Symbolic and Numerical Computation of Coquaternionic Functions in *Mathematica*

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Abstract. This paper presents an updated version of the *Mathematica* package `Coquaternions`, originally introduced by the authors in 2019. The new release extends the package's functionality by incorporating the definition and manipulation of elementary functions within the algebra of coquaternions. In addition to their implementation in *Mathematica*, several properties of these functions are formally introduced and proved.

Keywords: Coquaternions · Elementary Functions · Computer Algebra System

1 Introduction

Coquaternions, also known as split quaternions, form a number system introduced by James Cockle in 1849 [5] as a generalization of complex numbers. Over time, they have attracted increasing attention and have found applications in various fields, including geometry and physics [1–4,11–14,16–19].

In 2019, the authors of [9] introduced a *Mathematica* add-on application, `Coquaternions`, designed to define rules for the symbolic manipulation of coquaternions. Later, in [8], the package functionalities have been extended with a collection of functions, `CoqPolynomial`, for working with polynomials. This paper aims to further update the `Coquaternions` package by incorporating a class of functions of a coquaternionic variable, particularly some elementary functions.

The paper is organized as follows. Section 2 presents the fundamental results on the coquaternion algebra. In Sect. 3, we introduce elementary functions in the coquaternionic context and establish some of their properties. Finally, Sect. 4 details the *Mathematica* implementation of the new functionalities added to the package `Coquaternion`.

2 Basic Results and Notation

In this section, we provide a brief introduction to the algebra of coquaternions, recall some of their key properties, and establish the notation. For a more in-depth discussion of this nondivision algebra, we refer the interested reader to the references [8,10,15,18].

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Let $\{1, \mathbb{i}, \mathbb{j}, \mathbb{k}\}$ be an orthonormal basis for the Euclidean vector space $\mathbb{R}^4$. The algebra of real coquaternions, denoted by $\mathbb{H}_{\text{coq}}$, is generated by the product defined according to the following rules:

$$\mathbb{i}^2 = -1,\ \mathbb{j}^2 = \mathbb{k}^2 = 1,\ \ \mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i} = \mathbb{k}.$$

Consider a coquaterion of the form

$$q = q_0 + q_1\mathbb{i} + q_2\mathbb{j} + q_3\mathbb{k}.$$

The *conjugate* of q, denoted by $\overline{q}$, is given by

$$\overline{q} = q_0 - q_1\mathbb{i} - q_2\mathbb{j} - q_3\mathbb{k}.$$

Analogous to the complex case, the scalar q_0 is called the *real part* of q, and is denoted by $\operatorname{Re} q$, while the expression

$$\underline{q} = q_1\mathbb{i} + q_2\mathbb{j} + q_3\mathbb{k}$$

is referred to as the *vector part* of q. The subset of elements in $\mathbb{H}_{\text{coq}}$ with zero vector part is identified with $\mathbb{R}$, the set of real numbers.

The *trace* and *determinant* of q are defined, respectively, as

$$\operatorname{tr} q = q + \overline{q} = 2\operatorname{Re} q, \quad \det q = q\overline{q} = q_0^2 + q_1^2 - q_2^2 - q_3^2.$$

We equip $\mathbb{H}_{\text{coq}}$ with the semi-norm

$$\|q\| = \sqrt{|\det q|}$$

and classify a coquaternion as *non-singular* if $\|q\| \neq 0$, and as a *unit coquaternion* if $\|q\| = 1$.

The sign of $\det q$ determines the type of the coquaternion q: q is called *space-like, light-like* or *time-like*, if $\det q < 0$, $\det q = 0$ or $\det q > 0$, respectively; the sets of such coquaternions will be denoted by $\mathbb{S}$, $\mathbb{L}$ and $\mathbb{T}$, respectively.

It is easy to see that any coquaternion q with a non-singular vector part (i.e., $\underline{q} \in \mathbb{S} \cup \mathbb{T}$) can be written in the form

$$q = q_0 + \boldsymbol{\omega}_{\underline{q}}\|\underline{q}\| \tag{1}$$

where

$$\boldsymbol{\omega}_{\underline{q}} = \boldsymbol{\omega}(\underline{q}) = \frac{\underline{q}}{\|\underline{q}\|}$$

is a unit coquaternion satisfying

$$\boldsymbol{\omega}_{\underline{q}}^2 = \begin{cases} 1, & \text{if } \underline{q} \in \mathbb{S} \\ -1, & \text{if } \underline{q} \in \mathbb{T} \end{cases} \tag{2}$$

If $\underline{q} \in \mathbb{L}$, then it can be written in the form

$$q = q_0 + \boldsymbol{\omega}_{\underline{q}} \tag{3}$$

where $\boldsymbol{\omega}_{\underline{q}} = \underline{q}$ satisfies

$$\boldsymbol{\omega}_{\underline{q}}^2 = 0. \tag{4}$$

Remark 1. The following properties of $\boldsymbol{\omega}(\underline{q}) = \boldsymbol{\omega}_{\underline{q}}$ will be useful in the subsequent sections, and their proofs follow immediately from the definition.

