

The Mandelbrot Set for a Coquaternionic Family of Quadratics

Maria Irene Falcão, Fernando Miranda, and Ricardo Severino(✉)

CMAT and Departamento de Matemática, Universidade do Minho, Braga, Portugal
{mif,fmiranda,ricardo}@math.uminho.pt
http://www.cmat.uminho.pt

Abstract. The Mandelbrot set stands as one of the most fascinating and visually striking mathematical objects ever discovered, representing a fundamental milestone in dynamical systems theory and fractal geometry. Extending this set into the coquaternionic domain introduces significant mathematical challenges and unveils structures of remarkable complexity. In this work, we investigate the generalization of the Mandelbrot set from the quadratic mapping family $x^2 + b\,x$ in the complex plane to the coquaternionic space $\mathbb{H}_{\mathrm{coq}}$, examining its mathematical properties, visualization techniques, and theoretical implications.

Keywords: Coquaternions · Iteration of quadratic maps · Mandelbrot set

1 Introduction

The Mandelbrot set, first studied by French mathematicians Pierre Fatou and Gaston Julia in the early 20th century, gained significant attention in the 1980s when Benoit Mandelbrot used computers to visualize it. The resulting structure revealed infinite complexity, with patterns repeating at progressively smaller scales, exemplifying the self-similarity characteristic of fractals [7]. These properties, combined with its aesthetic appeal and the ability to generate intricate visual complexity from simple iterative rules, contributed to its popularity beyond mathematical circles.

Given its profound impact and the rich mathematical structures it unveiled, researchers soon began exploring generalizations of the Mandelbrot set beyond the complex plane, aiming to extend its principles to higher-dimensional spaces. To understand how such generalizations can be pursued, we first examine the definition of the Mandelbrot set.

Traditionally, the Mandelbrot set is defined as the set of points c in the complex plane for which the iteration of $z = 0$ under the quadratic map $z^2 + c$ remains bounded. Here, $z = 0$ serves as the critical point of the mapping. This specific choice within the one-parameter family of quadratic maps arises from a fundamental equivalence in dynamical behavior: every quadratic mapping in the complex plane is dynamically conjugate to $z^2 + c$ for some complex parameter

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c. This equivalence establishes the complex Mandelbrot set as a unique object of study, encapsulating essential dynamical properties of all complex quadratic mappings.

The first generalization of this set to higher-dimensional spaces was achieved for a family of quaternionic quadratics, see [1,2,8]. While this yielded interesting results, it did not introduce fundamentally new dynamics.

In this work, we propose studying the Mandelbrot set for a family of quadratic mappings in the four-dimensional space of coquaternions $\mathbb{H}_{\text{coq}}$, also referred to as split-quaternions in literature. This endeavor is inherently challenging due to the non-commutative nature of coquaternionic multiplication. Nevertheless, we are motivated by the diversity and complexity observed in coquaternionic dynamics, see [3–5], which provide strong justification for this study. Specifically, we focus on a family of quadratic mappings previously shown to be dynamically non-equivalent to simpler families [5].

This paper is organized as follows: Sect. 2 provides an overview of fundamental concepts related to coquaternions. Section 3 introduces key concepts from discrete dynamical systems in coquaternionic contexts. Section 4 presents a generalization of the complex Mandelbrot set to coquaternions and discusses visualization methods. Finally, Sect. 5 summarizes our findings and comments on their implications.

2 Basic Definitions and Results on Coquaternions

Let $\{1, \mathbb{i}, \mathbb{j}, \mathbb{k}\}$ form an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$, endowed with a bilinear product defined by:

$$\begin{cases} \mathbb{i}^2 = -1, & \mathbb{j}^2 = \mathbb{k}^2 = 1, \\ \mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i} = \mathbb{k} \end{cases}$$

A straightforward computation shows that this product generates an associative but non-commutative real algebra, denoted $\mathbb{H}_{\text{coq}}$, whose elements are called coquaternions. Notably, unlike Hamilton quaternions, $\mathbb{H}_{\text{coq}}$ is not a division algebra, as it contains both zero divisors and nilpotent elements.

We identify $\mathbb{R}^4$ with $\mathbb{H}_{\text{coq}}$ through the correspondence

$$(q_0, q_1, q_2, q_3) \leftrightarrow q_0 + q_1\mathbb{i} + q_2\mathbb{j} + q_3\mathbb{k}.$$

For $\mathsf{q} = q_0 + q_1\mathbb{i} + q_2\mathbb{j} + q_3\mathbb{k} \in \mathbb{H}_{\text{coq}}$, we define:

the *conjugate* $\overline{\mathsf{q}} = q_0 - q_1\mathbb{i} - q_2\mathbb{j} - q_3\mathbb{k}$;

the *real part* $\text{re}(\mathsf{q}) = q_0$;

the *vector part* $\text{vec}(\mathsf{q}) = q_1\mathbb{i} + q_2\mathbb{j} + q_3\mathbb{k}$.

