Mathematical and Statistical Frameworks Fostering Advances in AI Systems and Computing

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Abstract. The rapid expansion of artificial intelligence (AI) areas and applications has become an everyday social reality and the development of computing methods and of technology, in order to accomplish for the respective needs, emerge as one of the biggest challenges of this century. However, a strong background in Mathematics and Statistical fundamentals is crucial to foster the robustness of AI Systems and to attain for the correspondent generalization capabilities. Thus, recognizing the importance and benefits of integrate multiple mathematical methodologies that can leverage the unique strengths of each to create synergistic effects, is here discussed. In the literature it is possible to find diverse connections of Mathematical and Statistical frameworks with AI Systems and Computing. For illustration: Algebra - can be used to manipulate data in high-dimensional spaces; Statistical Analysis - helps on providing confidence in decision-making; Linear Regression- can be used on data modelling and accuracy assessment; and Bayesian statistics - can be used in probabilistic programming for AI applications. This work particularly highlights the evolving role of Hadamard Matrices and Coding Theory, illustrating their synergy with AI Systems and Computing, and how such synergy is driving innovation in various fields and approaching the new reality of Quantic paradigms.

Keywords: computational mathematics · geometric modeling · incomplete block designs · combinatorial optimization · hadamard matrices · AI · quantum computing

1 Introduction

In 1956 John McCarthy coined originally the term Artificial Intelligence, defining it as the science and engineering of creating intelligent machines [1]. Within the ever-evolving landscape of technological progression, Artificial Intelligence (AI) emerges

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O. Gervasi et al. (Eds.): ICCSA 2024 Workshops, LNCS 14816, pp. 235–246, 2024.
https://doi.org/10.1007/978-3-031-65223-3_16

as a groundbreaking frontier and a paramount force driving innovation, namely by revolutionizing sectors, restructuring financial systems and, more than ever, reconfiguring human potential. The author of [2] mentions that the developments of AI and the life sciences are intertwined and that the ultimate goal of AI is to achieve human-like intelligence, as the human brain is capable of multi-tasking, learning with minimal supervision, and generalizing learned skills, all accomplished with high efficiency and low energy cost. However, besides all the beneath the veneer of AI remarkable utilities, resides a deep-seated dependence on the principles of mathematics.

From the genesis of algorithms to the intricate workings of neural networks [3–6], mathematics serves as the backbone that enables AI to interpret, infer, and evolve. In addition to the technical advantages of using a combination of multiple mathematical methodologies, there are also practical benefits to consider [7]. For example, by leveraging different approaches, researchers can gain a more holistic perspective on their research questions and explore different angles of analysis. This can lead to more robust findings and a deeper understanding of the underlying phenomena being studied. Furthermore, using a combination of methodologies can also help researchers to validate their results and ensure the robustness of their conclusions, adding an additional layer of credibility to their research. For example, some methodologies may be better suited for handling large data sets, while others may excel at capturing complex relationships within the data. On the other hand, while a statistical model may be able to provide precise estimates for certain parameters, a machine learning algorithm may be better at capturing nonlinear relationships within the data [8].

By understanding the unique strengths of each methodology, researchers can effectively leverage a combination of approaches to gain a more comprehensive understanding of the data and generate more accurate insights. For example, Hadamard matrices are mathematical structures that have properties suitable for error correction and data encryption. Hence, by integrating Hadamard matrices into AI systems, researchers can develop more robust and secure algorithms for data processing and analysis as in the case of cybersecurity, where Hadamard matrices can be used to encrypt sensitive data and protect it from unauthorized access [3,9–12].

This work is divided in five sections. An introduction starts the manuscript; in Sect. 2 some details about exploring connections of mathematical and Statistical frameworks with AI systems and computing are described. Some concepts about Hadamard matrices appear in Sect. 3. Also, the advantages of applying matrices in the most varied areas are highlighted in the Sect. 3. In Sect. 4 are debated the risks, questions and queries of AI in the quantum paradigm. Section 5 details the conclusions and displays some final remarks.

