32}$$

Again, by Lemma 1,

$$\begin{aligned} |\boldsymbol{x} - \boldsymbol{q}_3 \boldsymbol{q}_4| &\leq (r_4 - r_2) r_1 + r_1 r_2 + |\boldsymbol{q}_1|\, r_4 + |\boldsymbol{q}_4|\, r_3 \\ &= r_4 r_1 + |\boldsymbol{q}_1|\, r_4 + |\boldsymbol{q}_4|\, r_3. \end{aligned} \tag{33}$$

Since $\boldsymbol{Q}_1 \subseteq \boldsymbol{Q}_3$, we know that $|\boldsymbol{q}_3 - \boldsymbol{q}_1| \leq r_3 - r_1$. Thus

$$|\boldsymbol{q}_1| - |\boldsymbol{q}_3| \leq |\boldsymbol{q}_3 - \boldsymbol{q}_1| \leq r_3 - r_1 \tag{34}$$

from where we conclude that

$$|\boldsymbol{q}_1| \leq r_3 - r_1 + |\boldsymbol{q}_3|\,. \tag{35}$$

Substituting in Eq. (33), we obtain

$$\begin{aligned} |\boldsymbol{x} - \boldsymbol{q}_3 \boldsymbol{q}_4| &\leq r_4 r_1 + (r_3 - r_1 + |\boldsymbol{q}_3|) r_4 + |\boldsymbol{q}_4|\, r_3 \\ &= |\boldsymbol{q}_4|\, r_3 + |\boldsymbol{q}_3|\, r_4 + r_3 r_4. \end{aligned} \tag{36}$$

Hence, $\boldsymbol{x} \in \boldsymbol{Q}_3 \boldsymbol{Q}_4$, and $\boldsymbol{Q}_1 \boldsymbol{Q}_2 \subseteq \boldsymbol{Q}_3 \boldsymbol{Q}_4$. □

3 Polynomial Equations in Closed Quaternion Balls

In this section, we present our main results concerning quaternion polynomial equations with imprecisely defined coefficients. We focus on determining conditions under which a closed quaternion ball contains all solutions of such equations, and we provide explicit bounds for the radius of the solution ball. We also derive a formula analogous to the de Moivre's theorem, suitable for computing powers and roots of closed quaternion balls.

Solving such equations involves subtleties that may not be immediately apparent. For instance, the solution set of equation

$$\boldsymbol{A}_2 X^2 = \boldsymbol{A}_1 X + \boldsymbol{A}_0, \tag{37}$$

where $\boldsymbol{A}_2, \boldsymbol{A}_1, \boldsymbol{A}_0 \in \mathcal{H}$, is not the same solution set of equation

$$\boldsymbol{A}_2 X^2 - \boldsymbol{A}_1 X = \boldsymbol{A}_0. \tag{38}$$

Since only closed quaternion balls with zero radius, i.e., scalar quaternions, possess additive inverses in $\mathcal{H}$, only such terms can be freely moved from one side of an equation to the other without altering the solution set. As a result, caution is required when manipulating these expressions, particularly when rearranging terms. Nonetheless, within this limitation, it is still possible to obtain meaningful results for certain specific cases.

Proposition 10. *Consider the equation*

$$\boldsymbol{A}_n X^n + \boldsymbol{A}_{n-1} X^{n-1} + \cdots + \boldsymbol{A}_1 X + \boldsymbol{A}_0 = 0_\alpha, \tag{39}$$

where $\boldsymbol{A}_i = \langle \boldsymbol{a}_i, r_i \rangle \in \mathcal{H}$, *for* $i = 0, \ldots, n$ *and* $0_\alpha = \langle 0, \alpha \rangle \in \mathcal{H}$. *Then* $\langle \boldsymbol{q}, r \rangle$ *is a solution of* (39) *if and only if*

$$|\boldsymbol{q}|^n r_n + |\boldsymbol{q}|^{n-1} r_{n-1} + \cdots + |\boldsymbol{q}| r_1 + r_0 \leq \alpha, \tag{40}$$

where $\boldsymbol{q}$ *is a zero of the quaternion polynomial*

$$p(x) = \boldsymbol{a}_n x^n + \boldsymbol{a}_{n-1} x^{n-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0. \tag{41}$$