Coquaternions with zero vector part are naturally identified with real numbers.

An important result for the coquaternions is the fact that the algebra $\mathbb{H}_{\text{coq}}$ is isomorphic to the algebra of real 2×2 matrices via:

$$\Phi(q_0 + q_1\mathbb{i} + q_2\mathbb{j} + q_3\mathbb{k}) = \begin{pmatrix} q_0 + q_3 & q_1 + q_2 \\ q_2 - q_1 & q_0 - q_3 \end{pmatrix}$$

The *determinant* of q, denoted $\det \mathsf{q}$, is defined as:

$$\det \mathsf{q} = q_0^2 + q_1^2 - q_2^2 - q_3^2 = \mathsf{q}\overline{\mathsf{q}}$$

A coquaternion q is invertible if and only if $\det \mathsf{q} \neq 0$, with inverse:

$$\mathsf{q}^{-1} = \frac{\overline{\mathsf{q}}}{\det \mathsf{q}}.$$

Definition 1. *Two coquaternions* q *and* p *are* similar *(*$\mathsf{q} \sim \mathsf{p}$*) if there exists an invertible* h *such that* $\mathsf{p} = \mathsf{hqh}^{-1}$.

This equivalence relation partitions $\mathbb{H}_{\text{coq}}$ into similarity classes, denoted by $[\mathsf{q}]$. One can show that, $[\mathsf{q}] = \{\mathsf{q}\}$ if and only if q is real.

For invertible h, the *similarity map* $\phi_{\mathsf{h}} : \mathbb{H}_{\text{coq}} \to \mathbb{H}_{\text{coq}}$ defined by:

$$\phi_{\mathsf{h}}(\mathsf{q}) = \mathsf{h}^{-1}\mathsf{qh} \tag{1}$$

constitutes an algebra automorphism.

Unlike Hamilton quaternions, coquaternions require three distinct representative forms for non-real elements, see [3] and the references cited therein:

Theorem 1. *For a non-real coquaternion* $\mathsf{q} = q_0 + \operatorname{vec}\mathsf{q}$ *with* $r = \det(\operatorname{vec}\mathsf{q}) = q_1^2 - q_2^2 - q_3^2$*, we have:*

if $r > 0$, $\exists\phi_{\mathsf{h}}$ *such that* $\phi_{\mathsf{h}}(\mathsf{q}) = q_0 + \sqrt{r}\mathbb{i}$;

if $r < 0$, $\exists\phi_{\mathsf{h}}$ *such that* $\phi_{\mathsf{h}}(\mathsf{q}) = q_0 + \sqrt{-r}\mathbb{j}$;

if $r = 0$, $\exists\phi_{\mathsf{h}}$ *such that* $\phi_{\mathsf{h}}(\mathsf{q}) = q_0 + \mathbb{i} + \mathbb{j}$.

Next, we recall some basic definitions of discrete dynamical systems.

3 Basic Definitions on Coquaternionic Discrete Dynamical Systems

Let us consider a coquaternionic map $\mathsf{f} : \mathbb{H}_{\text{coq}} \to \mathbb{H}_{\text{coq}}$. For $k \in \mathbb{N}$, we shall denote by f^k the k-th iterate of f, inductively defined by

$$\begin{cases} \mathsf{f}^0 = \mathrm{id}_{\mathbb{H}_{\text{coq}}} \\ \mathsf{f}^k = \mathsf{f} \circ \mathsf{f}^{k-1}. \end{cases}$$

For a given initial point $\mathsf{q}_0 \in \mathbb{H}_{\text{coq}}$, the orbit of q_0 under the map f is the sequence

$$\mathcal{O}(\mathsf{q}_0) := \left(\mathsf{f}^k(\mathsf{q}_0)\right)_{k\in\mathbb{N}_0}.$$

In the context of dynamical systems, the orbit of a point under a map f represents the time evolution of a specific quantity. Next, we focus on particular types of orbits, which correspond to distinctive patterns in the system's time evolution.