1. $\boldsymbol{\omega}(\boldsymbol{\omega}(\underline{q})) = \boldsymbol{\omega}_{\underline{q}}$
2. If $\alpha \in \mathbb{R}$, then
$$\boldsymbol{\omega}(\alpha\,\underline{q}) = \begin{cases} \operatorname{sgn}(\alpha)\,\boldsymbol{\omega}_{\underline{q}}, & \text{if } \underline{q} \in \mathbb{T} \cup \mathbb{S} \\ \alpha\,\boldsymbol{\omega}_{\underline{q}}, & \text{if } \underline{q} \in \mathbb{L} \end{cases}$$
where sgn is the usual real sign function.

3 Functions of a Coquaternionic Variable

In the context of quaternionic functions, there are several possible approaches to defining elementary functions, depending on the properties we are seeking. For example, in the case of the exponential function, multiple definitions have been derived over the years. For $x = x_0 + x_1\mathbb{i} + x_2\mathbb{j} + x_3\mathbb{k} \in \mathbb{H}$, the following exponential functions can be found (see [7] and references therein):

$$\begin{aligned}
\operatorname{Exp}_1(x) &= e^{x_0}\left(\cos\frac{x_1+x_2+x_3}{\sqrt{3}} + \frac{\mathbb{i}+\mathbb{j}+\mathbb{k}}{\sqrt{3}}\sin\frac{x_1+x_2+x_3}{\sqrt{3}}\right);\\
\operatorname{Exp}_2(x) &= 2e^{x_0}\left(-\frac{\sin\|\underline{x}\|}{\|\underline{x}\|} + \boldsymbol{\omega}_{\underline{x}}\frac{\|\underline{x}\|\cos\|\underline{x}\| - \sin\|\underline{x}\|}{\|\underline{x}\|^2}\right);\\
\operatorname{Exp}_3(x) &= e^{x_1+x_2+x_3}\left(\cos(x_0\sqrt{3}) - \frac{\mathbb{i}+\mathbb{j}+\mathbb{k}}{\sqrt{3}}\sin(x_0\sqrt{3})\right).
\end{aligned}$$

In addition to the above definitions, one of the most commonly used approaches to defining the exponential function in applications is through the standard power series expansion, i.e.,

$$\operatorname{Exp}(x) := \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^{x_0}(\cos\|\underline{x}\| + \boldsymbol{\omega}_{\underline{x}}\sin\|\underline{x}\|).$$

In this section, we take a systematic approach to introduce elementary functions, starting from an analytic complex function f and examining the properties of the resulting functions. Since f is analytic, it admits a Taylor series expansion around any real point x. For $z = x + iy$, we write:

$$f(z) = f(x+iy) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}(iy)^n.$$

At this stage, we do not concern ourselves with the convergence of the series. To obtain the corresponding coquaternionic function F, we replace the complex variable z with the coquaternion variable $q = q_0 + \underline{q}$, yielding

$$F(q) = F(q_0+\underline{q}) = \sum_{n=0}^{\infty}\frac{f^{(n)}(q_0)}{n!}(\underline{q})^n. \tag{5}$$

Taking into account the nature of the coquaternion q, we can write (see (1))

$$F(q) = F(q_0 + \boldsymbol{\omega}_{\underline{q}}\|\underline{q}\|) = \sum_{n=0}^{\infty} \frac{f^{(n)}(q_0)}{n!}(\boldsymbol{\omega}_{\underline{q}}\|\underline{q}\|)^n, \text{ if } \underline{q} \in \mathbb{T} \cup \mathbb{S},$$

or (see (3))

$$F(q) = F(q_0 + \boldsymbol{\omega}_{\underline{q}}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(q_0)}{n!}(\boldsymbol{\omega}_{\underline{q}})^n, \text{ if } \underline{q} \in \mathbb{L}.$$

Using (2) and (4), the coquaternionic function F can be written as

$$F(q) = \sum_{n=0}^{\infty}(-1)^n \frac{f^{(2n)}(q_0)}{(2n)!}\|\underline{q}\|^{2n} + \boldsymbol{\omega}_{\underline{q}} \sum_{n=0}^{\infty}(-1)^n \frac{f^{(2n+1)}(q_0)}{(2n+1)!}\|\underline{q}\|^{2n+1}, \text{if } \underline{q} \in \mathbb{T}, \quad (6)$$

$$F(q) = \sum_{n=0}^{\infty} \frac{f^{(2n)}(q_0)}{(2n)!}\|\underline{q}\|^{2n} + \boldsymbol{\omega}_{\underline{q}} \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(q_0)}{(2n+1)!}\|\underline{q}\|^{2n+1}, \text{if } \underline{q} \in \mathbb{S}, \quad (7)$$

$$F(q) = f(q_0) + \boldsymbol{\omega}_{\underline{q}} f'(q_0), \text{if } \underline{q} \in \mathbb{L}. \quad (8)$$