2 Exploring Connections of Mathematical and Statistical Frameworks with AI Systems and Computing

Algorithms are the essential components that drive the functionality of Artificial Intelligence, enabling machines to carry out specific tasks with precision and speed and likewise improving processes. By harnessing mathematical concepts, such as calculus and linear algebra, algorithms can efficiently process data, optimize workflows, and extract

meaningful patterns. For instance optimization algorithms, like gradient descent, work tirelessly to minimize errors and enhance the performance of AI models by iteratively adjusting their parameters. This particular method was originally presented in 1847 by Augustin-Louis Cauchy, but in 1907 Jacques Hadamard independently proposed and developed a similar one [7]. Through mathematical calculations, gradient descent adjusts the parameters of a model using weights, step by step and following the gradient of the cost function [4,6,8]. The gradient guides the algorithm to move weights in the direction of the steepest decrease in surface. As the process continues, it moves closer to a desired low point where predictions align more closely with the true data. Accordingly to IBM *"Gradient descent is an optimization algorithm which is commonly-used to train machine learning models and neural networks. It trains machine learning models by minimizing errors between predicted and actual results."* Still for gradient descent algorithm, the authors of [5] refer that *"This method involves adjusting the values of the parameters of the network to minimize the loss function over the course of many iterations. By repeating this process over and over again, the network will learn the weights needed to produce the optimal output for a given input."*

Statistical underpinnings illuminate the path forward in the realm of artificial intelligence, offering methodologies to extract patterns from data, estimating probabilities, and making rational decisions proposals amidst uncertainty. Statistical analysis is a powerful methodology for extracting valuable information from data and providing confidence in decision-making. Using probability theory, hypothesis testing, and inferential statistics, statisticians are able to quantify uncertainty, identify patterns, and draw meaningful conclusions from empirical data. Moreover, the essence of probability theory is exemplified in probabilistic graphical models, enabling the representation of complex dependencies between variables in AI domains like natural language processing and computer vision. By employing rigorous statistical and visualization techniques, analysts can assess the reliability of their findings, make informed decisions, and formulate data-driven strategies with a high degree of certainty.

Linear regression is a widely used statistical technique for modeling relationships between variables, assessing accuracy, and making predictions based on empirical data. By fitting a linear model to observed data points, researchers can estimate the strength and direction of relationships between variables, predict future outcomes, and evaluate the accuracy of their models. Linear regression is particularly useful in fields such as economics, finance, and social sciences [13,14], where understanding and predicting relationships between variables is critical for decision-making. Through the application of linear regression analysis, experts can uncover valuable insights, test hypotheses, and make informed decisions based on empirical evidence.

Bayesian statistics represents an innovative approach analyzing data and making predictions in a wide range of applications, including probabilistic programming for artificial intelligence (AI) systems. Using Bayesian inference, researchers can quantify uncertainty, update beliefs, and infer probabilities based on observed data. In the context of AI, Bayesian statistics can be used to build sophisticated models that encapsulate uncertainty, adapted to change environments, and make rational decisions in complex scenarios. Through the integration of Bayesian methods with AI technologies, develop-

ers can create more robust, adaptive, and intelligent systems capable of learning from data, making predictions, and solving complex problems. Through Bayesian inference, AI systems can refine beliefs based on new evidence, fostering adaptability and continuous learning.

Algebra is a fundamental mathematical tool that can be applied to manipulate data in high-dimensional spaces. In the realm of data analysis, algebraic techniques such as matrix operations, linear equations, and vector spaces play a crucial role in transforming raw data into meaningful insights. By applying algebraic principles, researchers can uncover patterns, relationships, and trends hidden within complex datasets, thereby facilitating the process of data interpretation and decision-making. The utilization of matrix operations, including multiplication and convolution, plays a crucial role in the transmission of signals throughout the layers of neural networks. This process enables the completion of complex tasks such as image recognition, language translation, and autonomous driving. Moreover, advanced concepts like eigenvectors and singular value decomposition serve to deconstruct and analyze a vast number of datasets in neural networks, heightening their efficacy and interpretability. By leveraging these mathematical tools, neural network architectures are able to efficiently navigate and process high-dimensional data for various applications [3,5,6,15].

