Calculation and Geometric Interpretation of the Curvatures of Implicit Curves and Surfaces with Mathematica

Rolando E. Ipanaqué-Silva[1], Jhonny W. Eche-Antón[2], Elmer P. Díaz Contreras[2], Ricardo Velezmoro-León[2], and Robert Ipanaqué-Chero[2(✉)]

[1] Universidad Tecnológica del Perú, Av. Vice Cdra 1-Costado Real Plaza, Piura, Peru
c25335@utp.edu.pe

[2] Universidad Nacional de Piura, Urb. Miraflores s/n Castilla, Piura, Peru
jechea@posgrado.unp.edu.pe, {ediazc,ripanaquec}@unp.edu.pe
https://www.unp.edu.pe/pers/ripanaque/

Abstract. In this paper, based on the formulas proposed by Ronald Goldman, new functions are defined and encoded in the Wolfram Language using the Mathematica symbolic calculation system. With the coded functions, the curvature of implicit plane curves, the curvature and twist of implicit space curves, and the Gaussian and mean curvatures of implicit surfaces can be calculated. All of these functions are included in a Mathematica package. Finally, the package's functionality is shown by solving several exciting examples. The application of this package allows users to explore and understand the geometry of the analyzed implicit curves and surfaces. This results in a helpful computer tool of great value for educational purposes and to improve the visualization of these concepts.

Keywords: Curvature · implicit curves · implicit surfaces · Wolfram Language

1 Introduction

Formulas to calculate the curvature of curves and surfaces given in parametric form are common in classic Differential Geometry texts [1,4]. However, the same does not happen with the formulas to calculate the curvature of curves and surfaces given implicitly. The procedure to find the curvature of implicitly given plane curves can be found in the text of Bolgov and co-authors [5]. In this text, an algorithm is described to compute the curvature of spatial curves given implicitly; however, a specific formula is not shown. In the case of implicitly given surfaces, O'Neill presents two formulas (one for the Gaussian curvature and another for the average curvature) [4]. Still, obtaining the tangent vector fields to the said surface is not detailed.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2024
O. Gervasi et al. (Eds.): ICCSA 2024 Workshops, LNCS 14815, pp. 285–301, 2024.
https://doi.org/10.1007/978-3-031-65154-0_18

Eighteen years ago, Goldman published the paper [2] presenting curvature formulas for implicit curves and surfaces. He establishes a bridge in that article by deducing curvature formulas for implicit curves and surfaces from the already-known curvature formulas for parametric curves and surfaces.

Based on Goldman's work, it is proposed to develop functions in the Wolfram Language [3,7] to implicitly calculate the curvatures of curves and surfaces given and group them in a new package of functions, ImplicitCurvatures.

This package allows users to immerse themselves in the world of implicit curves and surfaces, allowing them to observe their geometric properties more clearly and effectively. Additionally, it facilitates the teaching of these topics by providing an interactive platform that can be used to illustrate complex mathematical concepts in a more accessible and didactic way.

2 Mathematical Preliminaries

The formulas presented in this section have been derived and proven by Goldman [2].

2.1 Notations

1. Let $A_{n\times n}$ be a matrix. Then

$$A^* = (\mathbf{adj}(A))^t \ .$$

2. Let $C : f(x, y) = 0$ be an implicit curve. Then

$$\nabla f = (f_x, f_y)\,.$$
$$\nabla\left(\nabla f\right) = \nabla^2 f = H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \ .$$

3. Let $M : F(x, y, z) = 0$ be an implicit surface. Then

$$\nabla F = (F_x, F_y, F_z)\,.$$
$$\nabla\left(\nabla F\right) = \nabla^2 F = H(F) = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yx} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix} \ ,$$
$$\Delta F = \mathbf{Tr}\left(H(F)\right) \ .$$

2.2 Parameterized Plane Curves

Let $\beta(s)$ be a plane curve parameterized by arc length. Then, the curvature κ is defined by

$$\kappa = -\frac{dN}{ds} \cdot T\,,$$

where T is the tangent, and N is the normal to the curve.

2.3 Parameterized Spatial Curves

Let $\beta(s)$ be a spatial curve parameterized by arc length. Then, the curvature κ and torsion τ are defined by

$$\kappa = \left\| \frac{dT}{ds} \times T \right\| ,$$
$$\tau = \frac{\mathbf{det}\left(T \ \frac{dT}{ds} \ \frac{d^2T}{ds^2}\right)}{\kappa^2} ,$$

where T is the tangent to the curve.