Remark 2. When $\underline{q} \in \mathbb{T}$ the coquaternion $q_0 + \boldsymbol{\omega}_{\underline{q}}\|\underline{q}\|$ behaves like a complex, in the sense that $\boldsymbol{\omega}_{\underline{q}}^2 = -1$. The elementary functions can be obtained at once (like in the quaternionic case) from the corresponding complex ones (cf. (6)). In fact, if we write the complex function f as

$$f(x + iy) = u(x, y) + iv(x, y),$$

where u and v are real functions, then the corresponding coquaternionic function is

$$F(q_0 + \boldsymbol{\omega}_{\underline{q}}\|\underline{q}\|) = u(q_0, \|\underline{q}\|) + \boldsymbol{\omega}_{\underline{q}} v(q_0, \|\underline{q}\|).$$

In the particular case where $q = q_0 + q_1\mathbb{i}$, i.e., $q \in \mathbb{C}$, then $F(q) = f(q)$, meaning that all the functions derived from (6) coincide with the corresponding complex functions.

As a consequence of these observations, it suffices to derive the expressions of the functions and analyze their properties for the cases where $\underline{q} \in \mathbb{S}$ or $\underline{q} \in \mathbb{L}$. However, for completeness, we also provide details on the case $\underline{q} \in \mathbb{T}$.

3.1 Exponential Function

Since the power series of the exponential function has an infinite radius of convergence, the use of (6)-(8) leads to the following coquaternionic exponential function.

$$\mathrm{Exp}(q) = \begin{cases} e^{q_0}\left(\cos\|\underline{q}\| + \boldsymbol{\omega}_{\underline{q}} \sin\|\underline{q}\|\right), & \text{if } \underline{q} \in \mathbb{T} \\ e^{q_0}\left(\cosh\|\underline{q}\| + \boldsymbol{\omega}_{\underline{q}} \sinh\|\underline{q}\|\right), & \text{if } \underline{q} \in \mathbb{S} \\ e^{q_0}(1 + \boldsymbol{\omega}_{\underline{q}}), & \text{if } \underline{q} \in \mathbb{L} \end{cases} \quad (9)$$

We note that the exponential function derived in [6, Theorem 4] coincides with the one presented here, as it was obtained using a similar approach.

We list some properties of the coquaternionic exponential function, which hold regardless of the nature of the coquaternion.

Property 1. If p and q are coquaternions then:

1. $\operatorname{Exp}(q) \neq 0$.
2. $\|\operatorname{Exp}(q)\| = e^{q_0}$.
3. $\operatorname{Exp}(q_0 + \underline{q}) = e^{q_0} \operatorname{Exp}(\underline{q})$.
4. If p and q commute, then
$$\operatorname{Exp}(p+q) = \operatorname{Exp}(p) \operatorname{Exp}(q).$$
In particular,
$$\operatorname{Exp}(q) \operatorname{Exp}(-q) = 1 \quad \text{and} \quad (\operatorname{Exp}(q))^n = \operatorname{Exp}(nq)$$

Proof. The first three identities are directly derived from (9) and the properties (2) and (4) of $\boldsymbol{\omega}_{\underline{q}}$. Consider now the coquaternions $p = p_0 + p_1\mathbb{i} + p_2\mathbb{j} + p_3\mathbb{k}$ and $q = q_0 + q_1\mathbb{i} + q_2\mathbb{j} + q_3\mathbb{k}$. The use of (5) allows to write
$$\operatorname{Exp}(p+q) = e^{p_0+q_0} \sum_{n=0}^{\infty} \frac{\left(\underline{p} + \underline{q}\right)^n}{n!}.$$
Since p and q commute, their vector parts also commute, which means that
$$\operatorname{Exp}(p+q) = e^{p_0+q_0} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{\underline{p}^k \underline{q}^{n-k}}{n!} = e^{p_0+q_0} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\underline{p}^k}{k!} \frac{\underline{q}^{n-k}}{(n-k)!}.$$
By recognizing the last expression as the Cauchy product of two series, we can rewrite it as
$$\operatorname{Exp}(p+q) = e^{p_0+q_0} \sum_{n=0}^{\infty} \frac{\underline{p}^k}{k!} \sum_{n=0}^{\infty} \frac{\underline{q}^k}{k!} = e^{p_0+q_0} \operatorname{Exp}(\underline{p}) \operatorname{Exp}(\underline{q}).$$
The result follows now at once by using identity 3. of Property 1. □