Graph theory is a branch of discrete mathematics that finds ample utility in AI applications, particularly in optimization and network analysis. Graph algorithms like breadth-first search, Greedy Algorithm and Dijkstra's algorithm enable efficient traversal and path finding in complex networks, facilitating tasks such as route planning and resource allocation. Furthermore, optimization techniques - like integer programming and convex optimization - empower AI systems to maximize efficiency, allocate resources, and minimize costs across diverse domains, from logistics to finance. As artificial intelligence progresses, the respective connections with mathematics became increasingly important, paving the way for groundbreaking advancements in intelligence and automation [6,16].

Quantum Computation. Quantum computation, a revolutionary field that harnesses the principles of quantum mechanics to perform calculations exponentially faster than classical computers, is pushing the boundaries of what is possible in terms of artificial intelligence. Studies have shown that quantum computers have the potential to dramatically speed up tasks such as machine learning, optimization, and cryptography, leading to breakthroughs in AI research [17]. Further considerations will be presented in Sect. 4.

3 Hadamard Matrices: Connecting Past to Future in AI Applications

Hadamard matrices play a crucial role in various sectors, including quantum information, communication, networking, cryptography, biometrics and security, which make them a special bone in Artificial Intelligence. They are common in signal processing and coding applications. More information and examples can be seen in the books by Agaian [9] and by Horadam [18].

These matrices are key components in the formation of Hadamard configurations, a specialized block design from Experimental Design area that became a standard feature in everyday activities. It is then clear that, Hadamard matrices offer versatile solutions in numerous scenarios, namely through their adaptability showcased and their integration into advanced technologies and the popularity of QR codes [3,9,12,19]. They establish a very clear, and crucial bridge between mathematics, statistics and artificial intelligence by taking advantage of known classical methodologies, bringing it to current days and establishing their importance to recent and new scientific advances in the digital Era.

Consider the Hadamard's determinant theorem:

Theorem 1. *Let $\boldsymbol{X} = (x_{ij})$ be an $n \times n$ real matrix whose entries satisfy $|x_{ij}| \leq 1$, for all $i, j \in \mathbb{N}$. Then $|det(\boldsymbol{X})| \leq n^{n/2}$.*

These matrices were first considered as Hadamard determinants because the determinant of a Hadamard matrix satisfies equality in Hadamard's determinant theorem and so the equality holds if and only if **X** is a Hadamard matrix.

Definition 1. *Hadamard Matrix is an $n \times n$ matrix $\boldsymbol{H}$ with all entries ± 1 and $\boldsymbol{H}\boldsymbol{H}^T = nI_n$ is called a Hadamard matrix of order n, where C^T is the transpose of $\boldsymbol{H}$ and I_n is the identity matrix of order n.*

Important properties of Hadamard matrices are that:

1. The order of a Hadamard matrix is 1, 2 or $4n$, to n integer.
2. If the rows and columns of a Hadamard matrix are permuted, the matrix remains Hadamard.
3. If any row or column is multiplied by -1, the Hadamard property is retained.

A Hadamard matrix having the first row and first column containing only $+1$ entries is known as *normalized Hadamard Matrix*. Some details about Hadamard matrices can be found in [10,20–22]

The authors of [11] present examples of Hadamard matrices of order 2, 4 and 8, as it appears in Fig. 1.

The Hadamard conjecture proposes that a Hadamard matrix of order $4k$ exists for every natural k. Many orders of Hadamard matrices have already been constructed. Accordingly to the literature, Sylvester's construction and also Paley's work are the most famous methods to construct such kind of matrices [23,24]. In [25], the authors mention that it is well known that n must be 1, 2 or a multiple of 4 for a Hadamard matrix to exist, but was unknown at that time, whether Hadamard matrices exist for every n which is a multiple of 4.