2.4 Parameterized Surfaces

Let $\mathbf{x}(u, v)$ be the parameterization of a surface. Let $\mathbf{x}_u$ and $\mathbf{x}_v$ be the first-order partial derivatives of $\mathbf{x}$ concerning u and v. Then, the normal N to the surface, the gaussian curvature K_G and mean curvature K_H are defined by

$$N(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} ,$$
$$K_G = \frac{\mathbf{x}_u \times \mathbf{x}_v \cdot N_u \times N_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|^2} ,$$
$$K_H = \frac{\mathbf{x}_u \times \mathbf{x}_v \cdot (\mathbf{x}_v \times N_u - \mathbf{x}_u \times N_v)}{2\|\mathbf{x}_u \times \mathbf{x}_v\|} ,$$

2.5 Implicit Plane Curves

Let $C : f(x, y) = 0$ be an implicit plane curve. Then, the curvature κ is defined by

$$\kappa = -\frac{\nabla f \cdot \left(\nabla^2 f \cdot \nabla f^t\right)}{\|\nabla f\|^3} .$$

2.6 Implicit Spatial Curves

Let $C : \begin{cases} f(x, y, z) = 0, \\ g(x, y, z) = 0. \end{cases}$ be an implicit spatial curve. Then, the curvature κ and the torsion τ are defined by

$$\kappa = \frac{\| (\nabla f \times \nabla g \cdot \nabla(\nabla f \times \nabla g)) \times (\nabla f \times \nabla g)\|}{\|\nabla f \times \nabla g\|^3} ,$$
$$\tau = \frac{\mathbf{det}\left(T^I \ T^{II} \ T^{III}\right)}{\|T^I \times T^{II}\|^2} ,$$

where

$$\begin{aligned} T^I &= \nabla f \times \nabla g\,, \\ T^{II} &= T^I \cdot \nabla T^I = \nabla f \times \nabla g \cdot \nabla(\nabla f \times \nabla g)\,, \\ T^{III} &= \left(T^I \cdot \nabla(\nabla T^I)\right) \cdot T^I + \left(T^I \cdot \nabla T^I\right) \cdot \nabla T^I \\ &= (\nabla f \times \nabla g \cdot \nabla(\nabla(\nabla f \times \nabla g))) \cdot \nabla f \times \nabla g + \\ &\quad + (\nabla f \times \nabla g \cdot \nabla(\nabla f \times \nabla g)) \cdot \nabla(\nabla f \times \nabla g)\,. \end{aligned}$$

2.7 Implicit Surfaces

Let $M : F(x, y, z) = 0$ be an implicit surface. Then, the gaussian curvature K_G and mean curvature K_H are defined by

$$\begin{aligned} K_G &= \frac{\nabla F \cdot \left(\nabla^2 F^* \cdot \nabla F^t\right)}{\|\nabla F\|^4}\,, \\ K_H &= -\frac{\nabla F \cdot \left(\nabla^2 F \cdot \nabla F^t\right) - \Delta F\,(\nabla F \cdot \nabla F)}{2\|\nabla F\|^3}\,. \end{aligned}$$

3 Geometric Interpretation of the Curvatures of Implicit Curves and Surfaces

3.1 Graph of a Function

Given the function $\Phi : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$, the graph of Φ is the set

$$\mathcal{G}(\Phi) = \{(\mathbf{p}, \Phi(\mathbf{p})) \,|\, \mathbf{p} \in \mathcal{D}\}\,.$$

This means that the graph of ϕ is in $\mathbb{R}^{n+1}$.

3.2 Implicit Plane Curves

Let $C : f(x, y) = 0$ be an implicit plane curve. Then $\kappa : C \subset \mathbb{R}^2 \to \mathbb{R}$ (see Fig. 1) and

$$\mathcal{G}(\kappa) = \{(\mathbf{p}, \kappa(\mathbf{p})) \,|\, \mathbf{p} \in C\}\,.$$

3.3 Implicit Spatial Curves

Let $C : \begin{cases} f(x, y, z) = 0, \\ g(x, y, z) = 0. \end{cases}$ be an implicit spatial curve. Then $\kappa : C \subset \mathbb{R}^3 \to \mathbb{R}$ (see Fig. 2) and

$$\mathcal{G}(\kappa) = \{(\mathbf{p}, \kappa(\mathbf{p})) \,|\, \mathbf{p} \in C\}\,.$$

Similarly, $\tau : C \subset \mathbb{R}^3 \to \mathbb{R}$ (see Fig. 2) and

$$\mathcal{G}(\tau) = \{(\mathbf{p}, \tau(\mathbf{p})) \,|\, \mathbf{p} \in C\}\,.$$

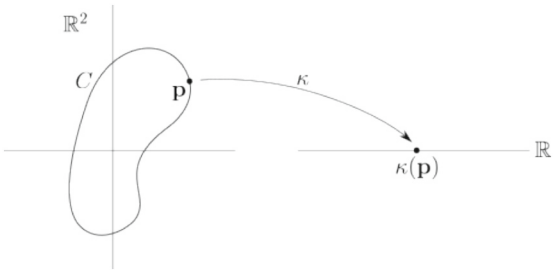

Fig. 1. Geometric interpretation of the curvature of an implicit plane curve.

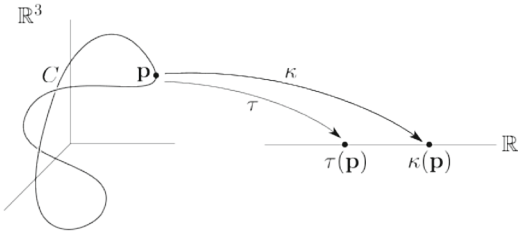

Fig. 2. Geometric interpretation of the curvature and torsion of an implicit space curve.

