Mean-Square Monotonicity Analysis of θ-Maruyama Methods

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Abstract. The paper focuses on the analysis of mean-square contractivity of the numerical dynamics arising from the application of θ-Maruyama methods to stochastic differential equations (SDEs) with linear affine drift and diffusion coefficients. We prove that the numerical deviation between two distinct solutions of the SDE is monotonically non-increasing under the same stepsize restrictions needed for mean-square stability or holds unconditionally for certain values of θ. A selection of numerical experiments complements the theoretical investigation.

Keywords: Stochastic differential equations · θ-Maruyama numerical methods · Numerical monotonicity

1 Introduction

This paper aims to analyse the behavior of the numerical dynamics resulting from discretizations of stochastic differential equations (SDEs) with linear affine drift and diffusion coefficients, i.e.,

$$\mathrm{d}X(t) = (\mu X(t) + \nu)\,\mathrm{d}t + (\sigma X(t) + \tau)\,\mathrm{d}W(t), \quad t \geq 0, \tag{1}$$

with $\mu, \nu, \sigma, \tau \in \mathbb{R}$ and $X(t) \in \mathbb{R}$. $\{W(t),\ t \geq 0\}$ is a Wiener process, with independent increments $W(\tau_{j+1}) - W(\tau_j)$, identically distributed as $\sqrt{h} \cdot \mathcal{N}(0,1)$, $\mathcal{N}(0,1)$ being a standard normal random variable.

The problem is intended in Itô sense, i.e., Eq. (1) is the shorthand notation for the integral equation

$$X(t) = \int_0^t (\mu X(s) + \nu)\,\mathrm{d}s + \int_0^t (\sigma X(s) + \tau)\,\mathrm{d}W(s), \quad t \geq 0,$$

where the integral of the drift is intended in Riemann sense along each path, while the integral of the diffusion is here meant in Itô sense, that is, for a given integrable function g,

$$\int_0^t g(X(s))\mathrm{d}W(s) = \lim_{n\to\infty} \sum_{j=0}^{n-1} g(X(\tau_j))\Big(W(\tau_{j+1}) - W(\tau_j)\Big).$$

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This limit is intended in mean-square sense, on the set of equidistant points

$$\mathcal{I}_h = \left\{ \tau_\ell = \ell h, \quad \ell = 0, 1, \ldots, n, \quad h = \frac{t}{n} \right\}.$$

For a general overview of the theory of SDEs, including Itô and Stratonovich calculi, results of existence and uniqueness of solutions in strong and weak senses, stability issues, the interested reader can consider, for instance, [3,17,25,28] and references therein.

It is worth observing that the existing literature on numerical methods for SDEs has addressed several significant aspects, such as convergence in strong and weak senses, the analysis of linear stability, principles of geometric numerical integration. For the presentation of a selection of relevant numerical methods for SDEs, as well as of their aforementioned aspects, one can consider [12,20,24,27] and references therein.

In particular, nonlinear stability analysis in exponential mean-square sense has been investigated in [6,12–14,21,22] and references therein. Stability analysis in exponential mean-square sense implies estimates for the mean-square deviation between two distinct solutions of (1), obtained by choosing two distinct initial values $X_0 \neq Y_0 \in \mathbb{R}$, leading to

$$\mathbb{E}\left[\|X(t) - Y(t)\|^2\right] \leq \mathbb{E}\left[\|X_0 - Y_0\|^2\right] \mathrm{e}^{\alpha t},$$

with exponential rate $\alpha = 2\mu + \sigma$. Supposing that α is negative, consequently the mean-square deviation $\mathbb{E}\left[\|X(t) - Y(t)\|^2\right]$ is monotone over each time interval $[t, t + \Delta t]$ and, specifically, non-increasing, i.e.,

$$\mathbb{E}\left[\|X(t + \Delta t) - Y(t + \Delta t)\|^2\right] \leq \mathbb{E}\left[\|X(t) - Y(t)\|^2\right]. \tag{2}$$

Analysing the monotonicity of mean-square deviations of (2) is useful, for instance, when approaching the numerical modelling of dissipative stochastic partial differential equations [5,15], semi-discretized along the spatial variables and leading to a system of SDEs of large dimension. Indeed, condition (2) formalizes the mean-square contractivity of the numerical dynamics, highlighting the mean-square dissipative character of the problem.