A point $\mathsf{q} \in \mathbb{H}_{\text{coq}}$ is said to be a periodic point of f, with period $n \in \mathbb{N}$, if we have $\mathsf{f}^n(\mathsf{q}) = \mathsf{q}$, with $\mathsf{f}^k(\mathsf{q}) \neq \mathsf{q}$ for $0 < k < n$; in this case, we say that the set

$$\mathcal{C} = \{\mathsf{q}, \mathsf{f}(\mathsf{q}), \ldots, \mathsf{f}^{n-1}(\mathsf{q})\}$$

is a n-cycle for f, usually written as

$$\mathcal{C} : \mathsf{q}_0 \xrightarrow{\mathsf{f}} \mathsf{q}_1 \xrightarrow{\mathsf{f}} \cdots \xrightarrow{\mathsf{f}} \mathsf{q}_{n-1}$$

with $\mathsf{q}_i = \mathsf{f}^i(\mathsf{q})$. Periodic points of period one are usually called fixed points.

Finally, there is one last definition relevant for the rest of the paper:

Definition 2. *We say that two maps* $\mathsf{f} : \mathbb{H}_{\text{coq}} \to \mathbb{H}_{\text{coq}}$ *and* $\mathsf{g} : \mathbb{H}_{\text{coq}} \to \mathbb{H}_{\text{coq}}$ *are conjugate if there exists an invertible map* $\phi : \mathbb{H}_{\text{coq}} \to \mathbb{H}_{\text{coq}}$ *such that*

$$\mathsf{f} \circ \phi = \phi \circ \mathsf{g}.$$

In this case, we say that the corresponding dynamical systems $(\mathbb{H}_{\text{coq}}, \mathsf{f})$ and $(\mathbb{H}_{\text{coq}}, \mathsf{g})$ are dynamically equivalent, since they share the same dynamical characteristics.

4 The Coquaternionic Mandelbrot Set of a Family of Quadratics

Prior to examining the coquaternionic generalization, it is instructive to review foundational results concerning the classical complex Mandelbrot set.

While the family of complex quadratic maps z^2+bz is topologically conjugate to the standard quadratic family $z^2 + c$, its associated Mandelbrot set remains underrepresented in canonical literature due to this structural equivalence. This conjugacy implies dynamical equivalence between the two families, rendering distinct visualizations of their Mandelbrot sets superfluous in most analyses. However, given the pedagogical and comparative objectives of this work, we deem it methodologically pertinent to explicitly illustrate the Mandelbrot set for the $z^2 + bz$ parametrization. Such a representation not only reinforces the foundational principles governing quadratic complex dynamics but also provides a conceptual bridge to the non-commutative coquaternionic extension discussed subsequently.

Consider the family of complex quadratic maps $R_b(z) = z^2 + bz$. It is known that the Möbius transformation $\phi(z) = z + b/2$ establishes a conjugacy between

R_b and the simpler quadratic family $Q_c(z) = z^2 + c$ (see the following diagram)

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{R_b} & \mathbb{C} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathbb{C} & \xrightarrow[Q_b]{} & \mathbb{C} \end{array}$$

Specifically, the inverse transformation $\phi^{-1}(z) = z - b/2$ facilitates the construction of the Mandelbrot set for R_b and enables the translation of properties from the classical complex Mandelbrot set to this parameterized family.

The mapping conjugacy induces a relationship between parameters b and c, given by

$$c = \frac{1}{4}b(2-b),$$

with the inverse transformation

$$b = 1 \pm \sqrt{1-4c}.$$

This result implies that the Mandelbrot set for the quadratics R_b bifurcates into two symmetric components along the parameter axis $\operatorname{re} b = 1$, intersecting at the point $b = 1$. These components are not mere geometric reflections but retain full dynamical equivalence to the classical Mandelbrot set under topological conjugacy.

A key consequence of the conjugacy is the derivation of the boundary curve for the Mandelbrot set of R_b, which takes the form of a Cassini oval:

$$|b\,(2-b)| = 8.$$

This curve encloses both symmetric components of the Mandelbrot set, forming a unified topological structure.

The conjugacy further induces an obvious correspondence between critical points: $\phi^{-1}(0) = -b/2$. Consequently, the Mandelbrot set for R_b can be characterized in two equivalent ways: either by mapping points from the classical Mandelbrot set via ϕ^{-1}, or by identifying parameters $b \in \mathbb{C}$ for which

$$\lim_{k\to\infty} \left|R_b^k\,(-b/2)\right| \nrightarrow \infty.$$

Figure 1 illustrates the resulting Mandelbrot set alongside its bounding Cassini oval.