3.4 Implicit Surfaces

Let $M : F(x, y, z) = 0$ be an implicit surface. Then $K_G : M \subset \mathbb{R}^3 \to \mathbb{R}$ (see Fig. 3) and

$$\mathcal{G}(K_G) = \{(\mathbf{p}, K_G(\mathbf{p})) \,|\, \mathbf{p} \in M\} \ .$$

Similarly, $K_H : C \subset \mathbb{R}^3 \to \mathbb{R}$ (see Fig. 3) and

$$\mathcal{G}(K_H) = \{(\mathbf{p}, K_H(\mathbf{p})) \,|\, \mathbf{p} \in M\} \ .$$

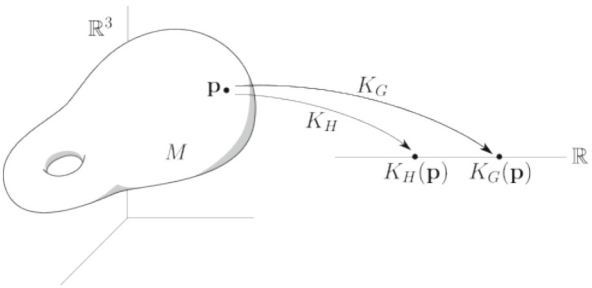

Fig. 3. Geometric interpretation of the Gaussian and mean curvatures of an implicit surface.

4 Implemented Code

According to [6], we will choose to use the three-dimensional model: $\{\mathcal{O}, \hat{\mathcal{B}}, \varphi\}$; where, $\mathcal{O} = (0,0,0)$, $\hat{\mathcal{B}} = \left\{\frac{3}{5\sqrt{3}}(-1,-1,-1),(1,0,0),(0,1,0),(0,0,1)\right\}$, and $\varphi(p_1,p_2,p_3,p_4) = \left(p_2 - \frac{3p_1}{5\sqrt{3}}, p_3 - \frac{3p_1}{5\sqrt{3}}, p_4 - \frac{3p_1}{5\sqrt{3}}\right)$. Based on this model, the following code will be used to visualize projections of curves and surfaces immersed in $\mathbb{R}^4$.

```
M={{-(3/(5*Sqrt[3])),-(3/(5*Sqrt[3])),-(3/(5*Sqrt[3]))},
 {1,0,0},{0,1,0},{0,0,1}};
P[{p1_,p2_,p3_,p4_}]:={p1,p2,p3,p4}.M
```

The following code has been created to show the coordinate axes that extend in the directions of the elements of $\hat{\mathcal{B}}$.

```
Vaxes[al_,M_,ah_:X]:=Graphics3D[{Gray,
 MapThread[Line[{{0,0,0},#1*#2}]&,{al,M}],
 MapThread[Text[Style[#1,Bold],#2]&,
 ({Array[Subscript[ah,#]&,Length[M]],al*M})]}]
```

The code for the five functions elaborated to calculate the curvatures is provided in the following subsections.

4.1 Implicit Plane Curves

```
Curvature[fxy_==0,{x_,y_}]:=
Module[{F=fxy,nablaF,HH,HHstar,nablaFt},
 nablaF=D[F,{{x,y}}];
 HH=D[F,{{x,y},2}];
 HHstar=Transpose[Adjugate[HH]];
 nablaFt=Transpose[nablaF];
 -nablaF.HHstar.nablaFt/Sqrt[nablaF.nablaF]^3//
 Simplify//Expand//Together
]
```

4.2 Implicit Spatial Curves

```
Curvature[{fxyz_==0,gxyz_==0},{x_,y_,z_}]:=
Module[{F=fxyz,G=gxyz,nablaF,nablaG,nablanablaFG,
  aux,norm},
 nablaF=D[F,{{x,y,z}}];
 nablaG=D[G,{{x,y,z}}];
 nablanablaFG=Transpose[D[Cross[nablaF,nablaG],{{x,y,z}}]];
 aux=Cross[Cross[nablaF,nablaG].nablanablaFG,Cross[nablaF,nablaG]];
 norm[v_]:=Sqrt[v.v];
 norm[aux]/norm[Cross[nablaF,nablaG]]^3//
 Simplify//Expand//Together//PowerExpand
]
```

```
Torsion[{fxyz_==0,gxyz_==0},{x_,y_,z_}]:=
Module[{F=fxyz,G=fxyz,nablaF,nablaG,T1,T2,T3,norm},
 nablaF=D[F,{{x,y,z}}];
 nablaG=D[G,{{x,y,z}}];
 T1=Cross[nablaF,nablaG];
 T2=T1.Transpose[D[T1,{{x,y,z}}]];
 T3=T1.Transpose[D[Transpose[D[T1,{{x,y,z}}]],{{x,y,z}}]].Transpose[T1]+
 T1.Transpose[D[T1,{{x,y,z}}]].Transpose[D[T1,{{x,y,z}}]];
 norm[v_]:=Sqrt[v.v];
 Det[Transpose[{T1,T2,T3}]]/norm[Cross[T1,T2]]^2//
 Simplify//Expand//Together
]
```