Later on, [21] presented one Hadamard matrix of order 268 and Neil Sloan established the library of Hadamard matrices, which is available at http://neilsloane.com/hadamard/ [26]. The author of [16] established that the necessary condition for the existence of a Hadamard matrix $\mathbf{H}_n$, $n > 2$ is that $n \equiv 0(mod 4)$, but whether this necessary condition is sufficient as well it is not known either. Later on, Cameron presented Hadamard Matrices at http://designtheory.org/library/encyc/topics/had.pdf [27], in the Encyclopedia of Design Theory [28]. The [12] mentioned that Hadamard matrices have their existence confirmed to all permissible values of $n \leq 424$. In the meanwhile, bigger

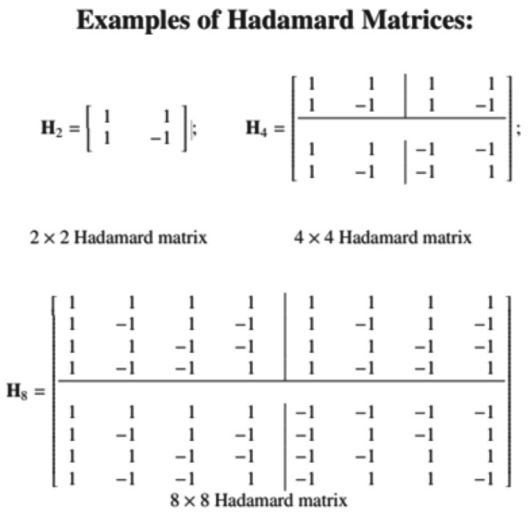

Fig. 1. Hadamard matrices of order 2, 4 and 8. Source [11].

Hadamard matrices were discovered, such as the one of order 428, which is available at http://neilsloane.com/hadamard/had.428.txt.

Research in the area of Hadamard matrices and their applications has steadily and rapidly grown, especially in this new century. The authors of [25] stated that in the area of Experimental Design, such matrices are particularly useful once they can be transformed to produce incomplete block designs, t-designs, Youden designs, orthogonal F-square designs, optimal saturated resolution III designs, optimal weighing designs, maximal sets of pairwise independent random variables with uniform measure, error correcting and detecting codes, Walsh functions, and other mathematical and statistical objects. Those authors also survey the existence of Hadamard matrices and many of their applications. Many other applications have emerged since then [1,3,9–11,15,18,25,29], and between a huge series of developments considering this matrices, the QR CODE from Coding Theory was selected for an example, thanks to the respective popularity and vast amount of real applications.

The QR Codes are 2-dimensional bar codes that can be easily read by a device capable of image capture, as is the case of most existing mobile phones. These codes can for example represent a file, an address for a web site (URL), a concert, a text, a GPS code, a phone number, an email, a contact or an SMS. QR codes have, as the base of their error correcting structure Hadamard matrices and there is a vast of number of links online for generating them, offering different features.

See, for example,

- a simple QR Code generator at https://www.qr-code-generator.com/;
- a dynamic QR code generator for the construction industry at https://www.flowcode.com/industry/construction;
- a QR Code for the Job Site at https://jobsitecodes.com/2023/11/06/using-qr-codes-on-the-job-site/;

- a QR Code for an event at https://www.qr-code-generator.com/solutions/event-qr-code/.

As an illustration, the QR Code for the 24th International Conference on Computational Science and Its Applications, ICCSA- 2024, was generated and is presented in Fig. 2.



Fig. 2. The QR Code for ICCSA 2024.

The mathematical properties jointly with the new technological advances allow simple figures to contain and transmit a huge amount of information. The mathematical properties that are intrinsic to these structures will allow great advances for error detection and correction, in huge amounts of information [12].