We aim to understand if monotonicity condition (2) is accurately preserved by θ-Maruyama numerical methods [4,7,12,19,20,29], whose expression, with reference to the discrete set

$$\mathcal{I}_{\Delta t} = \left\{ t_n = n\Delta t, \quad n = 0, 1, \ldots, N, \quad \Delta t = \frac{T}{N} \right\},$$

providing a discretisation of the interval $[0, T]$ and applied to problem (1), is given by

$$X_{n+1} = X_n + (1 - \theta)\Delta t(\mu X_n + \nu) + \theta \Delta t(\mu X_{n+1} + \nu) + (\sigma X_n + \tau)\Delta W_n, \tag{3}$$

for $\theta \in [0, 1]$. Here X_n represents the approximate value for $X(t_n)$ and ΔW_n is discretisation of the Wiener increment in the interval $[t_n, t_{n+1}]$. We observe

that, to make discretised Wiener increments computable in $\mathcal{I}_{\Delta t}$, usually Δt is taken proportional to the steplength h in concordance with the partition $\mathcal{I}_h$.

As proved in [19], the choice of the parameter $\theta \in [0,1]$ has a direct influence on the mean-square stability properties of the corresponding θ-Maruyama method (3). Indeed, for any $\theta \in [\frac{1}{2},1]$, the corresponding method is mean-square A-stable, i.e., it unconditionally preserves the mean-square stability properties of the problem. For $\theta \in [0,\frac{1}{2})$, this preservation occurs under a prescribed stepsize restriction.

In the context of SDEs with linear affine coefficients (1), in the following section we prove that condition (2) is well maintained along the numerical dynamics of θ-Maruyama methods (3), under a suitable stepsize restriction and, for some values of θ, the condition is preserved unconditionally.

2 Monotone Mean-Square Deviation

Let us apply the θ-Maruyama method (3) to the SDE with linear affine coefficients (1), for the approximation of two distinct solutions of (1), obtained with initial values $X_0 \neq Y_0$. Therefore, we obtain

$$X_{n+1} = X_n + (1-\theta)\Delta t(\mu X_n + \nu) + \theta \Delta t(\mu X_{n+1} + \nu) + (\sigma X_n + \tau)\Delta W_n,$$

$$Y_{n+1} = Y_n + (1-\theta)\Delta t(\mu Y_n + \nu) + \theta \Delta t(\mu Y_{n+1} + \nu) + (\sigma Y_n + \tau)\Delta W_n.$$

Subtraction of these two solutions yields

$$X_{n+1} - Y_{n+1} = \frac{1 + (1-\theta)\Delta t\mu + \sigma \Delta W_n}{1 - \theta \Delta t \mu}(X_n - Y_n),$$

and, passing to mean-square, gives

$$\mathbb{E}\left[\|X_{n+1} - Y_{n+1}\|^2\right] = \frac{(1 + (1-\theta)\Delta t\mu)^2 + \sigma^2 \Delta t}{(1-\theta\Delta t\mu)^2}\mathbb{E}\left[\|X_n - Y_n\|^2\right].$$

In other terms, focusing on a single step of the θ-Maruyama method, from the step point t_n to t_{n+1}, the numerical mean-square deviation is non-increasing if

$$\frac{(1 + (1-\theta)\Delta t\mu)^2 + \sigma^2 \Delta t}{(1-\theta\Delta t\mu)^2} \leq 1, \tag{4}$$

that is the condition of mean-square stability for stochastic θ-methods [12,19,20]. In summary, we have proved the following result.

Theorem 1. *For θ-Maruyama methods (3), monotonicity of the mean-square deviation of two distinct solutions of the SDE (1), obtained in correspondence to two distinct initial values, occurs over a single step as in (2) for all the values of the stepsize such that the method is mean-square stable.*

As a consequence of Theorem 1, we observe the following two cases.