Moreover, for each $\boldsymbol{q}$, *the solution is unique.*

Proof. Let $\boldsymbol{X} = \langle \boldsymbol{q}, r \rangle$ and $\boldsymbol{A}_k = \langle \boldsymbol{a}_k, r_k \rangle$. Calling upon Proposition 5, we have

$$\begin{aligned} \boldsymbol{A}_k \boldsymbol{X}^k &= \langle \boldsymbol{a}_k, r_k \rangle \left\langle \boldsymbol{q}^k, (|\boldsymbol{q}| + r)^k - |\boldsymbol{q}|^k \right\rangle \\ &= \left\langle \boldsymbol{a}_k \boldsymbol{q}^k, \left|\boldsymbol{q}^k\right| r_k + (|\boldsymbol{a}_k| + r_k) \left((|\boldsymbol{q}| + r)^k - |\boldsymbol{q}|^k \right) \right\rangle \\ &= \left\langle \boldsymbol{a}_k \boldsymbol{q}^k, (|\boldsymbol{a}_k| + r_k) (|\boldsymbol{q}| + r)^k - \left|\boldsymbol{a}_k \boldsymbol{q}^k\right| \right\rangle. \end{aligned} \tag{42}$$

Thus,

$$\begin{aligned}\sum_{k=0}^{n} \boldsymbol{A}_k \boldsymbol{X}^k &= \sum_{k=0}^{n} \left\langle \boldsymbol{a}_k \boldsymbol{q}^k, \left(|\boldsymbol{a}_k| + r_k\right)\left(|\boldsymbol{q}| + r\right)^k - \left|\boldsymbol{a}_k \boldsymbol{q}^k\right| \right\rangle \\ &= \left\langle \sum_{k=0}^{n} \boldsymbol{a}_k \boldsymbol{q}^k, \sum_{k=0}^{n} \left(\left(|\boldsymbol{a}_k| + r_k\right)\left(|\boldsymbol{q}| + r\right)^k - \left|\boldsymbol{a}_k \boldsymbol{q}^k\right| \right) \right\rangle \\ &= \left\langle \sum_{k=0}^{n} \boldsymbol{a}_k \boldsymbol{q}^k, \sum_{k=0}^{n} \left(\sum_{i=0}^{k} \left(|\boldsymbol{a}_k| + r_k\right) \binom{k}{i} |\boldsymbol{q}|^{k-i} r^i - \left|\boldsymbol{a}_k \boldsymbol{q}^k\right| \right) \right\rangle \end{aligned} \tag{43}$$

From equation $\sum_{k=0}^{n} \boldsymbol{A}_k \boldsymbol{X}^k = \langle 0, \alpha \rangle$, we get two equations, one for the center

$$\sum_{k=0}^{n} \boldsymbol{a}_k \boldsymbol{q}^k = 0, \tag{44}$$

and another for the radius

$$\alpha = \sum_{k=0}^{n} \left(\sum_{i=0}^{k} \left(|\boldsymbol{a}_k| + r_k\right) \binom{k}{i} |\boldsymbol{q}|^{k-i} r^i - |\boldsymbol{a}_k \boldsymbol{q}|^k \right). \tag{45}$$

The RHS of this last equation is a polynomial in r where all the coefficients and the independent term are positive. From $r \geq 0$, we conclude that

$$\alpha \geq \sum_{k=0}^{n} |\boldsymbol{q}|^k r_k. \tag{46}$$

Furthermore, if this final condition is satisfied, then for each quaternion root of Eq. (44), the associated Eq. (45) has all coefficients positive except for the constant term, which is negative. Consequently, by Descartes' Rule of Signs, the equation admits exactly one positive real solution. □

We next study second-degree quaternion polynomial equations of the form $f(x) = g(x)$, where both f and g have degree two.

