A New Probability Distribution: Theory, Simulation and Real Application

Nadia Hashim Al-Noor[1], Rafida M. Elobaid[2(✉)], and Suzan J. Obaiys[3]

[1] Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq
nadialnoor@uomustansiriyah.edu.iq
[2] Faculty of Communication, Arts and Sciences, Canadian University Dubai, Dubai, United Arab Emirates
rafida@cud.ac.ae
[3] Deparoment of Computer System and Technology, Faculty of Computer Science and Information Technology, University of Malaya, 50603 Kuala Lumpur, Malaysia

Abstract. Developing and implementing innovative statistical distributions is an exciting field of study where different statistical models are established and proposed for data modeling in various sectors. The prime goal of this paper is to propose a novel probability distribution for analyzing real datasets. The new distribution is named the truncated Rayleigh odd Weibull Lomax (TROWL) distribution. Several statistical properties along with the entropy and reliability measures are derived. The maximum likelihood estimators of the unknown parameters are obtained. A simulation study is also provided. To illustrate the usefulness of TROWL distribution, a practical application is presented, where the comparison of the TROWL distribution is made with common competing probability models. Based on four information criterion quantities, it is observed that the new distribution is the best competing model for analyzing the considered real data.

Keywords: Lomax Distribution · Truncated Rayleigh Odd Weibull · Statistical Properties

1 Introduction

Lomax [1] pioneered the Lomax or Pareto II (shifted Pareto) distribution for modeling business failure data. This distribution is widely used in various domains, including income and wealth disparity, actuarial science, medical and biological sciences, engineering, and reliability modeling. The distributional features, estimates, and inference of the Lomax distribution are presented in many studies, see [2]. There are several approaches to characterizing the Lomax distribution. It is classified as a variant of the Pearson type VI distribution and a hybrid of the exponential and gamma distributions. This distribution has been proposed as a heavy-tailed alternative to the exponential, Weibull, and Gamma distributions. It is also connected to the Burr distribution family and, as a specific case, compound gamma distributions. The Lomax model, which belongs to the decreasing failure rate family, limits residual lives at advanced ages (see

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2024
O. Gervasi et al. (Eds.): ICCSA 2024 Workshops, LNCS 14816, pp. 284–296, 2024.
https://doi.org/10.1007/978-3-031-65223-3_19

[2–4]). In recent years, there has been an increased interest in developing innovative generators for univariate continuous distributions. This entails adding one or more additional parameters to the existing distribution. These factors have been useful for investigating tail properties and improving the goodness-of-fit of the produced distribution. Several notable generator families can be found in [5–14]. The Lomax distribution has undergone substantial refining and extension to fit a wide range of situations and applications [15–18]. The main aim of this paper is to provide another version of extended Lomax distribution and to investigate several of its structural features. The novel distribution is proposed by adding two more parameters to the Lomax model. The rest of the paper is organized as follows. Section 2 contains the statistical methodology of the proposed distribution along with its cumulative distribution function (CDF), and probability density function (PDF). Section 3 discussed the distribution's essential statistical properties including reliability measures, linear form (expansion formula of the PDF), moments, moment-generating function, quantile function, and entropy. Further, estimates of the model's unknown parameters by maximum likelihood procedure are presented in Sect. 4 along with an empirical application. Section 5 provides a real application, while Sect. 6 offers some concluding remarks.

2 A New Model

In this section, the traditional Lomax is chosen as the baseline distribution to introduce a new member of the Truncated Rayleigh Odd Weibull-G (TROW − G) family proposed by Al-Noor and Sultan [19]. The CDF and PDF of the TROW − G with two parameters (scale parameter $\theta > 0$ and shape parameter $\beta > 0$) are given by

$$F(x;\theta,\beta,\psi) = \frac{1}{1-e^{-\theta/2}}\left(1-e^{-\frac{\theta}{2}\left(1-e^{-A^{\beta}}\right)^{2}}\right); A = \frac{G(x;\psi)}{\overline{G}(x;\psi)} \tag{1}$$

$$f(x;\theta,\beta,\psi) = \frac{\theta\beta g(x;\psi)G(x;\psi)^{\beta-1}}{\left(1-e^{-\theta/2}\right)\overline{G}(x;\psi)^{\beta+1}} e^{-A^{\beta}}\left(1-e^{-A^{\beta}}\right)e^{-\frac{\theta}{2}\left(1-e^{-A^{\beta}}\right)^{2}} \tag{2}$$

where $G(x;\psi)$ and $g(x;\psi)$ represent the CDF and PDF of any baseline distribution with vector parameter ψ and $\overline{G}(x;\psi) = 1 - G(x;\psi)$.