Corollary 1. *For any* $0 \leq \theta < \frac{1}{2}$*, monotonicity of the mean-square deviation over a single step of* (3) *applied to* (1) *occurs as in* (2) *if the following stepsize restriction is satisfied*

$$\Delta t \leq -\frac{2\mu + \sigma^2}{(1-2\theta)\mu^2}. \tag{5}$$

Corollary 2. *For any* $\frac{1}{2} \leq \theta \leq 1$*, monotonicity of the mean-square deviation over a single step of* (3) *applied to* (1) *occurs as in* (2) *without any restriction on the stepsize.*

Proof. Condition (4) is equivalent to $f(\Delta t) \leq 0$, with

$$f(\Delta t) = 2\mu + \sigma^2 + (1-2\theta)\mu^2 \Delta t.$$

We observe that $f'(\Delta t) = (1-2\theta)\mu^2 \leq 0$, for any value of $\theta \in [\frac{1}{2}, 1]$, and $f(0) = 2\mu + \sigma^2$, which is negative based on the hypothesis of mean-square stability for the SDE. Therefore, we have $f(\Delta t) \leq 0$, for any Δt, if $\theta \in [\frac{1}{2}, 1]$, leading to the thesis. □

As regards the unconditional conservation of monotonicity given by Corollary 2, we observe that all mean-square A-stable θ-Maruyama methods preserve both mean-square monotonicity and mean-square deviations between two distinct solutions of the linear problem (1). In the next section, we will illustrate these results using two examples.

3 Numerical Experiments

The literature contains numerous applications of SDEs with linear affine coefficients. Despite their simplicity, these models represent a valuable contribution to their respective application areas, often providing strong motivation to analyse their numerical behaviour. Before presenting examples that connect them with our theoretical results, we briefly present some of the most interesting applications discussed in the literature. Among the most widely considered are geometric Brownian motion as well as arithmetic Brownian motion, used, for instance, to model precipitation [2] as well as various financial models, including the Ornstein-Uhlenbeck process, mean-reverting models, and stochastic volatility models [23,25]. We will present two examples to illustrate the aforementioned theory: geometric Brownian motion and a mean-reverting model.

3.1 Geometric Brownian Motion

Geometric Brownian motion (GBM) is given by the following equation [25]:

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t), \tag{6}$$

where μ is the drift and σ is the volatility.

The most common application of GBM is in modelling stock prices. Therefore, the numerical experiments presented below will be conducted with this application in mind. GBM is preferred over the Brownian motion model in financial applications since it cannot take negative values and maintains stock movements proportional to the value of the stock—a consequence of its exponential nature—thus making it a more appropriate model of price dynamics. It is worth pointing out that, despite its simplicity and usefulness, this model is valid only over short time intervals, since it implies that the stock price over time follows a normal distribution, with mean μt and variance $\sigma^2 t$. It has been thoroughly investigated that stock price behaviour over long time does not follow a normal distribution, but rather a lognormal distribution [26].

In the context of financial applications, the modelled variable $X(t)$ represents the current asset price, μ is the expected return (usually represented as a percentage value) and σ is the standard deviation of returns [26]. To satisfy the monotonicity condition derived in the theoretical result, the drift must be negative. This special case will corresponds to negative market growth or bankruptcy. However, according to the properties of GBM, even though a negative drift drives the asset price downward, it will not fall below zero, as observed in real markets. To remain faithful to the model's statistical interpretation, volatility should not be assigned a negative value. Finally, when choosing parameters, it is important to ensure that the condition $2\mu + \sigma < 0$ is satisfied.

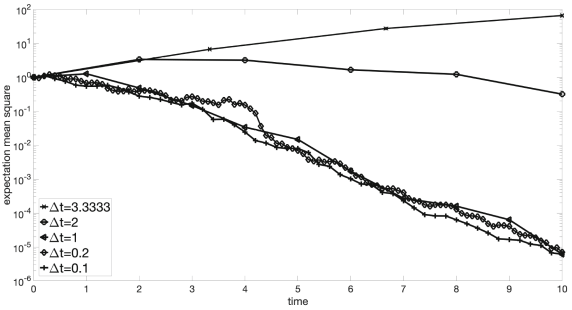
The parameters used in the numerical experiments are as follows: $\mu = -0.7$, $\sigma = 0.5$. In Fig. 1, the behaviour of the mean-square deviations is illustrated for three methods that are not A-stable, for $\theta = 0$, 0.2, 0.3, and hence have an upper bound on the timestep which, when surpassed, results in a blow-up. Under each subfigure, the method and the corresponding upper bound are indicated. In Fig. 1a, corresponding to Euler-Maruyama method, monotone behaviour of the mean-square deviation is observed for $\Delta t = 2$, since it is below the monotonicity bound 2.34. However, a blow-up occurs for $\Delta t = 3.33$ as this value goes over the bound. In Fig. 1b, corresponding to the method for $\theta = 0.2$, the value of $\Delta t = 3.33$ is below the monotonicity bound 3.91, but we can notice that even though no blow-up is observed, the obtained numerical deviation fails to exhibit the monotonicity characterizing the underlying exact dynamics. A similar situation is evident for the same timestep in Fig. 1c for $\theta = 0.3$. Overall, an improvement in monotonicity properties is observed as the value of θ increases.