Proposition 11. *Consider the second-degree closed quaternion ball equation*

$$\langle \boldsymbol{a}_2, r_2 \rangle X^2 + \langle \boldsymbol{a}_1, r_1 \rangle X + \langle \boldsymbol{a}_0, r_0 \rangle = \langle \boldsymbol{b}_2, r_2 \rangle X^2 + \langle \boldsymbol{b}_1, r_1 \rangle X + \langle \boldsymbol{b}_0, r_0 \rangle, \tag{47}$$

where $|\boldsymbol{b}_2| \neq |\boldsymbol{a}_2|$, *and let* $\boldsymbol{q}$ *be a zero of the quaternion polynomial*

$$p(x) = (\boldsymbol{a}_2 - \boldsymbol{b}_2)x^2 + (\boldsymbol{a}_1 - \boldsymbol{b}_1)x + \boldsymbol{a}_0 - \boldsymbol{b}_0. \tag{48}$$

For each $\boldsymbol{q}$, *the solutions of* (47) *are the closed quaternion balls* $\langle \boldsymbol{q}, 0 \rangle$, *and* $\langle \boldsymbol{q}, \boldsymbol{\beta} - 2|\boldsymbol{q}| \rangle$, *where*

$$\boldsymbol{\beta} = -\frac{|\boldsymbol{a}_1| - |\boldsymbol{b}_1|}{|\boldsymbol{a}_2| - |\boldsymbol{b}_2|} \tag{49}$$

provided that $\boldsymbol{\beta} \geq -2|\boldsymbol{q}|$, *which ensures the radius is non-negative and thus defines a valid closed quaternion ball.*

Proof. Applying the closed quaternion ball arithmetic

$$\langle \boldsymbol{a}_2, r_2 \rangle X^2 + \langle \boldsymbol{a}_1, r_1 \rangle X + \langle \boldsymbol{a}_0, r_0 \rangle = \langle \boldsymbol{b}_2, r_2 \rangle X^2 + \langle \boldsymbol{b}_1, r_1 \rangle X + \langle \boldsymbol{b}_0, r_0 \rangle , \quad (50)$$

and putting $X = \langle \boldsymbol{x}, r \rangle$, we get two equations: one for the center

$$\boldsymbol{a}_2 \boldsymbol{x}^2 + \boldsymbol{a}_1 \boldsymbol{x} + \boldsymbol{a}_0 = \boldsymbol{b}_2 \boldsymbol{x}^2 + \boldsymbol{b}_1 \boldsymbol{x} + \boldsymbol{b}_0; \quad (51)$$

and another for the radius, which, after canceling equal terms from both sides, takes the form

$$|\boldsymbol{a}_2| R^2 + |\boldsymbol{a}_1| R - \left|\boldsymbol{a}_2 \boldsymbol{q}^2\right| - |\boldsymbol{a}_1 \boldsymbol{q}| = |\boldsymbol{b}_2| R^2 + |\boldsymbol{b}_1| R - \left|\boldsymbol{b}_2 \boldsymbol{q}^2\right| - |\boldsymbol{b}_1 \boldsymbol{q}| \quad (52)$$

where $R = |\boldsymbol{q}| + r$. These equations can be expressed as follows:

$$p(x) = \boldsymbol{\alpha}_2 x^2 + \boldsymbol{\alpha}_1 x + \boldsymbol{\alpha}_0 \quad (53)$$

and

$$\boldsymbol{\alpha}_2 (R^2 - |\boldsymbol{q}|^2) + \boldsymbol{\alpha}_1 (R - |\boldsymbol{q}|) = 0, \quad (54)$$

where $\boldsymbol{\alpha}_i = |\boldsymbol{a}_i| - |\boldsymbol{b}_i|$.