Consider the baseline CDF ($G(x;\psi)$) and PDF ($g(x;\psi)$) related to the two parameters "shape (α) and scale(δ)" Lomax distribution that is given by [16]

$$F(x;\alpha,\delta) = 1-(1+\delta x)^{-\alpha}\ ;\ x>0,\alpha,\delta>0 \tag{3}$$

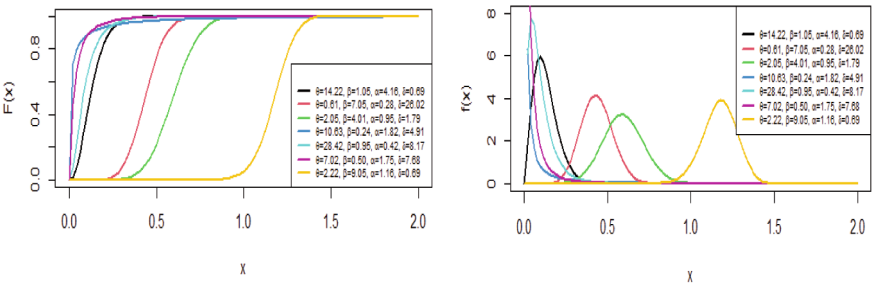
$$f(x;\alpha,\delta) = \alpha\delta(1+\delta x)^{-(\alpha+1)}\ ;\ x>0,\alpha,\delta>0 \tag{4}$$

Substituting (3) and (4) in (1) and (2), with $A = \frac{G(x;\alpha,\delta)}{\overline{G}(x;\alpha,\delta)} = (1+\delta x)^{\alpha}-1$, the CDF and PDF of the new extended Lomax namely Truncated Rayleigh Odd Weibull Lomax (TROWL) with four parameters $(\theta,\beta,\alpha,\delta)$ are given by

$$F(x;\theta,\beta,\alpha,\delta) = \frac{1}{1-e^{-\theta/2}}\left(1-e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)^{2}}\right); x>0,\theta,\beta,\alpha,\delta>0 \tag{5}$$

$$f(x;\theta,\beta,\alpha,\delta)=\frac{\theta\beta\alpha\delta}{\left(1-e^{-\theta/2}\right)}(1+\delta x)^{\alpha\beta-1}\left(1-(1+\delta x)^{-\alpha}\right)^{\beta-1}e^{-((1+\delta x)^{\alpha}-1)^{\beta}}$$
$$\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)^{2}} \tag{6}$$

Related to Fig. 1, the CDF's features are demonstrated in (a), and different PDF shapes are clearly shown in (b). As a result, it is noted that this new distribution is particularly suitable for modeling various positive data.



(a) (b)

Fig. 1. The TROWL (a) CDF (b) PDF plots with different parameter's values

3 Essential Statistical Properties

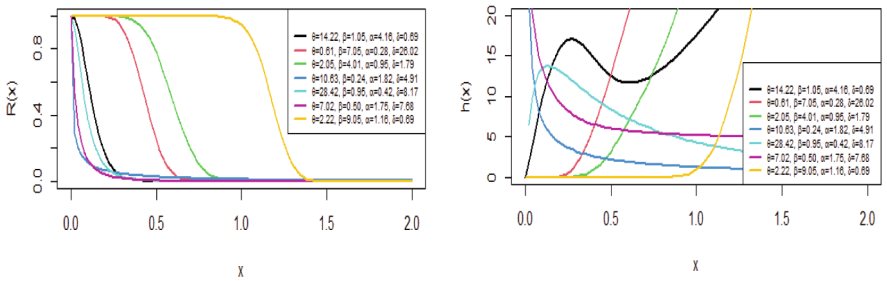
3.1 Reliability Measures

The reliability and hazard functions (RF and HF) are the most often used reliability measures in life data analysis and reliability engineering to assess a system's "component or item" operational performance. The forms of RF and HF of the TROWL are

$$R(x;\theta,\beta,\alpha,\delta)=1-F(x;\theta,\beta,\alpha,\delta)$$
$$=1-\frac{1}{1-e^{-\theta/2}}\left(1-e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)^{2}}\right) \tag{7}$$

$$h(x;\theta,\beta,\alpha,\delta)=\frac{f(x;\theta,\beta,\alpha,\delta)}{R(x;\theta,\beta,\alpha,\delta)}$$
$$=\frac{\theta\beta\alpha\delta(1+\delta x)^{\alpha\beta-1}\left(1-(1+\delta x)^{-\alpha}\right)^{\beta-1}}{e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)^{2}}-e^{-\frac{\theta}{2}}}\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)$$
$$e^{-((1+\delta x)^{\alpha}-1)^{\beta}-\frac{\theta}{2}\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)^{2}} \tag{8}$$