4.3 Implicit Surfaces

```
GaussianCurvature[fxyz_==0,{x_,y_,z_}]:=
Module[{F=fxyz,nablaF,HH,HHstar},
 nablaF=D[F,{{x,y,z}}];
 HH=D[F,{{x,y,z},2}];
 HHstar=Transpose[Adjugate[HH]];
 nablaF.HHstar.nablaF/Sqrt[nablaF.nablaF]^4//
 Simplify//Expand//Together
]

MeanCurvature[fxyz_==0,{x_,y_,z_}]:=
Module[{F=fxyz,nablaF,HH},
 nablaF=D[F,{{x,y,z}}];
 HH=D[F,{{x,y,z},2}];
 (nablaF.HH.nablaFt-Sqrt[nablaF.nablaF]^2 Tr[HH])/
  (2Sqrt[nablaF.nablaF]^3)//Simplify//Expand//Together
]
```

5 The New Package: `ImplicitCurvatures`

The new `ImplicitCurvatures` package incorporates the functions

`Curvature`, `Torsion`, `GaussianCurvature` and `MeanCurvature`.

The syntax of these functions is shown in Table 1.

Before describing the examples created with the help of the new package, it is necessary to initialize it.

Mathematica

Initialization of the `ImplicitCurvatures` package.

```
In[1]:= <<ImplicitCurvatures.m
```

Below are various and varied examples in which the new package is used.

Table 1. The syntax for functions is built into the `ImplicitCurvatures` package

`Curvature`$[f(x,y) == 0, \{x, y\}]$ gives the curvature of the implicit plane curve $C : f(x, y) = 0$.
`Curvature`$[\{f(x,y,z) == 0, g(x,y,z) == 0\}, \{x, y, z\}]$ gives the curvature of the implicit spatial curve $C : \begin{cases} f(x,y,z) = 0, \\ g(x,y,z) = 0. \end{cases}$.
`Torsion`$[\{f(x,y,z) == 0, g(x,y,z) == 0\}, \{x, y, z\}]$ gives the torsion of the implicit spatial curve $C : \begin{cases} f(x,y,z) = 0, \\ g(x,y,z) = 0. \end{cases}$.
`GaussianCurvature`$[F(x,y,z) == 0, \{x, y, z\}]$ gives the gaussian curvature of the implicit surface $M : F(x, y, z) = 0$.
`MeanCurvature`$[F(x,y,z) == 0, \{x, y, z\}]$ gives the mean curvature of the implicit surface $M : F(x, y, z) = 0$.

5.1 Implicit Plane Curves

The circumference $x^2 + y^2 - 1 = 0$.

Mathematica

We are obtaining the graph of the circumference.

```
In[2]:= f[x_,y_]:=x^2+y^2-1
        r=ImplicitRegion[f[x,y]==0,{x,y}];
        Graphics[mp=MeshPrimitives[DiscretizeRegion[r],1]]
Out[2]= See Fig. 4 (left).
```

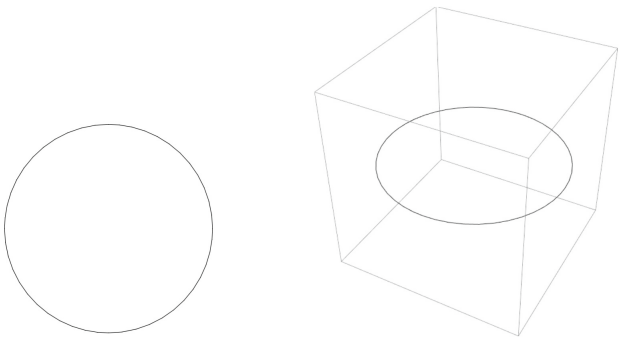

Fig. 4. The circumference and its curvature.

Mathematica

Function to obtain the value of the curvature of the circumference.

```
In[3]:= k=Curvature[f[x,y]==0,{x,y}]
```

Out[3]= $-\frac{1}{\sqrt{x^2+y^2}}$

Mathematica

We are obtaining the graph of the curvature of the circumference.

```
In[4]:= Graphics3D[mp/.Line[ps_]:>
          Line[{Sequence@@#,k/.MapThread[Rule,{{x,y},#}]}&/@ps]]
```

Out[4]= *See Fig. 4 (right).*

The fourth-degree Lamé curve $x^4 + y^4 - 1 = 0$.