If $\boldsymbol{q}$ is a solution of the first equation, then for the second equation

$$(\boldsymbol{\alpha}_2 (R + |\boldsymbol{q}|) + \boldsymbol{\alpha}_1)(R - |\boldsymbol{q}|) = 0 \iff (\boldsymbol{\alpha}_2 (r + 2\,|\boldsymbol{q}|) + \boldsymbol{\alpha}_1) r = 0, \quad (55)$$

there are two solutions: $r = 0$, and $r = -\boldsymbol{\alpha}_1 / \boldsymbol{\alpha}_2 - 2\,|\boldsymbol{q}|$, provided $\boldsymbol{\alpha}_1 / \boldsymbol{\alpha}_2 \leq 2\,|\boldsymbol{q}|$. □

Proposition 12. *Consider the closed quaternion ball polynomial equation*

$$\boldsymbol{A}_n X^n + \cdots + \boldsymbol{A}_1 X + \boldsymbol{A}_0 = \boldsymbol{B}_n X^n + \cdots + \boldsymbol{B}_1 X + \boldsymbol{B}_0, \quad (56)$$

where $\boldsymbol{A}_i = \langle \boldsymbol{a}_i, r_i \rangle$ *and* $\boldsymbol{B} = \langle \boldsymbol{b}_i, r_i \rangle$*, for* $i = 0, \ldots, n$*, and* $\boldsymbol{A}_n$ *and* $\boldsymbol{B}_n$ *are not concentric. Then the scalar* $\langle \boldsymbol{q}, 0 \rangle$ *is a solution, where* $\boldsymbol{q}$ *a zero of the quaternion polynomial*

$$p(x) = (\boldsymbol{a}_n - \boldsymbol{b}_n) x^n + \cdots + (\boldsymbol{a}_1 - \boldsymbol{b}_1) x + \boldsymbol{a}_0 - \boldsymbol{b}_0. \quad (57)$$

Furthermore, any other solution, if it exists, is independent of the r_i*'s.*

Proof. In the proof of Proposition 11, we note that the closed quaternion ball polynomial equation inherently produces two distinct equations: one for the center, which determines the roots of the corresponding quaternion polynomial, and another for the radius. The radii of the coefficients for terms of similar degree are the same on both sides of the equation, resulting in their cancellation and rendering the radius equation independent of these terms. Furthermore, it is evident that $r = 0$ consistently constitutes a valid solution for any root $\boldsymbol{q}$ of the central equation. As the radius equation is not dependent of r_i, it consequently follows that any other solution for the radius will also be independent of them. □

A classical problem associated with polynomial equations is the computation of n-th roots, here, in the context of closed quaternion balls. Specifically, an n-th root of a closed quaternion ball $\boldsymbol{A}$ is another closed quaternion ball $\boldsymbol{X}$ such that $\boldsymbol{X}^n = \boldsymbol{A}$. As shown in the following result, this leads to a quaternionic analogue of the de Moivre's formula, from which we derive an explicit expression for the n-th root of a closed quaternion ball.

Proposition 13. *Let $\boldsymbol{a} \in \mathbb{H}$ be given in the polar form*

$$\boldsymbol{a} = |\boldsymbol{a}| \left(\cos\theta + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\theta\right), \tag{58}$$

with $\hat{\boldsymbol{u}}_{\boldsymbol{a}} \equiv \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \neq 0$. Then $\boldsymbol{A} = \langle \boldsymbol{a}, r_{\boldsymbol{a}} \rangle \in \mathcal{H}$ has exactly n n-th roots given by

$$\sqrt[n]{\boldsymbol{A}} = \left\langle \sqrt[n]{|\boldsymbol{a}|} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \sqrt[n]{|\boldsymbol{a}| + r_{\boldsymbol{a}}} - \sqrt[n]{|\boldsymbol{a}|} \right\rangle, \tag{59}$$

for $k = 0, \ldots, n-1$.