Related to Fig. 2, the reliability features are demonstrated in (a), and an aspect of the hazard function that can take on different shapes, such as increasing, decreasing, bathtub, skewed, J, and reversed-J, is shown in (b), indicating that the new distribution can be applied to analyze a variety of life data.



(a) (b)

Fig. 2. The TROWL (a) RF (b) HF plots with different parameter's values

3.2 Linear Form

The helpful expansion formula for PDF of the TROWL can be obtained using the following expansion formulas with some simple mathematical steps:

$(1-z)^a = \sum_{m=0}^{\infty}(-1)^m\binom{a}{m}z^m$; $\frac{1}{(1-z)^a} = \sum_{m=0}^{\infty}\binom{m+a-1}{m}z^m$ that valid for $|z| < 1, a > 0$; and $e^{-z} = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}z^m$. Thus, the PDF can be expanded to take the following form

$$f(x;\theta,\beta,\alpha,\delta)^E = \sum_{k,m=0}^{\infty}\xi_{(k,m)}\alpha\delta(1+\delta x)^{-(\alpha+1)}\left(1-(1+\delta x)^{-\alpha}\right)^{m+\beta(k+1)-1} \quad (9)$$

where

$$\xi_{(k,m)} = \frac{\theta\beta}{1-e^{-\frac{\theta}{2}}}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+k}}{i!k!}(j+1)^k\binom{2i+1}{j}\left(\frac{\theta}{2}\right)^i\binom{m+\beta(k+1)}{m} \quad (10)$$

Further, the PDF in (9) can be expanded more by applying the Binomial expansion to be expressed as a linear representation of the baseline Lomax distribution, i.e.

$$\begin{aligned} f(x;\theta,\beta,\alpha,\delta)^E &= \sum_{k,m,s=0}^{\infty}\xi_{(k,m)}(-1)^s\binom{\beta(k+1)+m-1}{s}\alpha\delta(1+\delta x)^{-(\alpha(s+1)+1)} \\ &= \sum_{k,m,s=0}^{\infty}\xi_{(k,m)}\frac{(-1)^s}{s+1}\binom{\beta(k+1)+m-1}{s}\alpha(s+1)\delta(1+\delta x)^{-(\alpha(s+1)+1)} \\ &= \sum_{k,m,s=0}^{\infty}\xi_{(k,m)}\frac{(-1)^s}{s+1}\binom{\beta(k+1)+m-1}{s}g(x;\alpha(s+1),\delta)_L \end{aligned} \quad (11)$$

where $g(x;\alpha(s+1),\delta)_L$ represents the PDF of the traditional Lomax distribution with parameters $\alpha(s+1)$ and δ. That linear form plays an important role and will be used to obtain explicit expressions for the moments and moment generating function.

3.3 Moments and Moment Generating Function

The rth non-central moment can easily found regards linear form in (11)(11) as follows

$$E(X^r) = \sum_{k,m,s=0}^{\infty} \xi_{(k,m)} \frac{(-1)^s}{s+1} \binom{\beta(k+1)+m-1}{s} E(X^r)_{g(x;\alpha(s+1),\delta)_L}$$

where $E(X^r)_{g(x;\alpha(s+1),\delta)_L}$ represents the rth non-central moment of Lomax distribution with parameters $\alpha(s+1)$ and δ (see [16]). Therefore, the $r^{th}(r = 1,2,\ldots)$ moment of TROWL distribution for $\alpha(s+1) > r$ is given by

$$E(X^r) = \sum_{k,m,s=0}^{\infty} \xi_{(k,m)} \frac{(-1)^s}{s+1} \binom{\beta(k+1)+m-1}{s} \alpha(s+1)\delta^{-r} \frac{r!\Gamma(\alpha(s+1)-r)}{\Gamma(\alpha(s+1)+1)} \quad (12)$$