In subsequent analysis, we concentrate on the right component defined as the subset of parameters $b = b_0 + b_1\mathbb{i}$ with $b_0 \geq 1$. This component retains dynamical equivalence to the classical Mandelbrot set and serves as the foundation for our generalization to coquaternionic mappings.

Since the Mandelbrot set for R_b is topologically conjugate to the classical complex Mandelbrot set, we can conclude that it inherits the properties of being both bounded and connected. We now turn our focus to analyzing the Mandelbrot set arising from the generalization of this quadratic family to coquaternions.

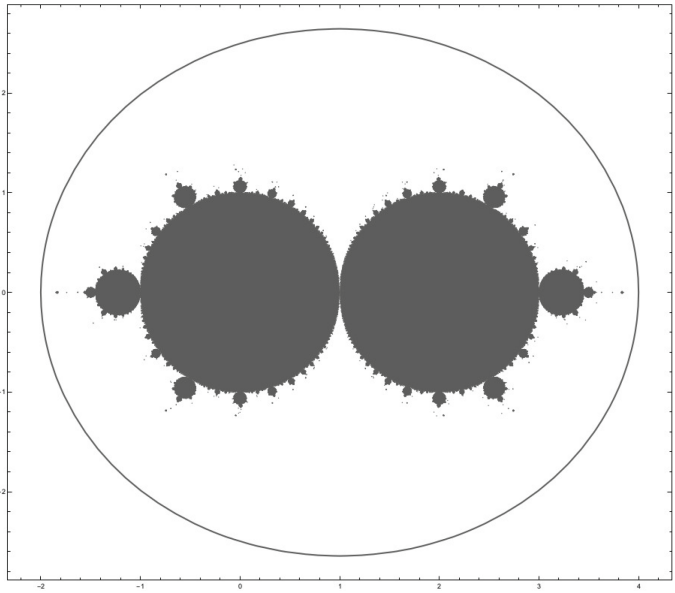

Fig. 1. The Mandelbrot set for the family of complex quadratics $R_b(z) = z^2 + bz$, bounded by the Cassini oval $|b(2-b)| = 8$. This Mandelbrot set bifurcates into two symmetric components along the parameter axis $\operatorname{re} b = 1$, intersecting at the point $b = 1$.

Consider the one-parameter family of quadratic maps

$$\begin{aligned} \mathsf{f}_\mathsf{b} : \mathbb{H}_{\mathrm{coq}} &\to \mathbb{H}_{\mathrm{coq}} \\ \mathsf{q} &\mapsto \mathsf{q}^2 + \mathsf{b}\,\mathsf{q} \end{aligned}$$

where b is a fixed parameter in $\mathbb{H}_{\mathrm{coq}}$, such that $\operatorname{re}\mathsf{b} > 1$. We introduce the following definition:

Definition 3. *The coquaternionic Mandelbrot set of the family* f_b *is the set of parameters* $\mathsf{b} \in \mathbb{H}_{\mathrm{coq}}$ *for which the map* f_b *has an attractor* Γ *such that the point* $-\mathsf{b}/2$ *belongs to its basin of attraction.*

We denote this coquaternionic Mandelbrot set as $\mathcal{M}_\mathsf{b}$.

Remark 1. The proposed definition of the coquaternionic Mandelbrot set for the family f_b differs from the standard one. The inclusion of a second restriction – requiring the existence of an attractor for the map f_b – is necessary due to results reported in [6], which indicate that there are complex attractors for R_b that are not attractors for f_b, where $\mathsf{b} = b_0 + b_1\mathbb{i}$. Including such parameters in the coquaternionic Mandelbrot set $\mathcal{M}_\mathsf{b}$ would be inconsistent. In the context of complex quadratic maps, this additional restriction would be excessive, as the existence of an attractor is guaranteed by the non-divergence of the orbit of the map's critical point.

The following results are crucial in guiding our approach to studying and obtaining the Mandelbrot set $\mathcal{M}_{\mathsf{b}}$.

Lemma 1. *Let ϕ_{h} be the similarity map defined by (1) for a given invertible coquaternion h. The dynamical system $(\mathbb{H}_{\text{coq}}, \mathsf{f}_{\mathsf{b}})$ is dynamically equivalent to the dynamical system $(\mathbb{H}_{\text{coq}}, \mathsf{f}_{\phi_{\mathsf{h}}(\mathsf{b})})$.*