Proof. Let $\boldsymbol{X} = \langle \boldsymbol{q}, r \rangle$ and $\boldsymbol{A} = \langle \boldsymbol{a}, r_{\boldsymbol{a}} \rangle$. The n-th roots of $\boldsymbol{A}$ can be obtained by the closed quaternion ball equation $\boldsymbol{X}^n = \boldsymbol{A}$. Substituting $\boldsymbol{X}$ and $\boldsymbol{A}$ in this equation, we obtain two equations:

$$\begin{cases} \boldsymbol{q}^n = \boldsymbol{a} \\ (|\boldsymbol{q}| + r)^n - |\boldsymbol{q}|^n = r_{\boldsymbol{a}} \end{cases}. \tag{60}$$

The solution for the first equation is wellknown and can be found in [9], and is given by

$$\boldsymbol{q} = \sqrt[n]{|\boldsymbol{a}|} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \tag{61}$$

for $k = 0, \ldots, n-1$.

For the second equation there is only one real positive solution, namely,

$$r = \sqrt[n]{|\boldsymbol{a}| + r_{\boldsymbol{a}}} - \sqrt[n]{|\boldsymbol{a}|}. \tag{62}$$

Hence, the n solutions for the n-th roots of the closed quaternion ball $\boldsymbol{A}$ are given by

$$\sqrt[n]{\boldsymbol{A}} = \left\langle \sqrt[n]{|\boldsymbol{a}|} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \sqrt[n]{|\boldsymbol{a}| + r_{\boldsymbol{a}}} - \sqrt[n]{|\boldsymbol{a}|} \right\rangle, \tag{63}$$

for $k = 0, \ldots, n-1$. □

4 Examples

In this section, we will present some examples illustrating our findings.

For the first three examples, the quaternion polynomial equation is the same. For the sake of clarity, we will illustrate how the zeros of this equation are computed using Niven's algorithm.

We need to solve the polynomial equation

$$p(x) = x^3 - (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})x^2 + (6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k})x + 6 = 0. \tag{64}$$

Following the zero-finding algorithm adapted from Niven's method (see Subsect. 2.1):

1. compute $n_p(x) = p(x)\overline{p}(x) = x^6 + 14x^4 + 49x^2 + 36$;
2. compute the complex zeros of $n_p(x)$: $\pm i, \pm 2i, \pm 3i$;
3. (a) for $z_1 = \pm i$, $\Delta_{z_1}(x) = x^2 + 1$; dividing $p(x)$ by $\Delta_{z_1}(x)$, we obtain

$$p(x) = q(x)\Delta_{z_1}(x) + \boldsymbol{\alpha}x + \boldsymbol{\beta} \tag{65}$$

where $\boldsymbol{\alpha} = -1 + 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{\beta} = 6 + \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, which gives the zero $\boldsymbol{q}_1 = -\boldsymbol{\alpha}^{-1}\boldsymbol{\beta} = \dfrac{1}{25}(12\boldsymbol{i} - 16\boldsymbol{j} + 15\boldsymbol{k})$;

(b) for $z_2 = \pm 2i$, $\Delta_{z_2}(x) = x^2 + 4$; dividing $p(x)$ by $\Delta_{z_2}(x)$, we obtain

$$p(x) = q(x)\Delta_{z_2}(x) + \boldsymbol{\alpha}x + \boldsymbol{\beta} \tag{66}$$

where $\boldsymbol{\alpha} = -4 + 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{\beta} = 6 + 4\boldsymbol{i} + 8\boldsymbol{j} + 12\boldsymbol{k}$, which gives the zero $\boldsymbol{q}_2 = -\boldsymbol{\alpha}^{-1}\boldsymbol{\beta} = \dfrac{1}{13}(-10\boldsymbol{j} + 24\boldsymbol{k})$;

(c) for $z_3 = \pm 3i$, $\Delta_{z_3}(x) = x^2 + 9$; dividing $p(x)$ by $\Delta_{z_3}(x)$, we obtain

$$p(x) = q(x)\Delta_{z_3}(x) + \boldsymbol{\alpha}x + \boldsymbol{\beta} \tag{67}$$

where $\boldsymbol{\alpha} = -9 + 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{\beta} = 6 + 9\boldsymbol{i} + 18\boldsymbol{j} + 27\boldsymbol{k}$, which gives the zero $\boldsymbol{q}_3 = -\boldsymbol{\alpha}^{-1}\boldsymbol{\beta} = 3\boldsymbol{k}$.