Based on (12), the moment generating function can be attained by applying its definition with the Maclaurin series as $\varphi_X(t) = E\left(e^{tX}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$, then

$$\varphi_X(t) = \sum_{k,m,s,r=0}^{\infty} \xi_{(k,m)} \frac{(-1)^s t^r}{s+1} \binom{\beta(k+1)+m-1}{s} \alpha(s+1)\delta^{-r} \frac{\Gamma(\alpha(s+1)-r)}{\Gamma(\alpha(s+1)+1)} \quad (13)$$

3.4 Quantile Function

The quantile function (QF), $Q(q)$ for $0 < q < 1$, is determined by solving the equation $F(Q(q)) = q$. Therefore, once calculating (5) for $Q(q)$ in terms of q yields the QF of the TROWL distribution as

$$Q(q) = x_q = \frac{1}{\delta}\left[\left(1 + \left\{-\ln\left(1 - \left(\frac{-2}{\theta}\ln\left(1 - \left(1 - e^{-\frac{\theta}{2}}\right)q\right)\right)^{\frac{1}{2}}\right)\right\}^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} - 1\right] \quad (14)$$

The TROWL random variable can be generated from (14) by replacing q with U; where U denotes a standard Uniform random variable. The median can be gained easily with setting $q = \frac{1}{2}$. Other quantities, such as the inter-quantile range (IQR), skewness (Sk) and kurtosis (Ku) measures can be also computed by

$$IQR = Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right); Sk. = \frac{Q\left(\frac{6}{8}\right) + Q\left(\frac{2}{8}\right) - 2Q\left(\frac{4}{8}\right)}{IQR}; Ku. = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{IQR}.$$

3.5 Entropy

The entropy of a variable is a measure of variation in uncertainty that is commonly used in physics, tumor molecular imaging, and engineering. Shannon is discussed here due to its present applications, such as in simulations of laser dynamics and emergent events [20].

The Shannon entropy is specified by $S(X) = -E(ln(f(x))$. For more details about this entropy (see [11] and [20]). Recall (6), then

$$S(X) = -E(\ln(f(x;\theta,\beta,\alpha,\delta)) = \ln\left(\frac{1-e^{-\theta/2}}{\theta\beta\alpha\delta}\right) - (\alpha\beta-1)E(\ln(1+\delta x)) - (\beta-1)E\left(\ln\left(1-(1+\delta x)^{-\alpha}\right)\right) + E\left((1+\delta x)^{\alpha}-1\right)^{\beta} - E\left(\ln\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)\right) - \frac{\theta}{2}E\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)^{2} \quad (15)$$

The mathematical expectations can be calculated as follows:
Based on (9),

$$E(\ln(1+\delta x)) = \sum_{k,m=0}^{\infty} \xi_{(k,m)} \int_{0}^{\infty} \ln(1+\delta x)\,\alpha\delta\, e^{-(\alpha+1)\ln(1+\delta x)} \left(1-(1+\delta x)^{-\alpha}\right)^{m+\beta(k+1)-1} dx$$

Using the transformation $y = -(\alpha+1)\ln(1+\delta x)$ and then the expansion series $(1-z)^{a}$ with some mathematical simplification steps, the outcome is

$$E(\ln(1+\delta x)) = \sum_{k,m,s=0}^{\infty} \xi_{(k,m)} \frac{(-1)^{s}}{\alpha(s+1)^{2}} \binom{\beta(k+1)+m-1}{s} \quad (16)$$

Using series $\ln(1-z) = \sum_{m=1}^{\infty} \frac{1}{m} z^{m}$; $|z| < 1$ and the negative Binomial series, then

$$E\left(\ln\left(1-(1+\delta x)^{-\alpha}\right)\right) = \sum_{m=1}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{m} \delta^{r} \binom{r+\alpha m-1}{r} E\left(X^{r}\right) \quad (17)$$

Using the Binomial series and expansion series $(1-z)^{a}$ with some mathematical simplification steps, then

$$E\left((1+\delta x)^{\alpha}-1\right)^{\beta} = \sum_{m,r=0}^{\infty} (-1)^{\beta-m} \delta^{r} \binom{\beta}{m} \binom{\alpha m}{r} E\left(X^{r}\right) \quad (18)$$

Using series $\ln(1-z)$, then

$$E\left(\ln\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)\right) = \sum_{m=1}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!} m^{s-1} E\left((1+\delta x)^{\alpha}-1\right)^{\beta s} \quad (19)$$

$$E\left(1-e^{-((1+\delta x)^{\alpha}-1)^{\beta}}\right)^{2} = \sum_{m,s=0}^{\infty} \frac{(-1)^{s+m}}{s!} m^{s} E\left((1+\delta x)^{\alpha}-1\right)^{\beta s} \quad (20)$$

where $E((1+\delta x)^{\alpha}-1)^{\beta s}$ as in (18) (18) with βs instead of β, and $E(X^{r})$ as given in (12).