Proof. We have

$$\begin{aligned}(\phi_{\mathsf{h}} \circ \mathsf{f}_{\mathsf{b}} \circ \phi_{\mathsf{h}}^{-1})(\mathsf{q}) &= (\phi_{\mathsf{h}} \circ \mathsf{f}_{\mathsf{b}})(\mathsf{hqh}^{-1}) \\ &= \phi_{\mathsf{h}}((\mathsf{hqh}^{-1})^2 + \mathsf{b}\,\mathsf{hqh}^{-1}) \\ &= \phi_{\mathsf{h}}(\mathsf{hq}^2\mathsf{h}^{-1} + \mathsf{b}\,\mathsf{hqh}^{-1}) \\ &= \mathsf{h}^{-1}(\mathsf{hq}^2\mathsf{h}^{-1} + \mathsf{b}\,\mathsf{hqh}^{-1})\mathsf{h} \\ &= \mathsf{q}^2 + \mathsf{h}^{-1}\mathsf{bh}\,\mathsf{q} = \mathsf{f}_{\phi_{\mathsf{h}}(\mathsf{b})}(\mathsf{q}).\end{aligned}$$

□

As a consequence of Lemma 1 above and of Theorem 1, we immediately obtain the result contained in the following theorem.

Theorem 2. *To study the dynamics of the quadratic map f_{b}, there is no loss of generality in assuming that the parameter b assumes one of the following forms, where $b_0, b_1, b_2 \in \mathbb{R}$:*

$\mathcal{B}_0$ $\quad \mathsf{b} = b_0$

$\mathcal{B}_1$ $\quad \mathsf{b} = b_0 + b_1\mathbb{i}, \quad b_1 > 0$

$\mathcal{B}_2$ $\quad \mathsf{b} = b_0 + b_2\mathbb{j}, \quad b_2 > 0$

$\mathcal{B}_3$ $\quad \mathsf{b} = b_0 + \mathbb{i} + \mathbb{j}$

We will refer to the four different forms of the parameter b given above as the *canonical forms*.

As an illustration of the dynamical implications of the existence of a conjugacy transformation between maps with similar parameters, consider the following example.

Example 1. Consider the set of parameters

$$\mathsf{b} = 2.632 + 1.2\,\mathbb{i} + b_2\mathbb{j} + b_3\mathbb{k}$$

such that $0 < b_2^2 + b_3^2 < 1.2$. Then, we computed the subsets of these parameters for which the orbit of the point $-\mathsf{b}/2$ converges to cycles of periods 6, 3, 14, and 1, respectively. The obtained results can be represent graphically on the $(\mathbb{j}, \mathbb{k})$ plane, see Fig. 2. As we expected, the subsets of parameters corresponding to the orbit of $-\mathsf{b}/2$ being attracted to a n-cycle are given as circular annuli in the $(\mathbb{j}, \mathbb{k})$ plane.

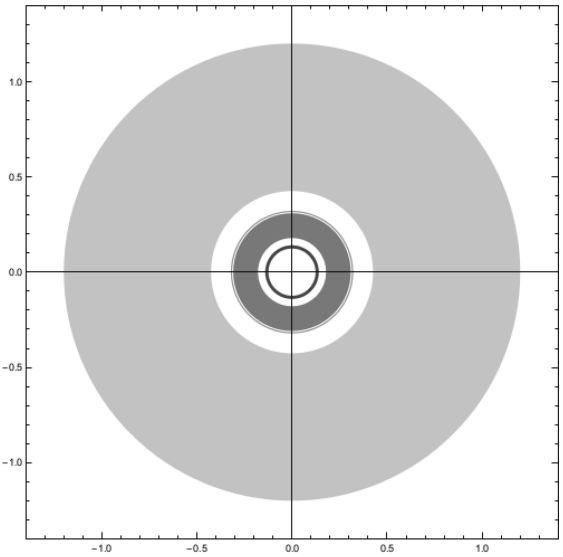

Fig. 2. The four gray circular annuli in the $(\mathbb{j}, \mathbb{k})$ plane indicate parameters $\mathsf{b} = 2.632 + 1.2\mathbb{i} + b_2\mathbb{j} + b_3\mathbb{k}$ for which the orbits of $-\mathsf{b}/2$ under f_b are attracted to cycles of period 6 (dark gray), 3 (medium gray), 14 (gray), and 1 (light gray), from the center outward, respectively. (Color figure online)

The circular parameter regions exhibiting equivalent dynamical behavior for orbits of the point $-\mathsf{b}/2$ directly result from the similarity transformation $\mathsf{f}_\mathsf{b} \sim \mathsf{f}_{\mathsf{b}'}$, where

$$\mathsf{b} = b_0 + b_1\mathbb{i} + b_2\mathbb{j} + b_3\mathbb{k}$$

and

$$\mathsf{b}' = b_0 + b_1\mathbb{i} + b_2'\mathbb{j} + b_3'\mathbb{k}$$

satisfy $b_2^2 + b_3^2 = b_2'^2 + b_3'^2$. This symmetry allows us to study $\mathcal{M}_\mathsf{b}$ by fixing $b_3 = 0$, with the understanding that these results extend to all possible b_3 values. The following result, a direct demonstration of map similarity, reveals the unbounded nature of the set $\mathcal{M}_\mathsf{b}$.