3.2 Trigonometric Functions

It is easy to see that the coquaternionic sine function obtained by the use of (6)-(8) is
$$\operatorname{Sin}(q) = \begin{cases} \sin q_0 \cosh \|\underline{q}\| + \boldsymbol{\omega}_{\underline{q}} \cos q_0 \sinh \|\underline{q}\|, & \text{if } \underline{q} \in \mathbb{T} \\ \sin q_0 \cos \|\underline{q}\| + \boldsymbol{\omega}_{\underline{q}} \cos q_0 \sin \|\underline{q}\|, & \text{if } \underline{q} \in \mathbb{S} \\ \sin q_0 + \boldsymbol{\omega}_{\underline{q}} \cos q_0, & \text{if } \underline{q} \in \mathbb{L} \end{cases} \tag{10}$$

while the cosine function is

$$\mathrm{Cos}(q) = \begin{cases} \cos q_0 \cosh \|\underline{q}\| - \boldsymbol{\omega}_{\underline{q}} \sin q_0 \sinh \|\underline{q}\|, & \text{if } \underline{q} \in \mathbb{T} \\ \cos q_0 \cos \|\underline{q}\| - \boldsymbol{\omega}_{\underline{q}} \sin q_0 \sin \|\underline{q}\|, & \text{if } \underline{q} \in \mathbb{S} \\ \cos q_0 - \boldsymbol{\omega}_{\underline{q}} \sin q_0, & \text{if } \underline{q} \in \mathbb{L} \end{cases} \tag{11}$$

A direct computation using (10) and (11) establishes several identities for the sine and cosine coquaternionic functions, valid for all coquaternions.

Property 2. If q is a coquaternion then:

1. $\mathrm{Sin}^2(q) + \mathrm{Cos}^2(q) = 1$.
2. $\mathrm{Sin}(q) = \sin q_0 \,\mathrm{Cos}(\underline{q}) + \cos q_0 \,\mathrm{Sin}(\underline{q})$.
3. $\mathrm{Cos}(q) = \cos q_0 \,\mathrm{Cos}(\underline{q}) - \sin q_0 \,\mathrm{Sin}(\underline{q})$.
4. The zeros of the coquaternionic sine and cosine functions reduce to the real ones.

3.3 Hyperbolic Functions

Applying the procedure described by (6)-(8) we obtain the following coquaternionic hyperbolic sine and cosine functions

$$\mathrm{Sinh}(q) = \begin{cases} \sinh q_0 \cos \|\underline{q}\| + \boldsymbol{\omega}_{\underline{q}} \cosh q_0 \sin \|\underline{q}\|, & \text{if } \underline{q} \in \mathbb{T} \\ \sinh q_0 \cosh \|\underline{q}\| + \boldsymbol{\omega}_{\underline{q}} \cosh q_0 \sinh \|\underline{q}\|, & \text{if } \underline{q} \in \mathbb{S} \\ \sinh q_0 + \boldsymbol{\omega}_{\underline{q}} \cosh q_0, & \text{if } \underline{q} \in \mathbb{L} \end{cases} \tag{12}$$

$$\mathrm{Cosh}(q) = \begin{cases} \cosh q_0 \cos \|\underline{q}\| + \boldsymbol{\omega}_{\underline{q}} \sinh q_0 \sin \|\underline{q}\|, & \text{if } \underline{q} \in \mathbb{T} \\ \cosh q_0 \cosh \|\underline{q}\| + \boldsymbol{\omega}_{\underline{q}} \sinh q_0 \sinh \|\underline{q}\|, & \text{if } \underline{q} \in \mathbb{S} \\ \cosh q_0 + \boldsymbol{\omega}_{\underline{q}} \sinh q_0, & \text{if } \underline{q} \in \mathbb{L} \end{cases} \tag{13}$$

The following properties follow directly from the explicit definitions of the hyperbolic functions (12) and (13).

Property 3. If q is a coquaternion then:

1. $\mathrm{Cosh}^2(q) - \mathrm{Sinh}^2(q) = 1$.

2. $$\mathrm{Sin}(q\,\boldsymbol{\omega}_{\underline{q}}) = \begin{cases} \mathrm{Sinh}(q)\,\boldsymbol{\omega}_{\underline{q}}, & \text{if } \underline{q} \in \mathbb{T} \\ \mathrm{Sin}(q)\,\boldsymbol{\omega}_{\underline{q}}, & \text{if } \underline{q} \in \mathbb{S} \\ q_0\,\boldsymbol{\omega}_{\underline{q}}, & \text{if } \underline{q} \in \mathbb{L} \end{cases}$$

3. $$\mathrm{Cos}(q\,\boldsymbol{\omega}_{\underline{q}}) = \begin{cases} \mathrm{Cosh}(q), & \text{if } \underline{q} \in \mathbb{T} \\ \mathrm{Cos}(q), & \text{if } \underline{q} \in \mathbb{S} \\ 1, & \text{if } \underline{q} \in \mathbb{L} \end{cases}$$

4. Beyond the real zero 0, the hyperbolic sine function has the time-like zeros $k\pi\boldsymbol{\omega}_{\underline{q}}$, while the hyperbolic cosine vanishes at $(k+\frac{1}{2})\pi\boldsymbol{\omega}_{\underline{q}}$, with $\underline{q} \in \mathbb{T}$.