Mathematica

We are obtaining the graph of the fourth-degree Lamé curve.

```
In[5]:= f[x_,y_]:=x^4+y^4-1
        r=ImplicitRegion[f[x,y]==0,{x,y}];
        Graphics[mp=MeshPrimitives[DiscretizeRegion[r],1]]
```

Out[5]= *See Fig. 5 (left).*

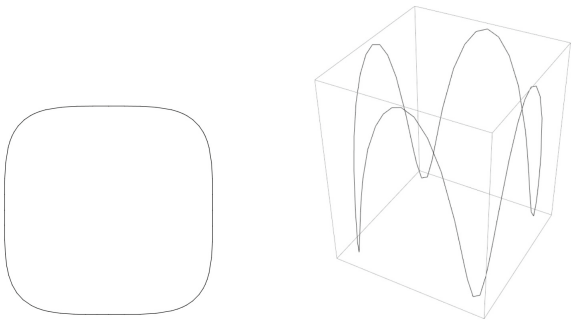

Fig. 5. Lamé's fourth-degree curve and its curvature.

Mathematica

Function to obtain the value of the curvature of the fourth-degree Lamé curve.

```
In[6]:= k=Curvature[f[x,y]==0,x,y]
```

Out[6]= $-\frac{3\left(x^6y^2+x^2y^6\right)}{(x^6+y^6)^{3/2}}$

Mathematica

We are obtaining the graph of the curvature of the fourth-degree Lamé curve.

```
In[7]:= Graphics3D[mp/.Line[ps_]:>
          Line[{Sequence@@#,k/.MapThread[Rule,{{x,y},#}]}&/@ps]]
```

Out[7]= *See Fig. 5 (right).*

The curve of lemniscatic trigonometry $x^2 + y^2 + x^2y^2 - 1 = 0$.

Mathematica

We are obtaining the graph of the lemniscatic trigonometry curve.

```
In[8]:= f[x_,y_]:=x^2+y^2+x^2y^2-1
        r=ImplicitRegion[f[x,y]==0,{x,y}];
        Graphics[mp=MeshPrimitives[DiscretizeRegion[r],1]]
```

Out[8]= *See Fig. 6 (left).*

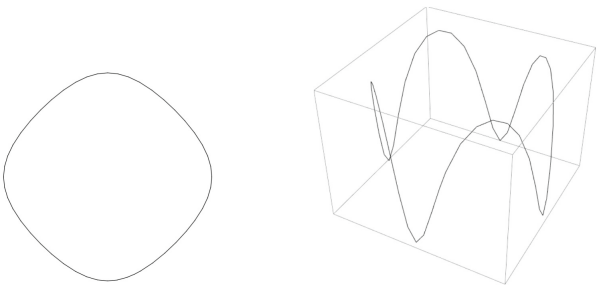

Fig. 6. The curve of lemniscatic trigonometry along with its curvature.

Mathematica

Function to obtain the value of the curvature of the lemniscatic trigonometry curve.

```
In[9]:= k=Curvature[f[x,y]==0,{x,y}]
```

Out[9]= $\frac{2x^4y^4+x^4y^2-x^4+x^2y^4-x^2-y^4-y^2}{(x^4y^2+x^2y^4+4x^2y^2+x^2+y^2)^{3/2}}$

Mathematica

We are obtaining the graph of the curvature of the lemniscatic trigonometry curve.

```
In[10]:= Graphics3D[mp/.Line[ps_]:>
           Line[{Sequence@@#,k/.MapThread[Rule,{{x,y},#}]}&/@ps]]
```

Out[10]= *See Fig. 6 (right).*

5.2 Implicit Spatial Curves

The curve of the circular and parabolic cylinders $\begin{cases} x^2+y^2-4=0, \\ z-x^2=0. \end{cases}$.

Mathematica

We are obtaining the graph of the curve of the circular and parabolic cylinders.

```
In[11]:= f[x_,y_,z_]:=x^2+y^2-4
         g[x_,y_,z_]:=z-x^2
         ContourPlot3D[{f[x,y,z]==0,g[x,y,z]==0},{x,-3,3},
           {y,-2,2},{z,0,4},Contours->0,ContourStyle->Opacity[0],
           Mesh->None,BoundaryStyle->{
            1->None,2->None,{1,2}->{
            AbsoluteThickness[1],Black}}]
Out[11]= See Fig. 7 (left).
```

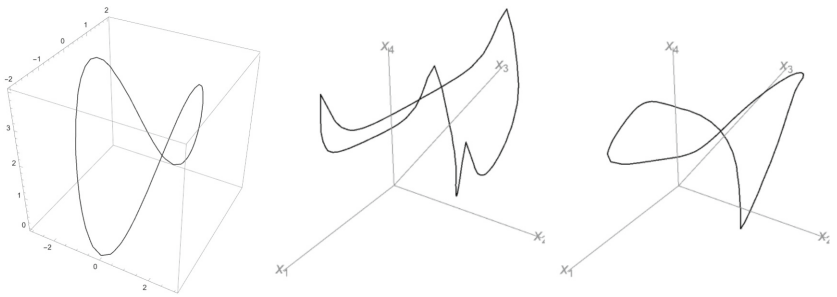

Fig. 7. The curve of circular and parabolic cylinders (left), their curvature (center), and torsion (right).