Lemma 2. *Given any parameter* $\mathsf{b} = b_0 + b_1\mathbb{i}$ *belonging to* $\mathcal{M}_\mathsf{b}$*, all parameters* $\mathsf{b}' = b_0 + b_1'\mathbb{i} + b_2'\mathbb{j} + b_3'\mathbb{k}$ *satisfying* ${b_1'}^2 - {b_2'}^2 - {b_3'}^2 = b_1^2$ *are similar to* b*. This infinite family of similar parameters demonstrates that the Mandelbrot set* $\mathcal{M}_\mathsf{b}$ *cannot be bounded.*

In Fig. 3 we can see the curve of points corresponding to similar parameters $\mathsf{b}' = 2.82 + b_1'\mathbb{i} + b_2'\mathbb{j}$ of a given parameter $\mathsf{b} = 2.82 + 0.59\mathbb{i} \in \mathcal{M}_\mathsf{b}$, being easy to conclude from this example that, unlike its complex counterpart, the Mandelbrot set $\mathcal{M}_\mathsf{b}$ is not bounded.

Given its role in analyzing $\mathcal{M}_\mathsf{b}$, next we introduce the set corresponding to its intersection with the complex plane (denoted $\mathcal{M}_0$), as shown in Fig. 4.

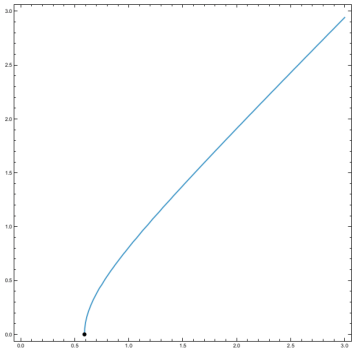

Fig. 3. Curve in the $(\mathbb{i}, \mathbb{j})$ plane showing parameters $\mathsf{b}' = 2.82 + b_1'\mathbb{i} + b_2'\mathbb{j}$ with ${b_1'}^2 - {b_2'}^2 = 0.59^2$. Since they are similar to $\mathsf{b} = 2.82 + 0.59\,\mathbb{i} \in \mathcal{M}_\mathsf{b}$, they also belong to $\mathcal{M}_\mathsf{b}$ and thus the coquaternionic Mandelbrot set cannot be bounded.

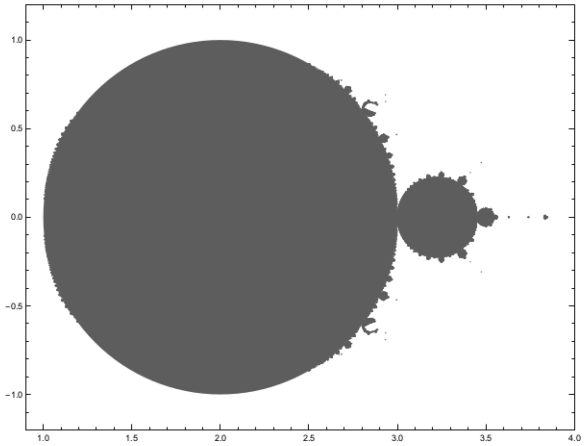

Fig. 4. Visualization of $\mathcal{M}_0$, the complex planar section of $\mathcal{M}_\mathsf{b}$.

As commonly recognized, the coquaternionic Mandelbrot set $\mathcal{M}_\mathsf{b}$ exists in 4-dimensional space, rendering complete visualization fundamentally impossible. However, our preceding analysis demonstrates that its projection onto the $(1, \mathbb{i}, \mathbb{j})$ subspace provides a structurally representative approximation. Nevertheless, even three-dimensional projections of such intricate mathematical objects inevitably obscure certain fine details. We therefore propose an alternative visualization strategy: by fixing $1 \leq b_0 \leq 4$, we can effectively represent parameters $\mathsf{b} = b_0 + b_1\mathbb{i} + b_2\mathbb{j}$ (within the $(\mathbb{i}, \mathbb{j})$ plane) belonging to $\mathcal{M}_\mathsf{b}$. Conceptually, the Mandelbrot set $\mathcal{M}_\mathsf{b}$ then becomes the topological union of all such cross-sectional slices.