Remark 3. If $\underline{q} \in \mathbb{T}$, the well-known relationship between the complex (or quaternionic) exponential function and the trigonometric and hyperbolic functions has the analogue:

$$\mathrm{Cos}(q) = \frac{\mathrm{Exp}(q\boldsymbol{\omega}_{\underline{q}}) + \mathrm{Exp}(-q\boldsymbol{\omega}_{\underline{q}})}{2}, \qquad \mathrm{Cosh}(q) = \frac{\mathrm{Exp}(q) + \mathrm{Exp}(-q)}{2},$$
$$\mathrm{Sin}(q) = -\boldsymbol{\omega}_{\underline{q}}\frac{\mathrm{Exp}(q\boldsymbol{\omega}_{\underline{q}}) - \mathrm{Exp}(-q\boldsymbol{\omega}_{\underline{q}})}{2}, \qquad \mathrm{Sinh}(q) = \frac{\mathrm{Exp}(q) - \mathrm{Exp}(-q)}{2}.$$

However, when $\underline{q} \in \mathbb{S}$, due to Property 3.3, the relationships involving the sine and cosine functions no longer hold. Only the relations between the exponential function and the hyperbolic functions remain valid.

3.4 Logarithm Function

We derive the expression for the coquaternionic logarithm function as the inverse of the exponential function defined in Sect. 3.1. Consequently, the following identities must hold:

$$\mathrm{Log}(\mathrm{Exp}(q)) = \mathrm{Exp}(\mathrm{Log}(q)) = q. \tag{14}$$

Property 4. The inverse of the exponential function (9) is the following logarithm function:

$$\mathrm{Log}(q) = \begin{cases} \log\|q\| + \boldsymbol{\omega}_{\underline{q}}\,\mathrm{atan2}(q_0, \|\underline{q}\|), & \text{if } \underline{q} \in \mathbb{T} \\ \log\|q\| + \boldsymbol{\omega}_{\underline{q}}\,\mathrm{arctanh}\,\frac{\|\underline{q}\|}{q_0}, & \text{if } \underline{q} \in \mathbb{S},\ q \in \mathbb{T},\ \text{and } q_0 > 0 \\ \log q_0 + \frac{1}{q_0}\boldsymbol{\omega}_{\underline{q}}, & \text{if } \underline{q} \in \mathbb{L},\ \text{and } q_0 > 0 \end{cases}$$

Proof. The case $\underline{q} \in \mathbb{T}$ follows directly from the complex case, as explained in Remark 2. Here, we use the function atan2 to denote the two-argument

arctangent, which determines the principal argument of a complex number. Consequently, the derived logarithm function represents the principal value of the coquaternionic logarithm.

If $\underline{q} = \boldsymbol{\omega}_{\underline{q}} \in \mathbb{L}$, then

$$\mathrm{Log}\left(\mathrm{Exp}(q)\right) = \mathrm{Log}\left(e^{q_0} + e^{q_0}\boldsymbol{\omega}_{\underline{q}}\right) = \log(e^{q_0}) + \tfrac{1}{e^{q_0}} e^{q_0}\boldsymbol{\omega}_{\underline{q}} = q$$

and

$$\mathrm{Exp}\left(\mathrm{Log}(q)\right) = \mathrm{Exp}\left(\log(q_0) + \tfrac{1}{q_0}\boldsymbol{\omega}_{\underline{q}}\right) = e^{\log(q_0)}(1 + \tfrac{1}{q_0}\boldsymbol{\omega}_{\underline{q}}) = q_0 + \boldsymbol{\omega}_{\underline{q}} = q.$$

Consider now the case where $\underline{q} = \boldsymbol{\omega}_{\underline{q}}\|\underline{q}\| \in \mathbb{S}$. Denoting by $\underline{\mathrm{Exp}}$ the vector part of the exponential function, and considering Remark 2, Property 1, and the condition $q_0 > 0$, we can write

$$\|\mathrm{Exp}(q)\| = e^{q_0};\ \boldsymbol{\omega}_{\underline{\mathrm{Exp}(q)}} = \boldsymbol{\omega}_{\underline{q}};\ \|\underline{\mathrm{Exp}(q)}\| = e^{q_0}\sinh\|\underline{q}\|. \tag{15}$$