After substituting (16)–(20) in (15), the Shannon entropy of the TROWL is attained.

4 Estimation and Empirical Illustration

The maximum likelihood methodology is the most often used parameter estimate method due to its desirable features such as consistency, asymptotic efficiency, and invariance. Here, the maximum likelihood estimates (MLEs) of the unknown TROWL parameters are obtained and then investigated their performance via a simulation study.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n taken from the TROWL distribution with parameters $(\theta, \beta, \alpha, \delta)$. According to $f(\theta, \beta, \alpha, \delta)$ in (6), the likelihood and natural logarithm likelihood functions are given by

$$L = \prod_{i=1}^{n} \frac{\theta\beta\alpha\delta}{\left(1 - e^{-\theta/2}\right)} (1 + \delta x_i)^{\alpha\beta - 1} \left(1 - (1 + \delta x_i)^{-\alpha}\right)^{\beta - 1} e^{-((1+\delta x_i)^{\alpha} - 1)^{\beta}} \left(1 - e^{-((1+\delta x_i)^{\alpha} - 1)^{\beta}}\right) e^{-\frac{\theta}{2}\left(1 - e^{-((1+\delta x_i)^{\alpha} - 1)^{\beta}}\right)^2}$$

and

$$\begin{aligned} \ell = \ln(L) =& n \ln\left(\frac{\theta\beta\alpha\delta}{1 - e^{-\theta/2}}\right) + (\alpha\beta - 1) \sum_{i=1}^{n} \ln(1 + \delta x_i) \\ &+ (\beta - 1) \sum_{i=1}^{n} \ln\left(1 - (1 + \delta x_i)^{-\alpha}\right) - \sum_{i=1}^{n} \left((1 + \delta x_i)^{\alpha} - 1\right)^{\beta} \\ &+ \sum_{i=1}^{n} \ln\left(1 - e^{-((1+\delta x_i)^{\alpha} - 1)^{\beta}}\right) - \frac{\theta}{2} \sum_{i=1}^{n} \left(1 - e^{-((1+\delta x_i)^{\alpha} - 1)^{\beta}}\right)^2 \end{aligned}$$

To obtain the MLEs $\widehat{\theta}, \widehat{\beta}, \widehat{\alpha}$, and $\widehat{\delta}$, set the partial derivatives $\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \delta}$ to zero and solve numerically. After collecting the MLEs for the TROWL parameters, the next step is to conduct a simulation study to evaluate the estimate's performance. The empirical investigation is conducted with various combinations of parameter default settings and a 3000 i.i.d random sample created using formula (14) with $q = u$; $u \in U(0,1)$, and sample sizes $n = 15, 30, 60$, and 120. To quantify the performance of the MLEs of four parameters $(\Delta = \theta, \beta, \alpha, \delta)$, the mean square error (MSE) is used, where:

$MSE\left(\widehat{\Delta}\right) = \frac{1}{3000} \sum_{i=1}^{3000} \left(\widehat{\Delta}_i - \Delta\right)^2$. The simulation outcomes are offered in Tables 1 and 2. All numerical illustrations are made using the programming language R. From the empirical results, it is noted that as sample size increases, the MSE values seem to decrease as expected, demonstrating the consistency of the estimators.