For fixed $1 \leq b_0 \leq 4$, consider the set of parameters $(b_0, b_1) \in \mathcal{M}_0$. These manifest along the $b_2 = 0$ axis in the $(\mathbb{i}, \mathbb{j})$ plane. Through the equivalence

of maps with similar parameters, we can say that points (b_1', b_2') satisfying ${b_1'}^2 - {b_2'}^2 = b_1^2$ correspond to parameters $\mathsf{b}' = b_0 + b_1'\mathbb{i} + b_2'\mathbb{j}$ belonging to the coquaternionic Mandelbrot set $\mathcal{M}_\mathsf{b}$. Noticeably, all such solutions reside in the region $b_2 \leq b_1$.

Following a similar reasoning, let us compute the set of values b_2 such that parameters $\mathsf{b} = b_0 + b_2\mathbb{j}$ belong to $\mathcal{M}_\mathsf{b}$. This time, these parameters correspond to points appearing along the $b_1 = 0$ axis. Then, the points (b_1', b_2') satisfying ${b_1'}^2 - {b_2'}^2 = -b_2^2$ represent parameters $\mathsf{b}' = b_0 + b_1'\mathbb{i} + b_2'\mathbb{j}$ similar to $\mathsf{b} = b_0 + b_2\mathbb{j}$, therefore also belonging to $\mathcal{M}_\mathsf{b}$. We can easily conclude that these points are constrained to the region $b_2 \geq b_1$.

Thus, each slice of $\mathcal{M}_\mathsf{b}$ is formed by combining both parameter sets through union operation. An illustration of such slices are given in Fig. 5, Fig. 6, and Fig. 7.

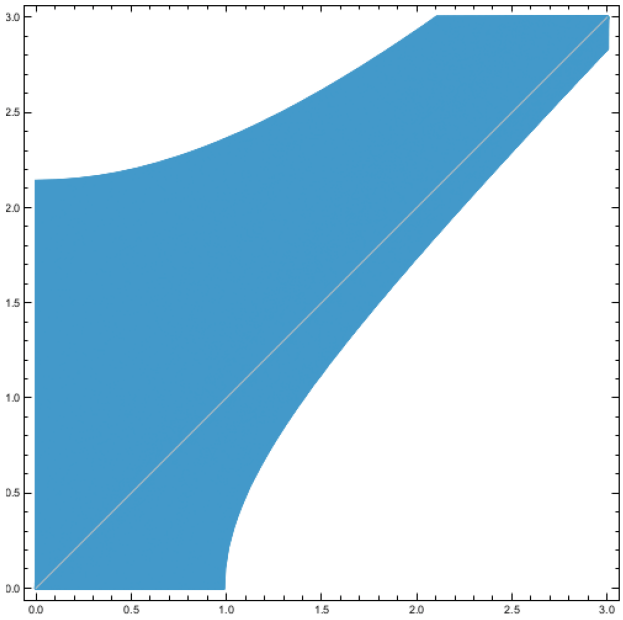

Fig. 5. The slice of the Mandelbrot set $\mathcal{M}_\mathsf{b}$ in the $(\mathbb{i}, \mathbb{j})$ plane corresponding to parameters $\mathsf{b} = 1.850 + b_1\mathbb{i} + b_2\mathbb{j}$.

The visualization of three coquaternionic Mandelbrot set $\mathcal{M}_\mathsf{b}$ slices yields two key observations: the first one concerns the complexity amplification: the intricate structure observed in the complex Mandelbrot set of quadratic maps $z^2 + bz$ becomes inherently amplified when generalized to coquaternions. This amplification arises because each point in the classical Mandelbrot set generates a curve of equivalent parameters in the $(\mathbb{i}, \mathbb{j})$ plane, preserving the original complexity while expanding its geometric representation.

