Pascal Trapezoids as Wigner Numbers and Some Combinatorial Properties

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Abstract. Despite their entirely different origins, the entries of Pascal trapezoids are closely related to Wigner numbers. In this paper, we bridge both approaches, rediscover recurrences using alternative techniques, and establish new combinatorial identities.

Keywords: Pascal trapezoids · Wigner numbers · Combinatorial identities

1 Introduction

A few years ago, some of the authors of this paper generalized the concept of Appell polynomials to the hypercomplex context (cf. [5]), resulting in a class of multidimensional polynomials that can be expressed in various forms using hypercomplex variables. Later, in [3], the authors investigated bijections between the coefficients of two different representations of these generalized Appell polynomials. In that study, they introduced the specific class of integers (Theorems 6 and 7),

$$\sigma_{i,j}^{k} = \sum_{m=0}^{i} (-1)^m \binom{i}{m} \binom{k-i}{j-m}, \; k = 0, 1, 2, \ldots; j = 0, \ldots, k, \tag{1}$$

that was shown to possess interesting properties, warranting a more in-depth study of its own, which was initiated in that paper. It was proven that these numbers satisfy a Pascal-like relation, forming arrangements in trapezoidal arrays, with the classical Pascal triangle emerging as a particular case within this collection. Moreover, the Catalan triangle, as discussed by Miana et al. [9], is also included. Several combinatorial properties of those numbers were proven, regarding symmetries, closed-form sums, connections to well-known numbers such as the central binomial coefficient as well as a generating function. Closely related numbers appeared later in the theory of angular momentum in quantum

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mechanics, particularly in the development of efficient evaluation methods for the reduced Wigner matrices (cf. [1]). Indeed, for the rotations of particles and systems the so-called Wigner rotation matrices

$$D^J_{M,M'}(\alpha,\beta,\gamma) = e^{-\mathrm{i}M\alpha} d^J_{M,M'}(\beta) e^{-\mathrm{i}M'\gamma} \tag{2}$$

play a central role. Here, (α,β,γ) are the Euler rotation angles, the index $J \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots\}$ stands for the angular momentum and (M, M') are the Z-components of the angular momentum, M, $M' = -J, -J+1, \ldots, J-1, J$. The calculation of the Wigner reduced matrices $d^J_{M,M'}(\beta)$ presents several challenges. Although explicit formulas and recurrence relations were known, as observed by Allen [1] none offered a completely satisfactory method for determining the reduced Wigner rotation matrices for large quantum numbers J. Aiming to establish an efficient and numerically stable method for arbitrary J, Allen noticed the emergence of the set of numbers (cf. [1, Eq. 97])

$$W^J_{m,n} = \sum_{\sigma=\max\{0,m-n\}}^{\min\{m,2J-n\}} (-1)^{m+\sigma} \binom{2J-n}{\sigma} \binom{n}{m-\sigma}, \; m, n = 0, 1, 2, \ldots, 2J,$$

that he called Wigner numbers. He recognized and explored a myriad of combinatorial properties of these numbers in the paper.

As it was noticed in [2, Eq. 2], this formulation is equivalent to

$$W^J_{m,n} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{2J-n}{m-i}, \tag{3}$$

for $J \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots\}$, $m, n \in [0, 2J]$, establishing a connection with the entries of the Pascal trapezoids (1). Also, in [2], the class of Wigner numbers was extended to an ℓ parameter-dependent class of numbers, called ℓth order Wigner numbers, of which the entries of Pascal trapezoids are of first order and the numbers introduced by Allen are of second order.

Although these sets of numbers emerged in two such distinct contexts, they were recognized in both cases as having intrinsic importance due to their interesting combinatorial properties and as worthy of deeper investigation beyond the context in which they arose. In this paper, for the sake of clarity, we begin by revisiting some properties previously established in [3] and introducing new properties of the Wigner numbers of the first order. These new properties include additional recurrence formulas that relate entries in consecutive rows of each trapezoid, as well as consecutive entries within the same row. Some of these properties exhibit structural similarities to those obtained by Allen for the Wigner numbers of the second order, whereas others have not been explored before.