It is well known that in this digital Era, the paradigm of modern communications systems is heavily reliant on statistical techniques in order to recover information in the presence of noise and interferences and, moreover, to look for minimizing the risks of losing information. A historical note about AI can be found in [7]. Nowadays, AI is playing a critical role in revolutionizing experimental design by facilitating the automation of repetitive tasks, enabling real-time monitoring and control, and enhancing the overall efficiency of experimental processes. With the aid of AI technologies such as machine learning and deep learning, researchers can predict outcomes, optimize parameters, and identify trends in data with unprecedented accuracy and speed [4–6, 24].

The interdisciplinary mode of STAT-MATH-AI promotes taking advantage of joining strengths and provide to achieve the mentioned goals using the Hadamard matrices, which offer a very powerful and proper mathematical structure. This will not only accelerate the pace of scientific discovery but also opens up new opportunities for interdisciplinary collaboration and innovation in experimental design and AI [10].

4 AI in the Quantum Paradigm: Risks, Questions and Queries

Quantum computation presents an opportunity to catalyze AI innovation by leveraging the principles of quantum mechanics to execute computations at unparalleled velocities. Quantum algorithms, rooted in mathematical constructs such as superposition and entanglement, hold the promise of addressing intricate optimization challenges and simulating molecular configurations with unparalleled precision. The introduction of qubits and quantum computing represents a paradigm shift in the field of information processing. Unlike classical bits, which can only exist in a single state at a time, qubits

can be in a superposition of states, enabling the exploration of multiple states simultaneously. This feature allows for the development of quantum algorithms that can exploit the parallelism inherent in quantum systems to solve complex problems with a higher efficiency than classical algorithms.

A quantum computer can accommodate both analog and digital approaches, using qubits and, in the digital case, also relying on so-called quantum logical gates. Like bits, qubits can be in quantum states of 1 or 0, the difference being that qubits can assume a superposition of states 1 and 0, i.e., any linear combination of solutions is a solution. The fact that a set of bits does not admit a unique state, but rather a state function, represents a new paradigm that enables the creation of algorithms capable of handling multiple possible states simultaneously, and to solve problems in less time complexity, see [30]. This author mentions that the quantum state space can be described in terms of vectors and matrices or using a standard notation called bracket notation, also known as Dirac notation. This terminology arises from the fact that the inner product of two states is represented by a bracket $\|x>=|x>$, resulting a complex number. The external product of vectors $|x>$ and $y|$ is represented by $|x>|$ and can be represented by a matrix.

Accordingly with the Dirac's notation, during the processing of quantum information, the logical quantum state is a quantum superposition of the "base states" $|0>$ and $|1>$, with the quantum state $|0>$ always being converted to zero, just as $|1>$ is always converted to 1, in classical logic. Quantum operations must always be unitary, that is, leading to transformations that preserve $\sum E\,|\Psi_E|^2 = 1$ and are consequently reversible. For example, in Classical Computing, a set of two *bits* can admit the states 00, 01, 10 or 11, in Quantum Computing the state space includes a superposition of states $\Psi_{00}|00> + \Psi_{01}|01> + \Psi_{10}|10> + \Psi_{11}|11>$, where

$$|\Psi_{00}|^2 + |\Psi_{01}|^2 + |\Psi_{10}|^2 + |\Psi_{11}|^2 = 1,$$

with probability $|\Psi E|^2$ associated to the set in a certain state E.

The quantum phenomenon cannot be indifferent to all research considered for current binary systems (Fig. 3).

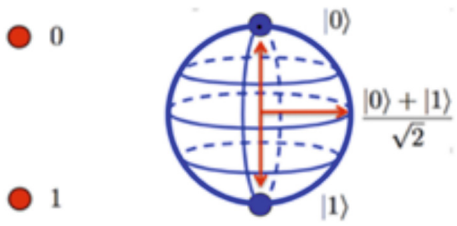

Fig. 3. Representation of bits (left side) and qubits (right side) [30]; inspired on [31].