Mathematica

Function to obtain the value of the curvature of the curve of the circular and parabolic cylinders.

```
In[12]:= k=Curvature[{f[x,y,z]==0,g[x,y,z]==0},{x,y,z}]
```

$$\text{Out[12]}= \frac{\sqrt{4x^6+x^4+2x^2y^2+4y^6+y^4}}{(4x^2y^2+x^2+y^2)^{3/2}}$$

Mathematica

Function to obtain the value of the torsion of the curve of the circular and parabolic cylinders.

```
In[13]:= t=Torsion[{f[x,y,z]==0,g[x,y,z]==0},{x,y,z}]
```

$$\text{Out[13]}= -\frac{2\left(4x^5y-x^3y+4xy^5-xy^3\right)}{4x^6+x^4+2x^2y^2+4y^6+y^4}$$

Mathematica

We are obtaining the graph of the curvature of the parabolic cylinder curve.

```
In[14]:= al={6,4,6,4};
         Show[ContourPlot3D[{f[x,y,z]==0,g[x,y,z]==0},{x,-3,3},
            {y,-2,2},{z,0,4},Contours->{0},
            ContourStyle->Opacity[0],Mesh->None,
            BoundaryStyle->{1->None,2->None,
            {1,2}->{AbsoluteThickness[1],Black}},RegionFunction->
            Function[{x,y,z},x^2+y^2+4x^2y^2>0.1]]/.
            Graphics3D[{GraphicsComplex[p_,q_,r_],
            GraphicsComplex[p2_,q2_,r2_,t2_],{}},s_]:>
            Graphics3D[{GraphicsComplex[P[{Sequence@@#,k/.
            MapThread[Rule,{{x,y,z},#}]}]&/@p,q],{}},
            Boxed->False,PlotRange->All],Vaxes[al,M]]
Out[14]= See Fig. 7 (center).
```

Mathematica

We are obtaining the graph of the torsion of the parabolic cylinder curve.

```
In[15]:= al={6,4,6,4};
         Show[ContourPlot3D[{f[x,y,z]==0,g[x,y,z]==0},{x,-3,3},
            {y,-2,2},{z,0,4},Contours->{0},
            ContourStyle->Opacity[0],Mesh->None,
            BoundaryStyle->{1->None,2->None,
            {1,2}->{AbsoluteThickness[1],Black}},RegionFunction->
            Function[{x,y,z},x^4+4x^6+2x^2y^2+y^4+4y^6>0.1]]/.
            Graphics3D[{GraphicsComplex[p_,q_,r_],
            GraphicsComplex[p2_,q2_,r2_,t2_],{}},s_]:>
            Graphics3D[{GraphicsComplex[P[{Sequence@@#,t/.
            MapThread[Rule,{{x,y,z},#}]}]&/@p,q],{}},
            Boxed->False,PlotRange->All],Vaxes[al,M]]
Out[15]= See Fig. 7 (right).
```

5.3 Implicit Surfaces

The sphere $x^2 + y^2 + z^2 - 1 = 0$.

Mathematica

We are obtaining the graph of the sphere.

```
In[16]:= F[x_,y_,z_]:=x^2+y^2+z^2-1
         R=ImplicitRegion[F[x,y,z]==0,{x,y,z}];
         Graphics3D[MP=MeshPrimitives[DiscretizeRegion[R],2]]
Out[16]= See Fig. 8 (left).
```

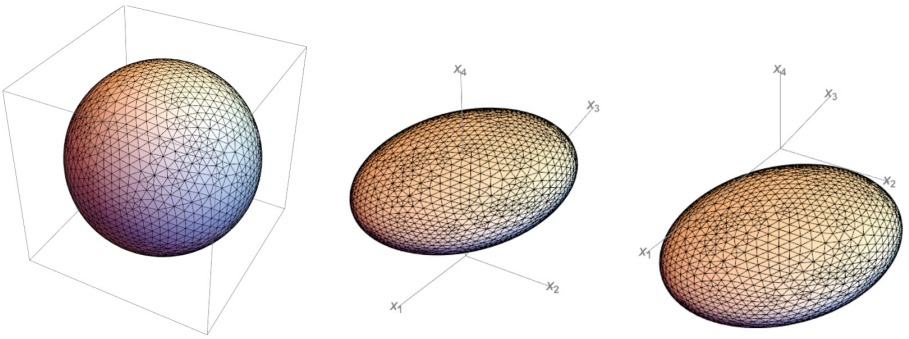

Fig. 8. The sphere and its Gaussian and mean curvatures.

Mathematica

Function to obtain the value of the gaussian curvature of the sphere.

In[17]:=
```
KG=GaussianCurvature[F[x,y,z]==0,{x,y,z}]
```

Out[17]= $\frac{1}{x^2+y^2+z^2}$

Mathematica

Function to obtain the value of the mean curvature of the sphere.

In[18]:=
```
KH=MeanCurvature[F[x,y,z]==0,{x,y,z}]
```

Out[18]= $-\frac{1}{\sqrt{x^2+y^2+z^2}}$

Mathematica

We are obtaining the graph of the gaussian curvature of the sphere.