The paper is organized as follows. In Sect. 2, we recall the extension of the Wigner numbers as defined in [2] and identify the Pascal trapezoids (1) as Wigner numbers of the first order, expressing them in terms of Gauss' hypergeometric function. Section 3 focuses on recurrence relations, some of which were previously established in [3], while others resemble those obtained by Allen in [1], but

are derived using different techniques. Finally, in Sect. 4, we explore additional combinatorial properties, including symmetries, closed summation formulas, and central coefficients of the Pascal trapezoids.

2 The Entries of Pascal Trapezoids as Wigner Numbers

Following [2], we consider the extension of the Wigner numbers to the ℓ-parameter class defined by

$$W_{m,n}^{J,\ell} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{\ell J - n}{m - i},$$

for $\ell \in \mathbb{N}$, $J \geq 0$ and $m, n \in [0, \ell J]$.

In this view, the integers (3) are 2nd order Wigner numbers, while the entries of the Pascal trapezoids (1) are 1st order Wigner numbers, by replacing (i, j, k) by (n, m, J), for $\ell = 1$.

As mentioned previously, the numbers in equation (1) emerged from the connection between two distinct representations of a multidimensional polynomial sequence, as briefly outlined below. By extending the concept of Appell polynomials within the framework of hypercomplex function theory in $\mathbb{R}^{n+1}$, the following infinite array of numbers arises as coefficients [5]:

$$T_s^k(n) = \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k}, \ n, k = 1, 2, \ldots, ; s = 0, 1, \ldots, k,$$

where $(a)_r := a(a+1)\ldots(a+r-1)$, for any integer $r \geq 1$ is the Pochhammer symbol with $(a)_0 := 1, a \geq 0$. In [6], various properties of these numbers were studied, namely their relationship with the generalized Vietoris numbers (cf. [4]),

$$c_k(n) = \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{\left(\frac{n}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}, \ n \in \mathbb{N}, \ k = 0, 1, 2, \ldots,$$

that can be given by

$$c_k(n) = \sum_{s=0}^{k} (-1)^k T_s^k(n).$$

Later, new relations between $T_s^k(n)$ and $c_k(n)$ were explored, giving rise to (cf. [3], Theorems 6 and 7)

$$\frac{2^k}{\binom{k}{s}} T_s^k(n) = \sum_{j=0}^{k} \sigma_{s,j}^k c_j(n)$$

and

$$\binom{k}{j} c_{k-j}(n) = \sum_{s=0}^{k} (-1)^s \sigma_{s,j}^k T_s^k(n), \ k = 0, 1, 2, \ldots; \ j = 0, 1, \ldots, k,$$

where the coefficients $\sigma_{s,j}^k$ are precisely the 1st order Wigner numbers (1).

We highlight that for arbitrary nonnegative integers i, j, k with $j \leq k$, the numbers $\sigma_{i,j}^k$ can be arranged, for each fixed i, in a triangular array. In this arrangement, the rows associated with k ($k = 0, 1, \dots$) are ordered from $j = 0$ to $j = k$ (see Table 1). The case $i = 0$ corresponds to the ordinary Pascal triangle because $\sigma_{0,j}^k = \binom{k}{j}$. In the other cases we obtain trapezoids, where for each row, the Pascal recurrence is satisfied, justifying the name of Pascal trapezoids (cf. Theorem 1 below). We remark that the case $i = 1$ corresponds to the Catalan triangle numbers (cf. [9]).

Similarly, for each fixed j, $\sigma_{i,j}^k$, can also be expressed as

$$\sigma_{i,j}^k = \sum_{m=0}^{j} (-1)^m \binom{i}{m}\binom{k-i}{j-m}, \tag{4}$$

because $\binom{k-i}{j-m} = 0$, for $m > j$, while $\binom{i}{m} = 0$ when $m > i$. In Table 2, these numbers are displayed as an arithmetic triangle, ordered from $i = 0$ to $i = k$.