It is possible that many developments of expansion and broadening of theories unknown to this new quantum context will be triggered. Experts predict that, among other things, the quantum revolution will allow the development of main issues like:

- ultra-precise sensors, envisioning a new era in medicine;
- new security systems based on Quantum Cryptography, which will be crucial for the protection of digital data of Web users;
- the evolution towards new forms of communication.
- More optimistic and future visionary researchers even consider the Quantum paradigm as the first step for tele-transportation.

Some interesting ideas can be found in [32] that presents a work about quantum risk analysis, comparing some traditional methods such as Monte Carlo with quantum computing. In [33] the authors conclude that recent work produced by quantum algorithms could act as the building blocks of machine learning programs, but the hardware and software challenges are still considerable. At [34], the authors claim that to understanding quantum machine learning requires rethinking generalization. In the same line, the author of [35] provides an explicit description of the reproducing kernel Hilbert space by embedding the Generalized Gaussian Radial Basis Function as an L2 measure that is used in the in the implementation of the analysis support vector machine. The empirical experiments described in the same work [35] suggest the possible future directions in terms of spectral decomposition of the Generalized Gaussian Radial Basis Functions.

5 Conclusions

The synergy between artificial intelligence and mathematics is becoming ever more essential as AI capabilities continue to progress, ushering in a new era of intelligence and automation. This symbiotic relationship is a testament to the synergistic potential inherent in their intersection and the fusion of mathematical principles with AI algorithms enables the creation of complex systems capable of processing vast amounts of data and deriving insights previously unimaginable. At the core of this symbiosis lies the foundational concepts of mathematics, providing the logical framework upon which AI systems operate. Whether through the application of algebra, statistical methods or optimization of neural networks, mathematics serves as the guiding force shaping the evolution of artificial intelligence.

Quantum computing stands out as a game-changing technology that can accelerate AI innovation by exploiting the principles of quantum mechanics to perform computations at unparalleled speeds. Quantum algorithms, grounded in mathematical concepts such as superposition and entanglement, have the potential to revolutionize AI by addressing complex optimization problems and enabling precise simulations. Quantum computation, is known as an emergent and revolutionary field that harnesses the principles of quantum mechanics to perform calculations exponentially faster than classical computers, is pushing the boundaries of what is possible in terms of artificial intelligence. Studies have shown that quantum computers have the potential to dramatically speed up tasks such as machine learning, optimization, and cryptography, leading to breakthroughs in AI research. The intersection of quantum computation and AI holds immense promise for unraveling the mysteries of the universe and unlocking new realms of innovation. Delving deeper into the intricate complexities of AI, the power

of mathematical abstraction becomes increasingly evident. The ability to distill complex problems into mathematical formulations not only enhances the efficiency of AI algorithms but also opens up new avenues of exploration and discovery.

Venturing into uncharted territories and being guided by the abstraction and the beauty of mathematical concepts, the horizon of possibilities in the field of artificial intelligence expands exponentially. The future of AI can be seen as a puzzle of human intellect and mathematical truths, where the ingenuity of the human mind is amplified and enhanced by the timeless principles of mathematics. Together, statistics, mathematics and AI will unlock new doors to innovation, revolutionizing industries, and transform the way of interacting with technology. In summary, considering the previous features, while navigating the quantum AI Era, requires a mindset shift towards embracing change, fostering innovation, and nurturing a symbiotic relationship between technology and humanity. By envisioning a future where quantum advancements amplify human capabilities, drive sustainability, and promote global cooperation, organizations and societies can pave the way for an equitable, sustainable, and prosperous future.

Acknowledgements. The first author was supported by Portuguese funds *The Portuguese Foundation for Science and Technology* (FCT), through the *project* UIDB/Multi/00006/2020.

The second author was supported by Portuguese funds *The Portuguese Foundation for Science and Technology* (FCT), through the *Center for Computational and Stochastic Mathematics* (CEMAT), University of Lisbon, Portugal, project UIDB/Multi/04621/2020, DOI: (https://doi.org/10.54499/UIDB/04621/2020) and through the *Center of Naval Research* (CINAV), Naval Academy, Portugal.

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