Therefore

$$\mathrm{Log}\left(\mathrm{Exp}(q)\right) = \log(e^{q_0}) + \boldsymbol{\omega}_{\underline{q}}\,\mathrm{arctanh}\,\tfrac{e^{q_0}\sinh\|\underline{q}\|}{e^{q_0}\cosh\|\underline{q}\|} = q_0 + \boldsymbol{\omega}_{\underline{q}}\|\underline{q}\| = q.$$

Observe that

$$\boldsymbol{\omega}_{\underline{\mathrm{Log}(q)}} = \mathrm{sgn}\left(\mathrm{arctanh}\,\tfrac{\|\underline{q}\|}{q_0}\right)\boldsymbol{\omega}_{\underline{q}} = \boldsymbol{\omega}_{\underline{q}}$$

and

$$\|\underline{\mathrm{Log}(q)}\| = \left|\mathrm{arctanh}\,\tfrac{\|\underline{q}\|}{q_0}\right| = \mathrm{arctanh}\,\tfrac{\|\underline{q}\|}{q_0}.$$

Therefore we can write

$$\begin{aligned}\mathrm{Exp}\left(\mathrm{Log}(q)\right) &= \mathrm{Exp}\left(\log\|q\| + \boldsymbol{\omega}_{\underline{q}}\,\mathrm{arctanh}\,\tfrac{\|\underline{q}\|}{q_0}\right)\\ &= \mathrm{Exp}(\log\|q\|)\left(\cosh\left(\mathrm{arctanh}\,\tfrac{\|\underline{q}\|}{q_0}\right) + \boldsymbol{\omega}_{\underline{q}}\sinh\left(\mathrm{arctanh}\,\tfrac{\|\underline{q}\|}{q_0}\right)\right)\\ &= \|q\|\left(\cosh\left(\mathrm{arctanh}\,\tfrac{\|\underline{q}\|}{q_0}\right) + \boldsymbol{\omega}_{\underline{q}}\sinh\left(\mathrm{arctanh}\,\tfrac{\|\underline{q}\|}{q_0}\right)\right).\end{aligned}$$

The result follows by using the well-known identities $\cosh(\mathrm{arctanh}\,x) = \frac{1}{\sqrt{1-x^2}}$ and $\sinh(\mathrm{arctanh}\,x) = \frac{x}{\sqrt{1-x^2}}$ and the observation that for any $q \in \mathbb{S}$, we have $\sqrt{q_0^2 - \|\underline{q}\|^2} = \|q\|$. □

Remark 4. The condition $q_0 > 0$ in the definition of the logarithm for $\underline{q} \in \mathbb{S}$ guarantees that the second condition in Eq. (14) is satisfied. However, we can extend the given definition to the case $q_0 < 0$.

3.5 Other Elementary Functions

Other elementary functions can be derived from the coquaternionic functions defined in the previous sections by employing analogues of the corresponding complex relationships, replacing, if necessary, the imaginary unit by $\boldsymbol{\omega}_{\underline{q}}$. For example, we can define

$$\mathrm{Tanh}(q) = \frac{\mathrm{Sinh}(q)}{\mathrm{Cosh}(q)}$$

or

$$\mathrm{ArcTan}(q) = -\boldsymbol{\omega}_{\underline{q}} \,\mathrm{Log}\left(\frac{1+\boldsymbol{\omega}_{\underline{q}} q}{1-\boldsymbol{\omega}_{\underline{q}} q}\right).$$

Since the numerators and denominators in the previous divisions commute, both left and right divisions can be considered.

4 *Mathematica* Tools

The *Mathematica* package `Coquaternions` [9] provides a foundation for manipulating coquaternions. This work extends this framework by defining and implementing elementary functions in the coquaternionic setting, further enhancing the package's applicability.

All these computational tools, including the release version described in this section, are available and can be freely downloaded from the website, along with their support documentation.

https://w3.math.uminho.pt/Coquaternions/

In the `Coquaternions` package, a coquaternion $q = q_0 + q_1\mathbb{i} + q_2\mathbb{j} + q_3\mathbb{k}$ is an object of the form `Coquaternion[q0,q1,q2,q3]`. The package allows the use of symbolic entries, assuming that all symbols represent real numbers and adds rules to `Plus`, `Times`, `NonCommutativeMultiply`, and `Power`. It also extends several standard functions to support coquaternionic objects, including `Re`, `Abs`, `Norm`, `Det`, `Tr`, among others. Additionally, new functions are available, such as `Vec` for extracting the vector part of a coquaternion, `PolarForm` for obtaining its polar form, and `Nature` for classifying a coquaternion as time-like, light-like, or space-like (as defined in Sect. 2), using the boolean functions `TimelikeQ`, `LightlikeQ`, or `SpacelikeQ`, respectively.