The use of appropriate properties of the Pochhammer symbol, enables to express the Wigner numbers of 1st order (1) in terms of the Gauss' hypergeometric function

$${}_2F_1(a, b; c; z) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s s!} z^s,$$

as

$$\sigma_{i,j}^k = \binom{k-i}{j} {}_2F_1(-i, -j; k-i-j+1; -1). \tag{5}$$

Remark 1. We remark that representations of the Wigner numbers of 1st order in terms of certain values of Jacobi polynomials can be easily establish, considering (5) and known formulas for the Jacobi polynomials in terms of the hypergeometric function. For instance, the following representation of the Jacobi polynomials,

$$P_n^{(\alpha,\beta)}(x) = \left(\tfrac{x+1}{2}\right)^n \tfrac{(n+\alpha)!}{n!\alpha!} {}_2F_1\left(-n, -n-\beta; \alpha+1; \tfrac{x-1}{x+1}\right),$$

evaluated at $x = 0$ with (n, α, β) as $(i, k-i-j, j-i)$, yields

$$\sigma_{i,j}^k = 2^i \frac{\binom{k}{j}}{\binom{k}{i}} P_i^{(k-i-j,j-i)}(0).$$

3 Recurrence Relations

In this section, we present different ways to obtain $\sigma_{i,j}^k$ recursively, either by fixing the value of i or by fixing the value of j.

Table 1. Trapezoids associated with $\sigma_{i,j}^{k}$, $(i = 0, \ldots, 3)$

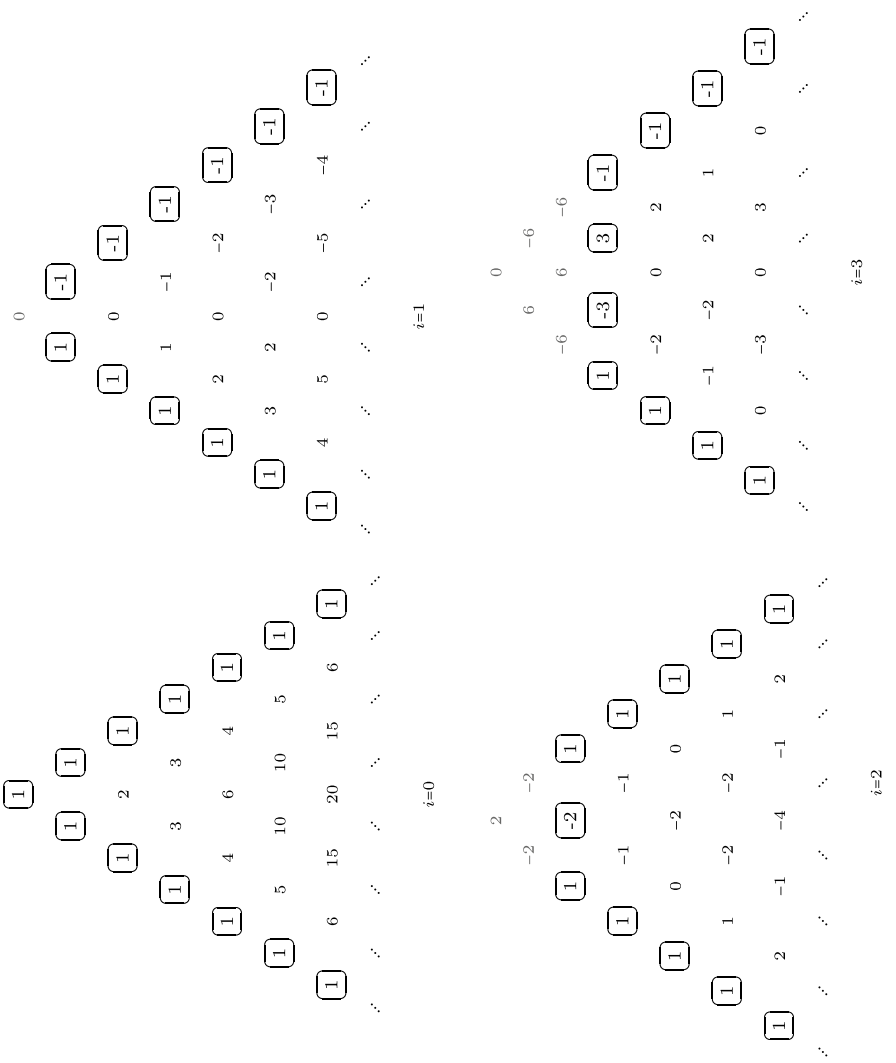
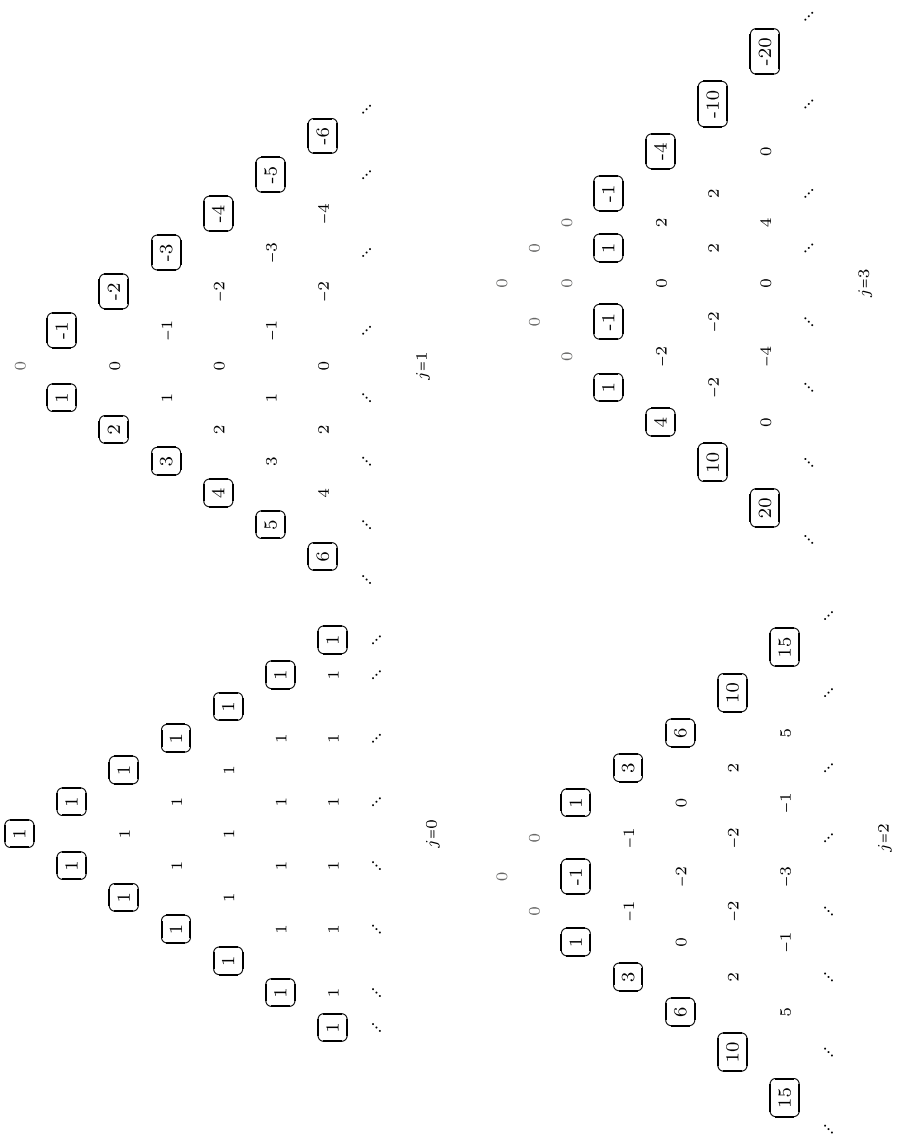