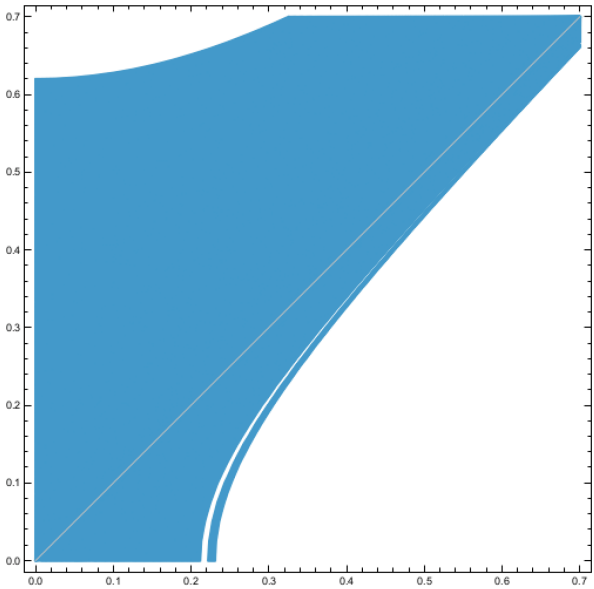

Fig. 6. The slice of the Mandelbrot set $\mathcal{M}_{\mathsf{b}}$ in the $(\mathbb{i}, \mathbb{j})$ plane corresponding to parameters $\mathsf{b} = 3.380 + b_1\mathbb{i} + b_2\mathbb{j}$.

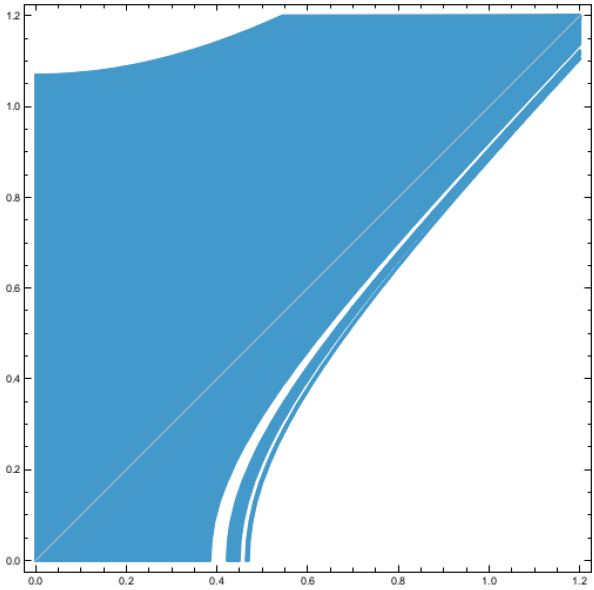

Fig. 7. The slice of the Mandelbrot set $\mathcal{M}_{\mathsf{b}}$ in the $(\mathbb{i}, \mathbb{j})$ plane corresponding to parameters $\mathsf{b} = 2.923 + b_1\mathbb{i} + b_2\mathbb{j}$.

The second one is about the upper-region simplicity: for parameters satisfying $b_2 > b_1$, upper regions correspond to coquaternionic parameters similar to $b_0 + b_2\mathbb{j} \in \mathcal{M}_{\mathsf{b}}$. These exhibit a direct correspondence with iterations of real quadratic maps, implying such regions derive from intervals of real parameters. Consequently, these upper subsets manifest as fundamentally simple geometric structures within the $(\mathbb{i}, \mathbb{j})$ plane.

5 Conclusions

The Mandelbrot set remains one of the most captivating and visually striking mathematical objects ever discovered. Defined by a simple criterion in the iteration of quadratic complex maps within the framework of complex discrete dynamical systems, it presents a significant challenge to generalize this concept to the four-dimensional space of coquaternions.

Extending the Mandelbrot set into the coquaternionic domain introduces profound mathematical challenges and reveals structures of remarkable complexity.

In this work, we explored the generalization of the Mandelbrot set from the quadratic mapping family $x^2 + b\,x$ in the complex plane to the coquaternionic space $\mathbb{H}_{\text{coq}}$. One of the first observations is the necessity of redefining the Mandelbrot set for this higher-dimensional context. Additionally, leveraging the equivalence of maps for similar parameters, we demonstrated that, unlike its complex counterpart, the coquaternionic Mandelbrot set is not a bounded set.

Using specialized visualization techniques, we concluded that, strictly speaking, the complexity of the coquaternionic Mandelbrot set $\mathbb{H}_{\text{coq}}$ mirrors that of its complex counterpart. While each point in the complex Mandelbrot set corresponds to a curve in the $(\mathbb{i}, \mathbb{j})$ plane for coquaternions, this geometric expansion preserves the same underlying complexity.

However, our final observation is that the Mandelbrot set does not fully capture the diversity and intricacy inherent in the dynamics generated by coquaternionic quadratic mappings. The apparent simplicity of upper-region slices conceals a deeper complexity in the dynamics associated with these parameter choices. This limitation arises because the only information encoded by the coquaternionic Mandelbrot set is whether or not the orbit of the critical point $-\mathsf{b}/2$ escapes to infinity. As such, it does not fully reflect the richness of dynamical behaviors present in this system.

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