A new function introduced in this release is the `Omega` function for obtaining $\boldsymbol{\omega}_{\underline{q}}$, which is essential for defining the elementary functions. It can be called with either one or two arguments, allowing for numerical or symbolic use, respectively.

```
Coquaternion/:Omega[q_Coquaternion,"Timelike"|"Spacelike"]:=
    Vec[q]Norm[Vec[q]]
```

```
Coquaternion/:Omega[q_Coquaternion,"Lightlike"]:= Vec[q]

Coquaternion /: Omega[q_Coquaternion]:= Module[{NatVecq = Nature@Vec@q},
  If[MatchQ[NatVecq, "Timelike"|"Spacelike"|"Lightlike"],
   Omega[q, NatVecq], Message[Coquaternion::UndefinedNature]]]
```

We now illustrate the use of the Omega function by considering some simple examples.

```
In[1]:= <<Coquaternions`
In[2]:= lq = {Coquaternion[1,2,0,0],Coquaternion[4,0,3,0],
          Coquaternion[1,2,2,0]};
In[3]:= lvq = Vec[lq]
Out[3]= {Coquaternion[0, 2, 0, 0], Coquaternion[0, 0, 3, 0], Coquaternion[0, 2, 2, 0]}
In[4]:= Nature@lvq
Out[4]= {Timelike, Spacelike, Lightlike}
In[5]:= Omega[lvq] // TraditionalForm
Out[5]= {i, j, 2i + 2j}
```

When q is a coquaternion with symbolic entries, using the `Omega` function with a single input parameter is not suitable. An additional input is required to clarify the nature of q.

```
In[6]:= q = Coquaternion[q0,q1,q2,q3];
         $Assumptions = Element[q0|q1|q2|q3,Reals];
In[7]:= Omega[q]
        ••• Coquaternion: Unable to determine the nature of the argument.
In[8]:= Omega[q,"Lightlike"]
Out[8]= Coquaternion[0, q1, q2, q3]
In[9]:= Omega[q,"Timelike"]
Out[9]= Coquaternion[0, q1/Sqrt[Abs[q1^2 - q2^2 - q3^2]], q2/Sqrt[Abs[q1^2 - q2^2 - q3^2]], q3/Sqrt[Abs[q1^2 - q2^2 - q3^2]]]
```

Following the same approach, we define each elementary function, using one input argument for numerical values and two for symbolic expressions, with the second parameter specifying the nature of the coquaternion. We do not provide the code for defining all the elementary functions, but instead illustrate only the case of the exponential function, according to the definition (9). The implementation details of the other functions can be found on the previously provided website.

```
Coquaternion/:Exp[q_Coquaternion,"Timelike"]:=
    Module[{Req=Re[q],NormVecq=Norm[Vec[q]],
    Exp[Req](Cos[NormVecq]+Omega[q,"Timelike"]Sin[NormVecq])]

Coquaternion/:Exp[q_Coquaternion,"Spacelike"]:=
    Module[{Req=Re[q],NormVecq=Norm[Vec[q]],
    Exp[Req](Cosh[NormVecq]+Omega[q,"Spacelike"]Sinh[NormVecq])]

Coquaternion/:Exp[q_Coquaternion,"Lightlike"]:=
    Exp[Re[q]](1+Omega[q,"Lightlike"])

Coquaternion /: Exp[q_Coquaternion]:= Module[{NatVecq = Nature@Vec@q},
  If[MatchQ[NatVecq, "Timelike"|"Spacelike"|"Lightlike"],
   Exp[q, NatVecq], Message[Coquaternion::UndefinedNature]]]
```

We now explore how this extended Exp function can be used to compute the coquaternionic exponential in selected examples and verify previously established properties. We use the data from the previous example.

– Numerical computations with Exp function:

```
In[10]:= Exp[1q]
```

$$\text{Out[10]}= \{\texttt{Coquaternion}[\mathbb{e}\,\texttt{Cos}[2], \mathbb{e}\,\texttt{Sin}[2], 0, 0],$$
$$\texttt{Coquaternion}[\mathbb{e}^4\texttt{Cosh}[3], 0, \mathbb{e}^4\texttt{Sinh}[3], 0], \texttt{Coquaternion}[\mathbb{e}, 2\mathbb{e}, 2\mathbb{e}, 0]\}$$