Table 2. Trapezoids associated with $\sigma_{i,j}^k$, $(j = 0, \ldots, 3)$



Theorem 1 (Pascal-like recurrences).

(i) *For each fixed value of i, the numbers $\sigma_{i,j}^k$ satisfy the following linear recurrence relation*

$$\sigma_{i,j+1}^{k+1} = \sigma_{i,j}^k + \sigma_{i,j+1}^k, \ (0 \le j \le k-1, k \ge i) \tag{6}$$

with boundary conditions

$$\sigma^k_{i,0} = 1, \qquad \sigma^k_{i,k} = (-1)^i, \ (k \geq i) \tag{7}$$

and initial values

$$\sigma^i_{i,j} = \binom{i}{j}(-1)^j, \ j = 1, \ldots, i-1 \tag{8}$$

(cf. [3, Theorem 10]).

(ii) *For each fixed value of j, the numbers $\sigma^k_{i,j}$ satisfy the following linear recurrence relation*

$$\sigma^{k+1}_{i+1,j} = 2\,\sigma^k_{i,j} - \sigma^{k+1}_{i,j}, \ (0 \leq i \leq k, \ k \geq j) \tag{9}$$

with boundary conditions

$$\sigma^k_{0,j} = \binom{k}{j}, \qquad \sigma^k_{k,j} = (-1)^j\binom{k}{j}, \ (k \geq j) \tag{10}$$

and initial values

$$\sigma^j_{i,j} = (-1)^i, \ i = 1, \ldots, j-1. \tag{11}$$

Proof.

(i) Recurrence (6) was obtained in Theorem 10 of [3].

(ii) Considering (5), we get for the numbers $\sigma^{k+1}_{i+1,j}$ and $\sigma^{k+1}_{i,j}$ the form

$$\sigma^{k+1}_{i+1,j} = \binom{k-i}{j}{}_2F_1(-i-1, -j; k-i-j+1; -1),$$
$$\sigma^{k+1}_{i,j} = \binom{k-i+1}{j}{}_2F_1(-i, -j; k-i-j+2; -1),$$

respectively.

The Gauss's relation for contiguous functions (cf. [10]),

$$c\ (1-z)\ {}_2F_1(a, b; c; z) - c\ {}_2F_1(a-1, b; c; z) + (c-b)z\ {}_2F_1(a, b; c+1; z) = 0,$$

with $z = -1$, leads to

$$c\ {}_2F_1(a-1, b; c; -1) = 2c\ {}_2F_1(a, b; c; -1) - (c-b)\ {}_2F_1(a, b; c+1; -1).$$

Setting $a = -i, b = -j$, and $c = k-i-j+1$, we obtain

$$(k-i-j+1)\ {}_2F_1(-i-1, -j; k-i-j+1; -1) =$$
$$2(k-i-j+1)\ {}_2F_1(-i, -j; k-i-j+1; -1) - (k-i+1)\ {}_2F_1(-i, -j; k-i-j+2; -1).$$

The result follows after multiplying both sides by $\frac{1}{k-i-j+1}\binom{k-i}{j}$. □

The following result, derived as a consequence of the latest theorem, corresponds to that presented in [1] for the 2nd order Wigner numbers, using different arguments (see [1, Eq. 66]).

Corollary 1. *The numbers* $\sigma_{i,j}^{k}$ *satisfy the relation*

$$\sigma_{i+1,j}^{k+1} = \sigma_{i,j}^{k} - \sigma_{i,j-1}^{k}, \ (0 \le i, j \le k)$$

with boundary conditions (7)-(10), *initial values* (8)-(11), *and* $\sigma_{i,-1}^{k} = 0$.

Proof. The result follows immediately from (6) and (9), since

$$\sigma_{i+1,j}^{k+1} = 2\sigma_{i,j}^{k} - (\sigma_{i,j-1}^{k} + \sigma_{i,j}^{k}) = \sigma_{i,j}^{k} - \sigma_{i,j-1}^{k}.$$

□

While the recurrences in Theorem 1 connect three elements from two consecutive rows of trapezoids (see Table 1 for fixed i and Table 2 for fixed j), the following results establish three-term recurrence relationships among elements within the same row.

Theorem 2. (i) *For each fixed value of* i, *the numbers* $\sigma_{i,j}^{k}$ *satisfy the following relation*

$$(j+1)\sigma_{i,j+1}^{k} + (2i-k)\sigma_{i,j}^{k} + (k-j+1)\sigma_{i,j-1}^{k} = 0, \tag{12}$$

$0 \le j \le k-1, \ k \ge i, \sigma_{i,0}^{k} = 1,$ *and* $\sigma_{i,-1}^{k} = 0$.

(ii) *For each fixed value of* j, *the numbers* $\sigma_{i,j}^{k}$ *satisfy the following relation*

$$(k-i)\sigma_{i+1,j}^{k} + (2j-k)\sigma_{i,j}^{k} + i\sigma_{i-1,j}^{k} = 0, \tag{13}$$

$0 \le i \le k-1, \ k \ge j,$ *and* $\sigma_{0,j}^{k} = \binom{k}{j}$.

Proof.