In Fig. 2, by contrast, mean-square contractivity is demonstrated for three A-stable methods with $\theta = 0.5$, 0.7, 1. These methods are unconditionally monotone for all general linear SDEs, and therefore impose no restriction on the timestep.

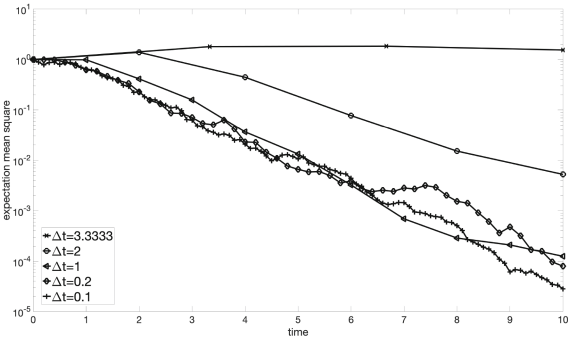
3.2 Mean Reverting Model

The mean-reverting model (MRM) is given by the following equation [25]:

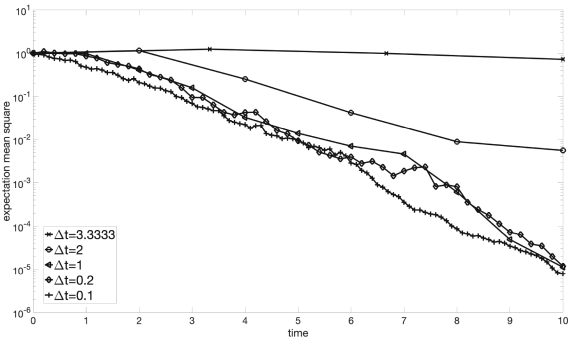
$$\mathrm{d}S(t) = r(\mu - S(t))\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \tag{7}$$



(a) $\theta = 0$; $\Delta t < 2.34$

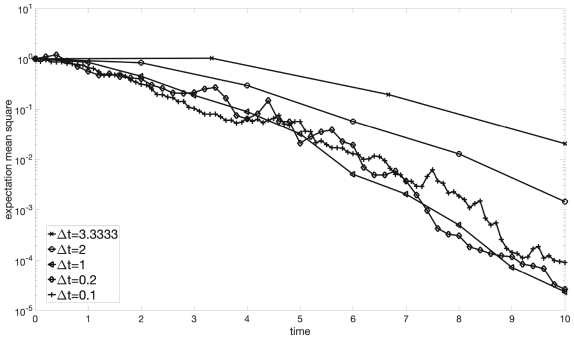


(b) $\theta = 0.2$; $\Delta t < 3.91$

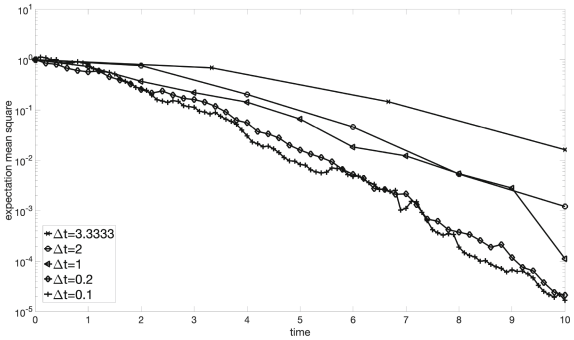


(c) $\theta = 0.3$; $\Delta t < 5.86$

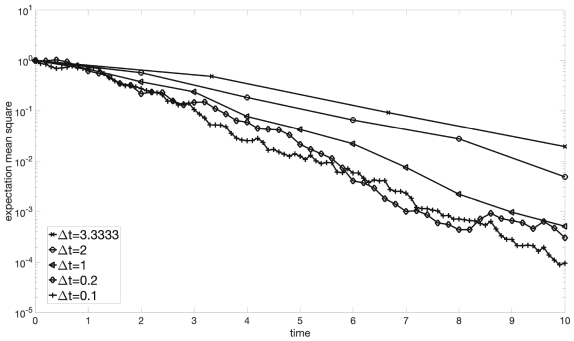
Fig. 1. Mean-square deviations associated to θ-Maruyama method (3) applied to GBM problem (6), for selected values of θ corresponding to methods that are not mean-square A-stable, emphasising the timestep restriction for each method to preserve monotonicity