The following example demonstrates the basic use of the solution from Proposition 10 for a cubic equation with uncertainty in the right-hand side.

Example 1. Let $X = \langle q, r \rangle$.

$$X^3 + \boldsymbol{a}_2 X^2 + \boldsymbol{a}_1 X = \langle -\boldsymbol{a}_0, 1 \rangle, \tag{68}$$

where

$$\begin{aligned} \boldsymbol{a}_2 &= -(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) \\ \boldsymbol{a}_1 &= 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k} \\ \boldsymbol{a}_0 &= 6 \end{aligned} \tag{69}$$

From the closed quaternion ball polynomial equation, we obtain two equations, one for the center and the other for the radius:

$$\begin{cases} \boldsymbol{q}^3 + \boldsymbol{a}_2\boldsymbol{q}^2 + \boldsymbol{a}_1\boldsymbol{q} + \boldsymbol{a}_0 = 0 \\ r^3 + (3\,|\boldsymbol{q}| + \sqrt{14})r^2 + (3\,|\boldsymbol{q}|^2 + 2\sqrt{14}\,|\boldsymbol{q}| + 7)r = 1 \end{cases}. \tag{70}$$

The zeros of the first equation where already calculated. It is easy to see that $\boldsymbol{q}_1 \in [\boldsymbol{i}]$, $\boldsymbol{q}_2 \in [2\boldsymbol{i}]$, and $\boldsymbol{q}_3 \in [3\boldsymbol{i}]$. Hence, $|\boldsymbol{q}_1| = 1$, $|\boldsymbol{q}_2| = 2$, and $|\boldsymbol{q}_3| = 3$. For each of these zeros, there exists only one radius. Solving the second equation, we obtain $r_1 \approx 0.055979$, $r_2 \approx 0.029195$, and $r_3 \approx 0.017644$, respectively. Thus, Eq. (68) has three solutions:

$$\begin{aligned} \boldsymbol{X}_1 &\approx \left\langle \frac{1}{25}\left(12\boldsymbol{i} - 16\boldsymbol{j} + 15\boldsymbol{k}\right), 0.055979 \right\rangle \\ \boldsymbol{X}_2 &\approx \left\langle \frac{1}{13}\left(-10\boldsymbol{j} + 24\boldsymbol{k}\right), 0.029195 \right\rangle \\ \boldsymbol{X}_3 &\approx \langle 3\boldsymbol{k}, 0.017644 \rangle . \end{aligned} \tag{71}$$

$\Diamond$

The following example illustrates Proposition 12. The equation is structurally similar to that in Example 1, except that the linear term has been moved to the opposite side, leading to a different solution due to the algebraic asymmetry of closed quaternion balls.

Example 2. Let $\boldsymbol{X} = \langle \boldsymbol{q}, r \rangle$.

$$\boldsymbol{X}^3 + \boldsymbol{A}_2\boldsymbol{X}^2 = -\boldsymbol{A}_1 X - \boldsymbol{B}, \tag{72}$$

where

$$\begin{aligned} \boldsymbol{A}_2 &= \langle -(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}), 0 \rangle \\ \boldsymbol{A}_1 &= \langle 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}, 0 \rangle \\ \boldsymbol{B} &= \langle -a_0, 1 \rangle \\ \boldsymbol{a}_0 &= 6 \end{aligned} \tag{73}$$