(i) Considering (5), we can write

$$\sigma_{i,j+1}^{k} = \binom{k-i}{j+1} \, {}_2F_1(-i, -j-1; k-i-j; -1),$$
$$\sigma_{i,j-1}^{k} = \binom{k-i}{j-1} \, {}_2F_1(-i, -j+1; k-i-j+2; -1),$$

and taking into account the well-known linear transformation (cf. [10]),

$${}_2F_1(a, b; c; z) = (1-z)^{-a} \, {}_2F_1\left(a, c-b; c; \tfrac{z}{z-1}\right),$$

we get, after making the appropriate substitutions, that

$${}_2F_1\left(-i, -j-1; k-i-j; -1\right) = 2^i \, {}_2F_1\left(-i, k-i+1; k-i-j; \tfrac{1}{2}\right),$$
$${}_2F_1\left(-i, -j; k-i-j+1; -1\right) = 2^i \, {}_2F_1\left(-i, k-i+1; k-i-j+1; \tfrac{1}{2}\right),$$
$${}_2F_1\left(-i, -j+1; k-i-j+2; -1\right) = 2^i \, {}_2F_1\left(-i, k-i+1; k-i-j+2; \tfrac{1}{2}\right).$$

Combining the previous relations, and the combinatorial identities

$$\binom{k-i}{j+1} = \frac{k-i-j}{j+1}\binom{k-i}{j} \quad \text{and} \quad \binom{k-i}{j-1} = \frac{j}{k-i-j+1}\binom{k-i}{j},$$

it follows that

$$(j+1)\sigma^k_{i,j+1} + (2i-k)\sigma^k_{i,j} + (k-j+1)\sigma^k_{i,j-1} = 2^i\binom{k-i}{j}\mathcal{A}^k_{i,j},$$

where

$$\begin{aligned}\mathcal{A}^k_{i,j} = (k-i-j)\,{}_2F_1\left(-i,k-i+1;k-i-j;\tfrac{1}{2}\right)\\ + (2i-k)\,{}_2F_1\left(-i,k-i+1;k-i-j+1;\tfrac{1}{2}\right)\\ + \tfrac{(k-j+1)j}{k-i-j+1}\,{}_2F_1\left(-i,k-i+1;k-i-j+2;\tfrac{1}{2}\right).\end{aligned}$$

Using the contiguity relation between hypergeometric functions (cf. [10])

$$\begin{aligned}c(c-1)(z-1)\ {}_2F_1(a,b;c-1;z) + c(c-1-(2c-a-b-1)z)\ {}_2F_1(a,b;c;z)\\ + (c-a)(c-b)z\ {}_2F_1(a,b;c+1;z) = 0,\end{aligned}$$

with $z = \frac{1}{2}$, i.e.,

$$\begin{aligned}c(c-1){}_2F_1(a,b;c-1;\tfrac{1}{2}) + c(-a-b+1){}_2F_1(a,b;c;\tfrac{1}{2})\\ + (c-a)(b-c){}_2F_1(a,b;c+1;\tfrac{1}{2}) = 0, \quad (14)\end{aligned}$$

and substituting $a = -i, b = k-i+1, c = k-i-j+1$, we get $\mathcal{A}^k_{i,j} = 0$, and the result follows.

(ii) Analogously to the proof of (i), this time using the linear transformation (cf. [10]),

$${}_2F_1(a,b;c;z) = (1-z)^{-b}\,{}_2F_1\left(c-a,b;c;\tfrac{z}{z-1}\right),$$

with appropriate substitutions, we obtain

$$\begin{aligned}{}_2F_1(-i-1,-j;k-i-j;-1) &= 2^j\,{}_2F_1(k-j+1,-j;k-i-j;\tfrac{1}{2}),\\ {}_2F_1(-i,-j;k-i-j+1;-1) &= 2^j\,{}_2F_1(k-j+1,-j;k-i-j+1;\tfrac{1}{2}),\\ {}_2F_1(-i+1,-j;k-i-j+2;-1) &= 2^j\,{}_2F_1(k-j+1,-j;k-i-j+2;\tfrac{1}{2}).\end{aligned}$$