Table 1. The MSE values of the parameters with $\theta = 0.8$

β	α	δ	n	$MSE\left(\hat{\theta}\right)$	$MSE\left(\hat{\beta}\right)$	$MSE(\hat{\alpha})$	$MSE\left(\hat{\delta}\right)$
0.8	0.8	0.8	15	0.15376	0.04660	0.08556	0.13058
			30	0.13982	0.02417	0.05873	0.10991
			60	0.11774	0.01410	0.03949	0.08264
			120	0.07146	0.00848	0.02328	0.05906
		3.8	15	2.54589	0.05636	0.04664	0.57484
			30	2.04800	0.02533	0.02507	0.48608
			60	1.56499	0.01299	0.01537	0.36576
			120	1.21595	0.00709	0.00968	0.25682
	3.8	0.8	15	2.54359	0.05585	1.03891	0.04524
			30	1.96029	0.02426	0.73120	0.02168
			60	1.53644	0.01253	0.45651	0.01415
			120	1.17183	0.00695	0.28833	0.00812
		3.8	15	4.77249	0.05847	1.51392	0.65831
			30	3.64465	0.02865	1.12342	0.47772
			60	2.40356	0.01453	0.58570	0.36005
			120	1.71234	0.00817	0.35195	0.24300
3.8	0.8	0.8	15	1.00856	0.67216	0.00455	0.00883
			30	0.95736	0.30337	0.00245	0.00512
			60	0.69499	0.14705	0.00143	0.00288
			120	0.56562	0.07244	0.00102	0.00222
		3.8	15	1.31644	0.73126	0.00244	0.10742
			30	1.21175	0.31548	0.00112	0.05771
			60	1.01482	0.15487	0.00064	0.03738
			120	0.81098	0.07549	0.00040	0.02300
	3.8	0.8	15	1.25079	0.82274	0.05770	0.00295
			30	1.16146	0.34628	0.03613	0.00163
			60	1.07692	0.16451	0.02338	0.00089
			120	0.97032	0.08258	0.01847	0.00066
		3.8	15	1.57937	0.88882	0.10691	0.10398
			30	1.55206	0.35128	0.06908	0.06453
			60	1.40266	0.17155	0.04836	0.04292
			120	1.08332	0.08965	0.03923	0.03283

Table 2. The MSE values of the parameters with $\theta = 3.8$

β	α	δ	n	$MSE\left(\hat{\theta}\right)$	$MSE\left(\hat{\beta}\right)$	$MSE\left(\hat{\alpha}\right)$	$MSE\left(\hat{\delta}\right)$
0.8	0.8	0.8	15	0.71852	0.05142	0.13726	0.20735
			30	0.53042	0.02325	0.07076	0.14500
			60	0.49501	0.01238	0.04169	0.10353
			120	0.40646	0.00754	0.02913	0.07431
		3.8	15	2.59504	0.04668	0.06750	1.02020
			30	2.16270	0.01871	0.03816	0.83027
			60	1.56751	0.00890	0.01811	0.64045
			120	1.44441	0.00496	0.01164	0.48159
	3.8	0.8	15	3.32293	0.04563	1.36966	0.05166
			30	2.33285	0.01866	1.11777	0.02744
			60	1.63680	0.00886	0.68167	0.01550
			120	1.24378	0.00430	0.41350	0.01032
		3.8	15	6.13892	0.04687	1.83802	0.86736
			30	4.39848	0.01992	1.39589	0.63768
			60	2.64507	0.00933	0.84043	0.46963
			120	1.79249	0.00486	0.51075	0.34381
3.8	0.8	0.8	15	0.90275	0.83419	0.00505	0.00923
			30	0.74397	0.32512	0.00253	0.00525
			60	0.61284	0.14825	0.00156	0.00325
			120	0.52579	0.07165	0.00113	0.00249
		3.8	15	1.14603	0.87652	0.00328	0.12181
			30	1.04420	0.34476	0.00110	0.05357
			60	0.92939	0.15582	0.00061	0.03218
			120	0.78274	0.07310	0.00036	0.02452
	3.8	0.8	15	1.07682	0.95370	0.07336	0.00355
			30	1.05458	0.37302	0.03491	0.00155
			60	0.87862	0.16460	0.02292	0.00085
			120	0.75269	0.07604	0.01579	0.00060
		3.8	15	1.73248	0.96797	0.11104	0.09848
			30	1.73136	0.36938	0.05497	0.05668
			60	1.58875	0.16517	0.03986	0.03819
			120	1.17608	0.07534	0.03401	0.02842

5 Real Application

The data represent a medical dataset that includes 42 values indicating deaths from leukemia, the most common type of cancer among children under the age of 15, as reported by the Iraqi Ministry of Health and Environment's Central Children's Teaching Hospital from January 1, 2021, to August 31, 2022 (for more details about the data, see [21]). These values are:

"264, 91, 21, 5, 2, 515, 33, 154, 17, 60, 65, 534, 1, 177, 15, 16, 562, 23, 4, 219, 22, 2, 431, 34, 485, 145, 29, 22, 180, 241, 25, 3, 158, 64, 111, 131, 5, 45, 17, 83, 258, 594".