From the closed quaternion ball polynomial equation, we obtain two equations, one for the center and the other for the radius:

$$\begin{cases} \boldsymbol{q}^3 + \boldsymbol{a}_2\boldsymbol{q}^2 + \boldsymbol{a}_1 q + \boldsymbol{a}_0 = 0 \\ r^3 + (3\,|\boldsymbol{q}| - \sqrt{14})r^2 + (3\,|\boldsymbol{q}|^2 - 2\sqrt{14}\,|\boldsymbol{q}| + 7)r - 1 = 0 \end{cases} . \tag{74}$$

The zeros of the first equation are the same as in Example 1. For each of these zeros, there exists only one radius. Solving the second equation, we obtain $r_1 \approx 0.41989$, $r_2 \approx 0.21859$, and $r_3 \approx 0.083365$, respectively. Thus, Eq. (72) has three solutions:

$$\begin{aligned} \boldsymbol{X}_1 &\approx \left\langle \frac{1}{25}\left(12\boldsymbol{i} - 16\boldsymbol{j} + 15\boldsymbol{k}\right), 0.41989 \right\rangle \\ \boldsymbol{X}_2 &\approx \left\langle \frac{1}{13}\left(-10\boldsymbol{j} + 24\boldsymbol{k}\right), 0.21859 \right\rangle \\ \boldsymbol{X}_3 &\approx \langle 3\boldsymbol{k}, 0.083365 \rangle . \end{aligned} \tag{75}$$

$\Diamond$

The third example also relates to Proposition 12. The equation is similar to that in Example 1, but the coefficients now have nonzero radii. This affects the radial part of the solution, altering the admissible values and causing one of the previously valid solutions to disappear.

Example 3. Let $\boldsymbol{X} = \langle \boldsymbol{q}, r \rangle$.

$$\boldsymbol{X}^3 + \boldsymbol{A}_2\boldsymbol{X}^2 + \boldsymbol{A}_1\boldsymbol{X} + \boldsymbol{A}_0 = \langle 0, 1 \rangle \,, \tag{76}$$

where

$$\begin{aligned} \boldsymbol{A}_2 &= \langle -(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}), 0.1 \rangle \\ \boldsymbol{A}_1 &= \langle 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}, 0.2 \rangle \\ \boldsymbol{A}_0 &= \langle 6, 0.1 \rangle \end{aligned} \tag{77}$$

From the closed quaternion ball polynomial equation, we obtain two equations, one for the center and the other for the radius:

$$\begin{cases} \boldsymbol{q}^3 + \boldsymbol{a}_2\boldsymbol{q}^2 + \boldsymbol{a}_1\boldsymbol{q} + \boldsymbol{a}_0 = 0 \\ R^3 + (|\boldsymbol{a}_2| + r_2)R^2 + (|\boldsymbol{a}_1| + r_1)R + r_0 - 1 - \left|\boldsymbol{q}^3\right| - \left|\boldsymbol{a}_2\boldsymbol{q}^2\right| - |\boldsymbol{a}_1\boldsymbol{q}| = 0 \end{cases}, \tag{78}$$

where $R = |\boldsymbol{q}| + r$.

The zeros of the first equation are the same as in Example 1. The condition (40) of Proposition 10 is not satisfied for $\boldsymbol{q}_3$, so it does not correspond to a valid solution. For $\boldsymbol{q}_2$ and q_1 the condition is verified. Solving the second equation, we obtain $r_1' \approx 0.0331229$, and $r_2' \approx 0.00289056$. Thus, Eq. (76) has two solutions:

$$\begin{aligned} \boldsymbol{X}_1 &\approx \left\langle \frac{1}{25}\left(12\boldsymbol{i} - 16\boldsymbol{j} + 15\boldsymbol{k}\right), 0.0331229 \right\rangle \\ \boldsymbol{X}_2 &\approx \left\langle \frac{1}{13}\left(-10\boldsymbol{j} + 24\boldsymbol{k}\right), 0.00289056 \right\rangle . \end{aligned} \tag{79}$$

$\Diamond$

The following example illustrates Proposition 11 by considering a quadratic equation in which corresponding terms on both sides have the same radius, allowing the radius components to cancel and yielding a purely algebraic condition on the centers.