Bearing in mind (5),

$$\begin{aligned}\sigma^k_{i+1,j} &= \binom{k-i-1}{j}{}_2F_1(-i-1,-j;k-i-j;-1),\\ \sigma^k_{i-1,j} &= \binom{k-i+1}{j}{}_2F_1(-i+1,-j;k-i-j+2;-1),\end{aligned}$$

and the identities

$$\binom{k-i-1}{j} = \frac{k-i-j}{k-i}\binom{k-i}{j} \quad \text{and} \quad \binom{k-i+1}{j} = \frac{k-i+1}{k-i-j+1}\binom{k-i}{j},$$

it follows that

$$(k-i)\sigma^k_{i+1,j} + (2j-k)\sigma^k_{i,j} + i\sigma^k_{i-1,j} = 2^j \binom{k-i}{j} \mathcal{B}^k_{i,j},$$

where

$$\begin{aligned}\mathcal{B}^k_{i,j} = (k-i-j)\,{}_2F_1\left(k-j+1,-j;k-i-j;\tfrac{1}{2}\right) \\ + (2j-k)\,{}_2F_1\left(k-j+1,-j;k-i-j+1;\frac{1}{2}\right) \\ + \frac{i(k-i+1)}{k-i-j+1}\,{}_2F_1\left(k-j+1,-j;k-i-j+2;\tfrac{1}{2}\right).\end{aligned}$$

Using (14) with $a = k-j+1$, $b = -j$, and $c = k-i-j+1$, we obtain $\mathcal{B}^k_{i,j} = 0$, thus proving the statement. □

The recurrence relations (12) and (13) correspond, respectively, to Eqs. 27 and 55 of [1], which were derived in the context of calculating unknown quantities (the 2nd order Wigner numbers), involved in the reduced Wigner matrices $d^J_{M,M'}(\beta)$ of rotation matrices (2).

4 Other Properties

In this section, we explore and prove additional properties linking the numbers in the previously introduced trapezoids, uncovering deeper connections between various combinatorial structures.

Property 1 (Patterns).

(i) Trapezoids of $i-$ and $j-$even order are symmetric, i.e.,

$$\sigma^k_{2i,j} = \sigma^k_{2i,k-j} \qquad \text{and} \qquad \sigma^k_{i,2j} = \sigma^k_{k-i,2j}; \tag{15}$$

(ii) Trapezoids of of $i-$ and $j-$odd order are anti-symmetric, i.e.,

$$\sigma^k_{2i+1,j} = -\sigma^k_{2i+1,k-j} \qquad \text{and} \qquad \sigma^k_{i,2j+1} = -\sigma^k_{k-i,2j+1}. \tag{16}$$

Proof. The identities on the left hand side of (15) and (16) come, respectively, from Property 11 (i) and (ii) derived in [3]. Using the identity (4), one can write

$$\sigma^k_{k-i,j} = \sum_{m=0}^{j} (-1)^m \binom{k-i}{m}\binom{i}{j-m} = \sum_{m=0}^{j} (-1)^{j-m} \binom{k-i}{j-m}\binom{i}{m} = (-1)^j \sigma^k_{i,j}$$

and the result is proved. □

In the next property, we present the central coefficients of the trapezoid of order i and j. These coefficients exhibit a distinct behavior depending on whether the indices i and j are even or odd. Specifically, we derive the explicit expressions for $\sigma_{i,k}^{2k}$ and $\sigma_{k,j}^{2k}$ based on the parity of i and j, revealing their dependency on binomial coefficients and alternating signs.

Property 2 (Central Coefficients). The central coefficients of the trapezoid of order i and j are given, respectively, by

$$\sigma_{i,k}^{2k} = \begin{cases} 0, & \text{if } i \text{ is odd;} \\ (-1)^{\frac{i}{2}} \dfrac{\binom{k}{\frac{i}{2}}}{\binom{2k}{i}} \binom{2k}{k}, & \text{if } i \text{ is even.} \end{cases}$$

and

$$\sigma_{k,j}^{2k} = \begin{cases} 0, & \text{if } j \text{ is odd;} \\ (-1)^{\frac{j}{2}} \binom{k}{\frac{j}{2}}, & \text{if j is even.} \end{cases}$$

Proof. The expression for $\sigma_{i,k}^{2k}$ was derived as Property 12 in [3]. The expression for $\sigma_{k,j}^{2k}$ follows from the expansion of $(1-x)^k(1+x)^k$ (cf. [7, Formula 1.19]) by taking identity (4). □

Next result presents summation formulas for the numbers in the trapezoids, including both the sum and alternating sum of the rows. These formulas depend on the parity of the indices and provide explicit expressions for different cases.

Property 3 (Summation formulas).