Table 3 exhibits the data's basic descriptive statistics, indicating data to be right-skewed and leptokurtic. The proposed distribution's fitting behavior is compared with the baseline and extended Gompertz linked to different families: Beta-G [5], Kumaraswamy-G [7], Exponentiated Generalized-G [8], Weibull-G [9], and Gompertz-G [10] represented by BL, KuL, EGL, WL, and GoL, respectively. It is worth noting that the p-values of KS (Kolmogorov Smirnov) goodness of fit for the competitive distributions are significant (more than 0.05). Furthermore, the fitted competitive distributions are evaluated using four information criteria (IC). The mathematical forms of ICs (Akaike, Consistent Akaike, Bayesian, and Quinn and Hanan) are given with the sample size (n), number of the parameters (p), and the natural log-likelihood function assessed at ML estimates ($\hat{\ell}$) by:

$$AIC = -2\left(\hat{\ell} - p\right); CAIC = -2\left(\hat{\ell} - np/(n-p-1)\right); BIC = -2\hat{\ell} + p\ln(n);$$

and $HQIC = -2\left(\hat{\ell} - p\ln(\ln(n))\right)$. The lower value for these IC indicates a better model. Table 4 displays the computed MLE values for all models' unknown parameters. The IC results are presented in Table 5. The TROWL distribution has the lowest IC values compared to other competitive distributions, giving it the best fit for the data under consideration. Furthermore, the plots of the estimated PDFs and CDFs in Fig. 3 demonstrate TROWL's best fitting.

Table 3. Basic descriptive statistics for the dataset

No. of Observations	Mean	Median	Sk	Ku	IQR	Min	Max
42	139.6	62	1.49	1.10	172.8	1	594

Table 4. The MLE values for real data

Dist.	$\widehat{\theta}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\delta}$
TROWL	1.873245142	0.408337976	2.226987477	0.005220733
BL	15.16697169	17.41461274	0.089978350	23.49742675
KuL	0.778873302	5.107866512	0.393857533	0.003247987
EGL	1.525472892	0.720553488	1.525472857	0.003677375
WL	0.774736320	0.026015526	0.123750038	0.002238511
GoL	0.150395300	2.452011600	0.263685900	0.760978600
L	—	—	1.118514210	0.015501820

Table 5. The IC for fitting real data

Dist.	NLL	AIC	CAIC	BIC	HQIC
TROWL	244.2122	496.4244	497.5054	503.3750	498.9721
BL	248.0107	504.0213	505.1024	510.9720	506.5690
KuL	245.4805	498.9609	500.0420	505.9116	501.5086
EGL	245.4544	498.9088	499.9899	505.8595	501.4565
WL	245.4091	498.8182	499.8993	505.7689	501.3659
GoL	244.2726	496.5451	497.6262	503.4958	499.0928
L	246.7595	497.5189	497.8266	503.9943	498.9928

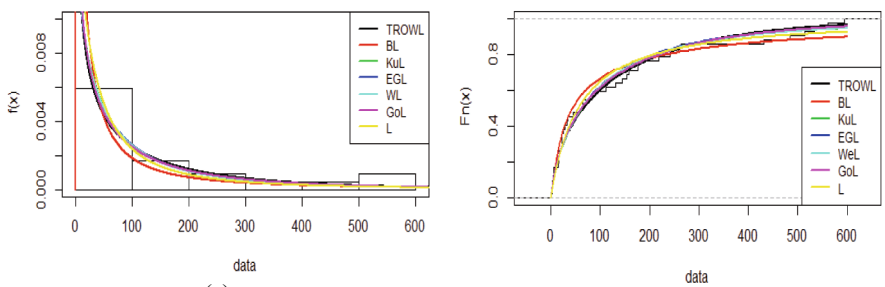


Fig. 3. The TROWL and competitive distributions (a) histogram (b) empirical CDF plots.

6 Conclusion

The Lomax distribution has been widely used in a variety of contexts. Although the Lomax distribution was originally designed to represent business failure data, it has since been utilized for dependability modeling and life testing. In contrast, the Lomax distribution has served as the basis for several generalizations and extensions. Here, a new extended Lomax distribution with four parameters, the Truncated Rayleigh Odd Weibull Lomax (TROWL), is introduced. Some important statistical functions and features are discussed, including the quantile function, linear representation, moments, moment-generating function, median, simulated data, reliability measures, and entropy. The maximum likelihood method is used to estimate the distribution parameter. The new distribution's abilities and flexibility are demonstrated through simulation and real-life applications. The estimators' performance is measured using mean square error. The simulation results demonstrated the flexibility and consistency of the maximum likelihood estimates, and for various information criteria, the real-life application results demonstrated that the proposed distribution outperformed other considered distributions for representing deaths from leukemia among children under the age of 15 in Iraq from January 1, 2021, to August 31, 2022. Because of this versatility, the TROWL distribution may be used in a larger range of applications. Furthermore, as an open subject for future research, various parameter estimate approaches such as moments, least squares, and Bayesian employing informative priors might be examined.