Example 4. Consider the equation

$$\langle \mathbf{2}, 2 \rangle X^2 + \langle \mathbf{1}, 10 \rangle X + \langle \mathbf{2}, 1 \rangle = \langle \mathbf{1}, 2 \rangle X^2 + \langle \mathbf{4}, 10 \rangle X + \langle \mathbf{5}, 1 \rangle \,. \tag{80}$$

From the closed quaternion ball polynomial equation, we obtain two equations, one for the center and the other for the radius:

$$\begin{cases} \boldsymbol{q}^2 - 3\boldsymbol{q} - 3 = 0 \\ (r + 2\,|\boldsymbol{q}| - 3)r = 0 \end{cases}. \tag{81}$$

From the first equation, we get

$$\boldsymbol{q}_1 = \frac{3-\sqrt{21}}{2} \quad \text{and} \quad \boldsymbol{q}_2 = \frac{3+\sqrt{21}}{2}. \tag{82}$$

From Proposition 11, we have $\boldsymbol{\beta} = 3$. For $\boldsymbol{q}_1$, the quantity $\boldsymbol{\beta} - 2\,|\boldsymbol{q}_1| = 6 - \sqrt{2} > 0$. Hence, for this root, there are two solutions:

$$\boldsymbol{X}_1 = \langle \boldsymbol{q}_1, 0 \rangle \quad \text{and} \quad \boldsymbol{X}_2 = \left\langle \boldsymbol{q}_1, 6 - \sqrt{21} \right\rangle. \tag{83}$$

For $\boldsymbol{q}_2$, the quantity $\boldsymbol{\beta} - 2\,|\boldsymbol{q}_2| < 0$. Therefore, for this root, there is only one solution:

$$\boldsymbol{X}_3 = \langle \boldsymbol{q}_2, 0 \rangle. \tag{84}$$

$\Diamond$

Finally, the next example illustrates Proposition 13 by applying the quaternionic de Moivre formula to compute the cube roots of a closed quaternion ball. The result demonstrates how both the center and radius transform under root extraction in this setting.

Example 5. Let $\boldsymbol{A} = \langle 4(1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}), 19 \rangle$. The quaternion $\boldsymbol{a} = 4(1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$ written in polar form is given by $\boldsymbol{a} = 8\,(\cos\theta + \hat{\boldsymbol{u}}_{\boldsymbol{a}} \sin\theta)$, where $\theta = \frac{\pi}{3}$ and $\hat{\boldsymbol{u}}_{\boldsymbol{a}} = \frac{1}{\sqrt{3}}(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$. Applying the de Moivre's formula, the cube roots of $\boldsymbol{A}$ are:

$$\begin{aligned}
\boldsymbol{Q}_1 &= \left\langle 8\left(\cos\left(\frac{\pi}{9}\right) + \sin\left(\frac{\pi}{9}\right)\frac{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{\sqrt{3}}\right), 1 \right\rangle \\
\boldsymbol{Q}_2 &= \left\langle 8\left(\cos\left(\frac{7\pi}{9}\right) + \sin\left(\frac{7\pi}{9}\right)\frac{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{\sqrt{3}}\right), 1 \right\rangle \\
\boldsymbol{Q}_3 &= \left\langle 8\left(\cos\left(\frac{13\pi}{9}\right) + \sin\left(\frac{13\pi}{9}\right)\frac{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{\sqrt{3}}\right), 1 \right\rangle.
\end{aligned} \tag{85}$$

It can easily be checked that $\boldsymbol{Q}_i^3 = \boldsymbol{A}$, for $i = 1, 2, 3$. $\Diamond$

Acknowledgments. R. Serôdio is supported by FCT (Fundação para a Ciência e a Tecnologia, Portugal), research project UIDB/00212/2020 of CMA-UBI (Centro de Matemática e Aplicações, Universidade da Beira Interior, Portugal).

Disclosure of Interests. The authors have no competing interests to declare that are relevant to the content of this article.

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