In[19]:=
```
al={1.5,1,3,2.25};
Show[Graphics3D[MP/.Polygon[ps_]:>
     Polygon[P[{Sequence@@#,KG/.
   MapThread[Rule,{{x,y,z},#}]}]&/@ps]],Vaxes[al,M],
   Boxed->False]
```

Out[19]= *See Fig. 8 (center).*

Mathematica

We are obtaining the graph of the mean curvature of the sphere.

In[20]:=
```
al={2.5,1,1,0.75};
Show[Graphics3D[MP/.Polygon[ps_]:>
     Polygon[P[{Sequence@@#,KH/.
   MapThread[Rule,{{x,y,z},#}]}]&/@ps]],Vaxes[al,M],
   Boxed->False]
```

Out[20]= *See Fig. 8 (right).*

The fourth-degree surface of Lamé $x^4 + y^4 + z^4 - 1 = 0$.

Mathematica

We are obtaining the graph of the fourth-degree Lamé surface.

```
In[21]:= F[x_,y_,z_]:=x^4+y^4+z^4-1
         R=ImplicitRegion[F[x,y,z]==0,x,y,z];
         Graphics3D[MP=MeshPrimitives[DiscretizeRegion[R],2]]
Out[21]= See Fig. 9 (left).
```

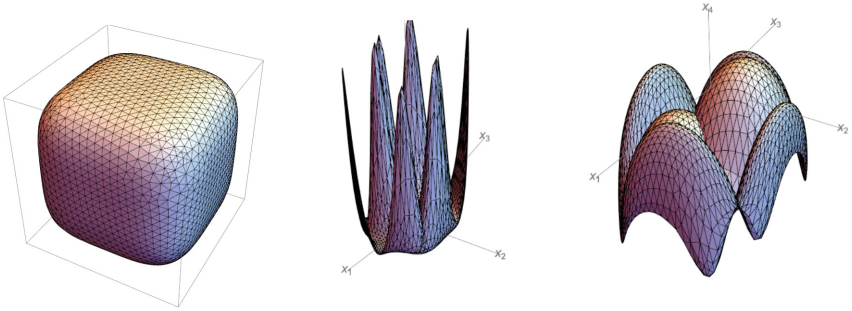

Fig. 9. The surface of Lamé and its Gaussian and mean curvatures.

Mathematica

Function to obtain the value of the gaussian curvature of the fourth-degree Lamé surface.

```
In[22]:= KG=GaussianCurvature[F[x,y,z]==0,{x,y,z}]
```

$$\text{Out[22]}= \frac{9\left(x^6y^2z^2+x^2y^6z^2+x^2y^2z^6\right)}{\left(x^6+y^6+z^6\right)^2}$$

Mathematica

Function to obtain the value of the mean curvature of the fourth-degree Lamé surface.

```
In[23]:= KH=MeanCurvature[F[x,y,z]==0,{x,y,z}]
```

$$\text{Out[23]}= -\frac{3\left(x^6y^2+x^6z^2+x^2y^6+x^2z^6+y^6z^2+y^2z^6\right)}{2\left(x^6+y^6+z^6\right)^{3/2}}$$

Mathematica

We are obtaining the graph of the gaussian curvature of the fourth-degree Lamé surface.

```
In[24]:= al={3.5,2.5,4,6.25};
         Show[Graphics3D[MP/.Polygon[ps_]:>Polygon[P[{Sequence@@#,KG/.
           MapThread[Rule,{{x,y,z},#}]}]&/@ps]],Vaxes[al,M],
           Boxed->False]
Out[24]= See Fig. 9 (center).
```

Mathematica

We are obtaining the graph of the mean curvature of the fourth-degree Lamé surface.

```
In[25]:= al={3.5,2,2,1.25};
         Show[Graphics3D[MP/.Polygon[ps_]:>Polygon[P[{Sequence@@#,KH/.
           MapThread[Rule,{{x,y,z},#}]}]&/@ps]],Vaxes[al,M],
           Boxed->False]
Out[25]= See Fig. 9 (right).
```

Descartes' surface $x^3 + y^3 + z^3 - 3xy - 3xz - 3yz = 0$.

Mathematica

We are obtaining the graph of the Descartes surface.

```
In[26]:= F[x_,y_,z_]:=x^3+y^3+z^3-3xy-3xz-3yz
         R=ImplicitRegion[F[x,y,z]==0,x,y,z];
         Graphics3D[MP=MeshPrimitives[DiscretizeRegion[R],2]]
Out[26]= See Fig. 10 (left).
```

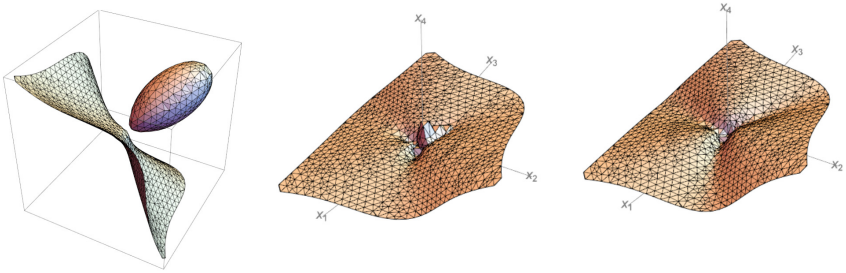

Fig. 10. The Descartes' surface and its Gaussian curvature.