References

1. Lomax, K.S.: Business failures: another example of the analysis of failure data. J. Am. Stat. Assoc. **49**(268), 847–852 (1954)
2. Tahir, M.H., Gauss, M.C., Mansoorz, M., Zubair, M.: The Weibull-Lomax distribution: properties and applications. Hacettepe J. Math. Statist. **44**(2), 455–474 (2015)
3. Chahkandi, M., Ganjali, M.: On some lifetime distributions with decreasing failure rate. Comput. Stat. Data Anal. **53**, 4433–4440 (2009)
4. El-Din, M.M., Okasha, H.M., Al-Zahrani, B.: Empirical bayes estimators of reliability performances using progressive type-ii censoring from lomax model. J. Adv. Res Appl. Math. **5**(1), 74–83 (2013)
5. Eugene, N., Lee, C., Famoye, F.: Beta-normal distribution and its applications. Commun. Statist.-Theory Meth. **31**, 497–512 (2002)
6. Zografos, K., Balakrishnan, N.: On the families of beta-and gamma-generated generalized distribution and associated inference. Statist. Methodol. **6**, 344–362 (2009)
7. Cordeiro, G.M., de Castro, M.: A new family of generalized distributions. J. Stat. Comput. Simul. **81**, 883–893 (2011)
8. Cordeiro, G.M., Ortega, E.M.M., da Cunha, D.C.C.: The exponentiated generalized class of distributions. J. Data Sci. **11**, 1–27 (2013)
9. Bourguignon, M., Silva, R.B., Cordeiro, G.M.: The Weibull-G family of probability distributions. J. Data Sci. **12**(1), 53–68 (2014)
10. Alizadeh, M., Cordeiro, G.M., Pinho, L.G.B., Ghosh, I.: The Gompertz-G family of distributions. J. Statist. Theory Pract. **11**(1), 179–207 (2017)
11. Hesham, R., Jamal, F., Othman, S., Hamedani, G.G.: The transmuted Gompertz-G family of distributions: properties and applications. Tbilisi Math. J. **11**, 47–67 (2018)

12. Abid, S.H., Al-Noor, N.H., Boshi, M.A.A.: On the generalized inverse Weibull distribution. AIP Conf. Proc. **2086**, 030002 (2019)
13. Dey, S., Nassar, M., Kumar, D., Alzaatreh, A., Tahir, M.H.: A new lifetime distribution with decreasing and upside-down bathtub-shaped hazard rate. Statistica (Bologna) **4**, 399–426 (2019)
14. Mandouh, R.M., Mahmoud, M.R., Abdelatty, R.E.: A new (T-Xθ) family of distributions: properties, discretization and estimation with applications. Sci. Rep. **14**, 1–22 (2024)
15. Cordeiro, G.M., Ortega, E.M., Popovi´c, B.V., Pescim, R.R.: The Lomax generator of distributions: properties, minification process and regression model. Appl. Math. Comput. **247**, 465–486 (2014)
16. Shabbir, M., Riaz, A., Gull, H.: Rayleigh Lomax distribution. J. Middle East North Africa Sci. **4**(12), 1–4 (2018)
17. Hadi, H.H., Al-Noor N.H.: Truncated exponential Marshall Olkin Lomax distribution: properties, entropies, and applications. In: 6th Local Scientific Conference-3rd Scientific International, pp. 1–12. AIP Conference Proceedings 2414, 040054, Iraq (2023)
18. Al-Zahrani, B., Sagorb, H.: The Poisson-Lomax distribution. Revista Colombiana de Estadística **37**(1), 223–243 (2014)
19. Al-Noor, N.H., Sultan, A.J.: Truncated Rayleigh odd Weibull exponential distribution with application to COVID-19 data. In: Contemporary Information Technology and Mathematics (ICCITM). 8th International Conference, pp. 410–416. IEEE, Iraq (2022)
20. Guisado, J.L., Jiménez-Morales, F., Guerra, J.M.: Application of Shannon's entropy to classify emergent behaviors in a simulation of laser dynamics. Math. Comput. Model. **42**, 847–854 (2005)
21. Ghasaq, H.S., Hassan I.: Study of new exponential-Weibull distribution about children with Leukemia. pp. 1–12. AIP Conference Proceedings 3036, 040030, Iraq (2024)