(a) $\theta = 0.5$

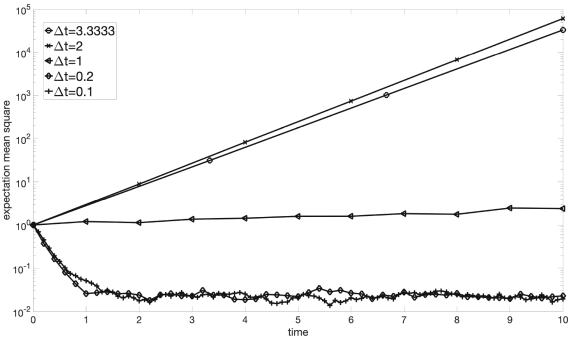


(b) $\theta = 0.7$

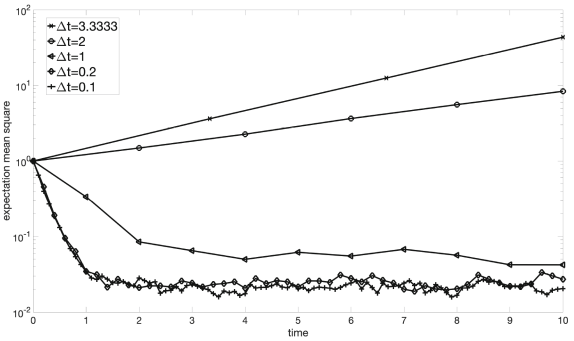


(c) $\theta = 1$

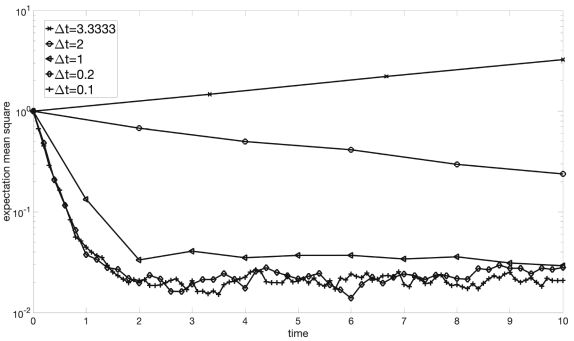
Fig. 2. Mean-square deviations associated to θ-Maruyama method (3) applied to GBM problem (6), for selected values of θ corresponding to methods that are mean-square A-stable