– Symbolic computations with Exp function:

```
In[11]:= Exp[q, "Spacelike"]
```

$$\text{Out[11]}= \texttt{Coquaternion}\left[\mathbb{e}^{q0}\texttt{Cosh}\left[\sqrt{\texttt{Abs}[q1^2 - q2^2 - q3^2]}\right], \frac{\mathbb{e}^{q0}q1\texttt{Sinh}\left[\sqrt{\texttt{Abs}[q1^2-q2^2-q3^2]}\right]}{\sqrt{\texttt{Abs}[q1^2-q2^2-q3^2]}},\right.$$
$$\left.\frac{\mathbb{e}^{q0}q2\texttt{Sinh}\left[\sqrt{\texttt{Abs}[q1^2-q2^2-q3^2]}\right]}{\sqrt{\texttt{Abs}[q1^2-q2^2-q3^2]}}, \frac{\mathbb{e}^{q0}q3\texttt{Sinh}\left[\sqrt{\texttt{Abs}[q1^2-q2^2-q3^2]}\right]}{\sqrt{\texttt{Abs}[q1^2-q2^2-q3^2]}}\right]$$

– Verification of Identities (15)

```
In[12]:= Simplify[Omega[Vec[Exp [q,"Spacelike"]],"Spacelike"]==
          Omega[Vec[q],"Spacelike"]]
```

$$\text{Out[12]}= \texttt{True}$$

```
In[13]:= Norm[Vec[Exp[q,"Spacelike"]]] // Simplify
```

$$\text{Out[13]}= \mathbb{e}^{q0}\texttt{Sinh}\left[\sqrt{\texttt{Abs}[q1^2 - q2^2 - q3^2]}\right]$$

– Verification of relation 4. of Property 1 for $\underline{q} \in \mathbb{T}$

We note that two coquaternions q_1 and q_2, whose vector parts do not vanish, commute if and only if $\boldsymbol{\omega}(q_1) = \pm\boldsymbol{\omega}(q_2)$, i.e., $q_2 = \alpha + \boldsymbol{\omega}(q_1)\beta$, with $\alpha, \beta \in \mathbb{R}$.

```
In[14]:= p = α + Omega[q,"Timelike"] β;
In[15]:= Assuming[Det[Vec[q] > 0 && Element[β,Reals], Simplify@
          Exp[p+q,"Timelike"]-Exp[p,"Timelike"]**Exp[q,"Timelike"]]
Out[15]= Coquaternion[0,0,0,0]
```

We conclude this section by presenting additional examples featuring other elementary functions.
– Numerical computations with Sin function:

```
In[16]:= Sin[lq] // TableForm
Out[16]/TableForm=
         Coquaternion[Sin[1] Cosh[2],Cos[1] Sinh[2],0,0]
         Coquaternion[Sin[4] Cos[3],0,Sin[3] Cos[4],0]
         Coquaternion[Sin[1],2 Cos[1],2 Cos[1],0]
```

– Verification of identity 2. Property 2, for $\underline{q} \in \mathbb{T}$

```
In[17]:= Sin[q,"Timelike"] - Sin[q0] Cos[Vec[q],"Timelike"] -
          Cos[q0] Sin[Vec[q],"Timelike"] // Simplify
Out[17]= Coquaternion[0, 0, 0, 0]
```

– Verification of identity 3. Property 3, for $\underline{q} \in \mathbb{H}_{\text{coq}}$

```
In[18]:= Assuming[Det[Vec[q]] > 0,
         Simplify[Cos[q Omega[q,"Timelike"],"Timelike"]]==
         Simplify[Cosh[q,"Timelike"]]]
Out[18]= True
In[19]:= Assuming[Det[Vec[q]] < 0,
         Simplify[Cos[q**Omega[q,"Spacelike"],"Spacelike"]]==
         Simplify[Cos[q,"Spacelike"]]]
Out[19]= True
In[20]:= Assuming[Det[Vec[q]] == 0,
         Simplify[Cos[q**Omega[q,"Lightlike"],"Lightlike"]]]
Out[20]= Coquaternion[1,0,0,0]
```

– Numerical computations with Log function:

```
In[21]:= Log[lq] // TableForm
```

Out[21]/TableForm=

$$\texttt{Coquaternion}\left[\frac{\texttt{Log[5]}}{2}, \texttt{ArcTan}[2], 0, 0\right]$$

$$\texttt{Coquaternion}\left[\frac{\texttt{Log[7]}}{2}, 0, \texttt{ArcTanh}\left[\frac{3}{4}\right], 0\right]$$

$$\texttt{Coquaternion}\,[0, 2, 2, 0]$$

– Numerical computations with Tan function:

```
In[22]:= Tan[lq] // N // TableForm
```

Out[22]/TableForm=

```
Coquaternion[0.0338128,1.01479,0.,0.]
Coquaternion[1.21443,0.,-0.34298,0.]
Coquaternion[1.55741,6.85104,6.85104,0.]
```

5 Conclusion

We have developed a systematic framework for defining and computing elementary coquaternionic functions through analytic extension. Our implemented *Mathematica* functions support both numerical and symbolic computations, and the examples presented throughout the article confirm the theoretical properties established earlier while underscoring the practical applicability of our approach.

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