(i) Sum of the rows:

$$\sum_{j=0}^{k} \sigma_{i,j}^{k} = 0, \qquad (k \geq i > 0); \tag{17}$$

$$\sum_{i=0}^{k} \sigma_{i,j}^{k} = \begin{cases} 0, & \text{if } j \text{ is odd;} \\ \binom{k+1}{j+1}, & \text{if } j \text{ is even.} \end{cases}, \qquad (k \geq j > 0). \tag{18}$$

(ii) Alternating sum of the rows:

$$\sum_{j=0}^{k} (-1)^j \sigma_{i,j}^{k} = 0, \qquad (k > i); \tag{19}$$

$$\sum_{i=0}^{k} (-1)^i \sigma_{i,j}^{k} = \begin{cases} 0, & \text{if } k-j \text{ is odd;} \\ \binom{k+1}{j}, & \text{if } k-j \text{ is even.} \end{cases}, \qquad (k > j).$$

Proof.

(i) Relation (17) is the identity obtained in Property 13 (i) of [3]. To prove (18), observe that

$$\sum_{i=0}^{k} \sigma_{i,j}^{k} = \sum_{i=0}^{k} \sum_{m=0}^{i} (-1)^m \binom{i}{m}\binom{k-i}{j-m} = \sum_{m=0}^{k} (-1)^m \sum_{i=0}^{k} \binom{i}{m}\binom{k-i}{j-m}.$$

Since $\binom{k-i}{j-m} = 0$, when $m > j$, one can write

$$\sum_{i=0}^{k} \sigma_{i,j}^{k} = \sum_{m=0}^{j} (-1)^m \sum_{i=0}^{k} \binom{i}{m}\binom{k-i}{j-m}.$$

Using now relation (5.26) of [8] we obtain

$$\sum_{i=0}^{k} \sigma_{i,j}^{k} = \sum_{m=0}^{j} (-1)^m \binom{k+1}{j+1} = \binom{k+1}{j+1} \sum_{m=0}^{j} (-1)^m$$

and the result follows at once.

(ii) Relation (19) is the identity obtained in Property 13 (ii) of [3]. Using the arguments of (i), we can write

$$\sum_{i=0}^{k} (-1)^i \sigma_{i,j}^{k} = \sum_{m=0}^{j} (-1)^m \sum_{i=0}^{k} (-1)^i \binom{i}{m}\binom{k-i}{j-m}.$$

Since $\binom{i}{m} = 0$, for $i < m$ and $\binom{k-i}{j-m} = 0$, when $i > k-j+m$, we can rewrite the inner sum as

$$\sum_{i=m}^{k-j+m} (-1)^i \binom{i}{m}\binom{k-i}{j-m} = \sum_{i=0}^{k-j} (-1)^{i+m} \binom{i+m}{m}\binom{k-i-m}{j-m}.$$

Substituting this into the original sum and rearranging the order of summation, we obtain

$$\begin{aligned}\sum_{i=0}^{k} (-1)^i \sigma_{i,j}^{k} &= \sum_{m=0}^{j} \sum_{i=0}^{k-j} (-1)^i \binom{i+m}{m}\binom{k-i-m}{j-m} \\ &= \sum_{i=0}^{k-j} (-1)^i \sum_{m=0}^{j} \binom{i+m}{i}\binom{k-i-m}{k-i-j}.\end{aligned}$$

Finally, applying identity (5.26) of [8], we conclude that

$$\sum_{i=0}^{k} (-1)^i \sigma_{i,j}^{k} = \sum_{i=0}^{k-j} (-1)^i \binom{k+1}{k-j+1} = \binom{k+1}{j} \sum_{i=0}^{k-j} (-1)^i$$

and the final result follows. □

5 Conclusion

As is often the case in mathematics, problems that were once considered completely independently and without knowledge of their practical application can also turn out to be practically significant.

The results obtained a few years ago in [3] for purely mathematical reasons in the search for intrinsic combinatorial connections between two different sets of coefficients have been supplemented by new ones, inspired by their connection to the Wigner numbers.

Although some of the results in the present paper correspond to rediscoveries of the "intriguing properties" identified by Allen [1], other cases establish entirely new identities. Regardless of the context or tools used to derive them, these properties are inherently interesting, revealing valuable connections that merit further exploration.

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