(a) $\theta = 0; \Delta t < 0.99$

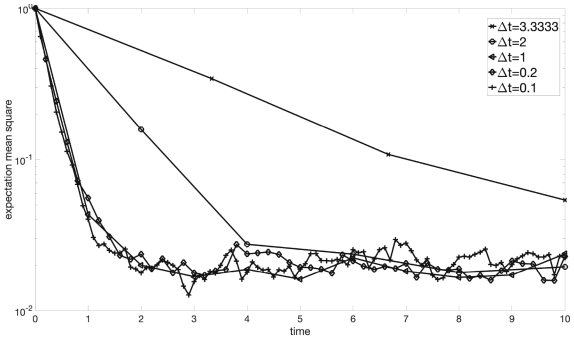


(b) $\theta = 0.2; \Delta t < 1.65$

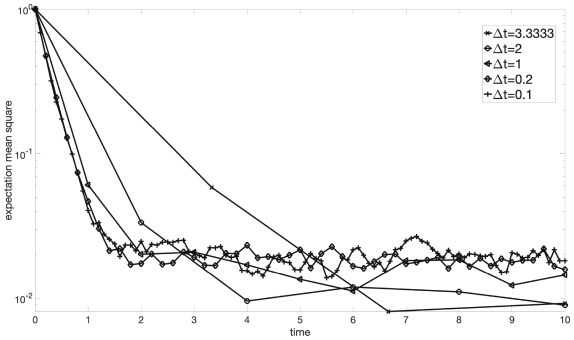


(c) $\theta = 0.3; \Delta t < 2.47$

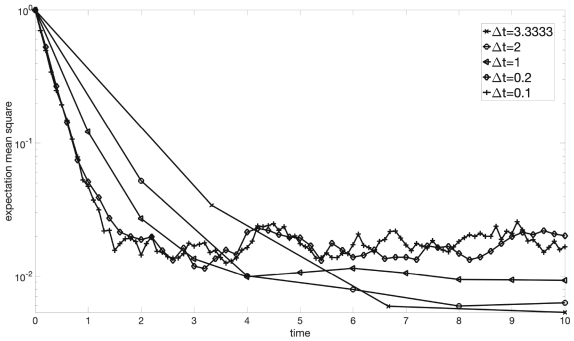
Fig. 3. Mean-square deviations associated to θ-Maruyama method (3) applied to MRM problem (7), for selected values of θ corresponding to methods that are not mean-square A-stable, emphasising the timestep restriction for each method to preserve monotonicity



(a) $\theta = 0.5$



(b) $\theta = 0.7$



(c) $\theta = 1$

Fig. 4. Mean-square deviations associated to θ-Maruyama method (3) applied to MRM problem (7), for selected values of θ corresponding to methods that are mean-square A-stable

where r is the reversion rate, μ is the long-run mean price and σ is the volatility as before.

This model is similar to GBM, but it additionally accounts for mean reversion, introduced through the reversion parameter r. Following the stochastic dynamics described previously, it includes a tendency for the process to revert toward the its mean. It is typically used to model interest rate dynamics.

To explain the role of reversion, we first examine the behaviour of the model. By construction, it is evident that when $S(t)$ rises above (or falls below) the mean value μ, the drift function becomes negative (or positive), causing $\mathrm{d}S(t)$ more likely to become negative (or positive) and therefore leading $S(t)$ to decrease (or increase) [25]. As the dynamics evolve, $S(t)$ will repeatedly revert toward the mean, but the previous deviations from the mean, known as excursions, are going to occur each time the process returns to the mean. The duration of excursions is governed by the reversion rate r [23].

The parameters used in the numerical experiments are as follows: $r = 2$, $\mu = 1$, $\sigma = 0.2$. Figure 3, illustrates the behaviour of mean-square deviations for three methods that are not A-stable with $\theta = 0,\ 0.2,\ 0.3$, and hence exhibit an upper bound on the timestep, beyond which numerical blow-up occurs. Below each subfigure, the method and the corresponding upper bound are indicated. We observe that, for decreasing values of Δt, the methods exhibit a monotone decrease of the mean-square deviation between two solutions up to a plateau, due to accuracy demands. For each subfigure, similar considerations to those discussed in the previous example apply with respect to both the blow-up behaviour and the monotone decay.

In Fig. 4, by contrast, monotonicity is demonstrated for three A-stable methods with $\theta = 0.5,\ 0.7,\ 1$. These methods are unconditionally monotone for all general linear SDEs, implying that the time step is unrestricted and the mean-square deviations remain monotone. As demonstrated for all three methods, this monotone behaviour is consistently preserved, along with the characteristic shape of the decay.

4 Conclusions

We have analysed the ability of stochastic θ-methods to provide monotone mean square deviations between two solutions of SDEs with linear affine drift and diffusion coefficients (1), obtained with two distinct initial values. The investigation revealed that the conservation of the monotone character of such mean-square deviations is given in terms of stepsize restrictions when $\theta < \frac{1}{2}$ and it occurs unconditionally when $\theta \geq \frac{1}{2}$. Moreover, the analysis revealed that all stepsize values ensuring mean-square stability also satisfy the monotonicity condition (2). This issue has been confirmed by selected numerical experiments.

Future developments of this research may establish the connection between analogous stability and monotonicity properties of θ-methods also for systems of SDEs, involving spectral or pseudospectral properties of the coefficient matrices. The results may also be extended to other evolution problems, also regarding

non-local problems with memory or stiff problems arising from the spatial discretization of reaction-diffusion partial differential equations [1,8–11,16,18,30].

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