Mathematica

Function to obtain the value of the gaussian curvature of the Descartes surface.

```
In[27]:= KG=GaussianCurvature[F[x,y,z]==0,{x,y,z}]
```

$$\text{Out[27]}= (4x^4yz - x^4 - 12x^2y^2z + 2x^2y^2 - 12x^2yz^2 + 4x^2y + 2x^2z^2 + 4x^2z + 4xy^4z - 12xy^2z^2 + 4xy^2 + 4xyz^4 + 12xyz + 4xy + 4xz^2 + 4xz - y^4 + 2y^2z^2 + 4y^2z + 4yz^2 + 4yz - z^4)/(x^4 - 2x^2y - 2x^2z + 2x^2 - 2xy^2 + 2xy - 2xz^2 + 2xz + y^4 - 2y^2z + 2y^2 - 2yz^2 + 2yz + z^4 + 2z^2)^2$$

Mathematica

Function to obtain the value of the mean curvature of the Descartes surface.

```
In[28]:= KH=MeanCurvature[F[x,y,z]==0,{x,y,z}]
```

Out[28]= $(x^4(-y) - x^4z + 3x^2y^2 + 4x^2yz - 2x^2y + 3x^2z^2 - 2x^2z - x^2 - xy^4 + 4xy^2z - 2xy^2 + 4xyz^2 - 3xy - xz^4 - 2xz^2 - 3xz - y^4z + 3y^2z^2 - 2y^2z - y^2 - yz^4 - 2yz^2 - 3yz - z^2)/(x^4 - 2x^2(y + z - 1) - 2x\left(y^2 - y + (z-1)z\right) + y^4 - 2y^2(z-1) - 2y(z-1)z + z^2\left(z^2+2\right))^{3/2}$

Mathematica

We are obtaining the graph of the gaussian curvature of the Descartes surface.

```
In[29]:= al={3.5,2.5,4,6.25};
         Show[Graphics3D[MP/.Polygon[ps_]:>Polygon[P[{Sequence@@#,KG/.
           MapThread[Rule,{{x,y,z},#}]}]&/@ps]],Vaxes[al,M],
           Boxed->False]
```

Out[29]= *See Fig. 10 (center).*

Mathematica

We are obtaining the graph of the mean curvature of the Descartes surface.

```
In[30]:= al={3.5,2,2,1.25};
         Show[Graphics3D[MP/.Polygon[ps_]:>Polygon[P[{Sequence@@#,KH/.
           MapThread[Rule,{{x,y,z},#}]}]&/@ps]],Vaxes[al,M],
           Boxed->False]
```

Out[30]= *See Fig. 10 (right).*

6 Conclusions

This paper has established a solid framework for defining and coding new features in the Wolfram Language. These functions allow the calculation of several vital geometric properties, such as the curvature of implicit plane curves, the curvature and twist of implicit space curves, and the mean and Gaussian curvatures of implicit surfaces. Including these functions in a Mathematica package gives users a powerful tool to explore and understand the geometry of the implicit curves and surfaces being analyzed. The resolution of several illustrative examples demonstrates this package's effectiveness and practical usefulness, both in the educational field and in improving the visualization of complex geometric concepts. Among the examples analyzed are the circumference, Lamé's fourth-degree curve, the curve of lemniscatic trigonometry, the curve of circular and parabolic cylinders, the sphere, Lamé's fourth-degree surface, and Descartes's surface.

References

1. Do Carmo, M.P.D.: Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition. Courier Dover Publications (2016). google-Books-ID: uXF6DQAAQBAJ
2. Goldman, R.: Curvature formulas for implicit curves and surfaces. Comput. Aided Geom. Des. **22**(7), 632–658 (2005). https://doi.org/10.1016/j.cagd.2005.06.005. https://linkinghub.elsevier.com/retrieve/pii/S0167839605000737

3. Maeder, R.E.: Programming in Mathematica. Addison-Wesley Professional, Reading (1997)
4. O'Neill, B.: Elementary Differential Geometry, 2nd edn. Elsevier Academic Press, Amsterdam (2006). oCLC: ocm62493605
5. Bolgov, V., et al.: Problemas de las matemáticas superiores, vol. I. Mir (1983)
6. Velezmoro, R., Ipanaqué, R., Mechato, J.A.: A mathematica package for visualizing objects inmersed in $\mathbb{R}^4$. In: Misra, S., et al. (eds.) ICCSA 2019. LNCS, vol. 11624, pp. 479–493. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-24311-1_35
7. Wolfram, S.: An Elementary Introduction to the Wolfram Language by Stephen Wolfram (2023). https://www.wolfram.com/language/elementary-introduction/3rd-ed/