

The Zeros of Quadratic Coquaternionic Polynomials Revisited

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Abstract. In this paper, we address the problem of solving the quadratic equation $x^2 + bx + c = 0$, where b and c are elements of the algebra of coquaternions. The nature of this algebra leads, as expected, to results very different from the ones obtained in the classical complex case, making the root-finding problem much more demanding. In this paper we derive straightforward conditions to characterize the number and form of the non-isolated zeros of monic quadratic coquaternionic equations. Through carefully constructed examples and by leveraging the Mathematica system, we illustrate the core findings and showcase the array of scenarios that may emerge.

Keywords: Coquaternions · Quadratic Equation · Non-isolated Zeros · Computer Algebra System

1 Introduction

In 1849, James Cockle [5] introduced a non-commutative algebra whose elements – the coquaternions, also referred to as split quaternions – generalize the complex numbers. Although coquaternions have often been overlooked, in recent years one can find a growing interest in the use of this four-dimensional real algebra, not only from the theoretical point of view, but also from the applications standpoint [1–3,10,11,14,15,18–21,24,25,27,28].

The motivation for this work came from previous studies on the dynamics of certain families of quadratic coquaternionic maps [8,11,12] which revealed that working on the coquaternion algebra setting leads to dynamics which are richer and even more extraordinary than the ones obtained in the classical real and complex cases. In fact, it was possible to show that, for some coquaternionic quadratics, non-isolated sets of fixed points appear: either lines, or even hyperboloids, of fixed points. Determining the fixed points of a quadratic dynamical system is inherently tied to the computation of the zeros of a quadratic polynomial. This connection underscores the importance of determining the zeros of generic quadratic polynomials and constructing polynomials with prescribed zeros.

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O. Gervasi et al. (Eds.): ICCSA 2024 Workshops, LNCS 14815, pp. 268–284, 2024.
https://doi.org/10.1007/978-3-031-65154-0_17

The study of the zero-set of generic monic unilateral polynomials[1] was previously undertaken by the authors of this work in [10]. The idea follows the pioneer work of Niven for the quaternions [22] and led to a comprehensive characterization of the zeros of a unilateral coquaternionic polynomial of degree n, revealing three distinct types of zeros - isolated, linear or hyperboloidal - with the total *number of zeros* not exceeding $n(2n-1)$. However, the method proposed in [10,22] involves computing the zeros of a $2n$ degree real polynomial, typically resulting in only numerical approximations. Moreover, accuracy issues associated with classical rootfinding methods can hinder the identification of the type of the zero. Indeed, even for the $n = 2$ case, it is not possible to obtain a general expression of the zeros in terms of its coefficients nor to know easily how many zeros of each nature a quadratic polynomial has. The work [13] marked an initial effort to investigate the coexistence of non-isolated zeros of the aforementioned types. While the study provided valuable insights into this phenomenon, it fell short of establishing a direct correlation between the zeros and the coefficients of the polynomials.

In [26], the factorizability of quadratic polynomials was explored, presenting results relevant to this problem. Furthermore, explicit formulas for the roots of such polynomials were provided in [4]. It is worth noting that our approach diverges significantly from those in these two works, as well as from the methodology employed in [10] to establish results on polynomial zeros of arbitrary degree. Here, we draw inspiration from the work on quaternions presented in [17] to determine both the number and nature of zeros that a quadratic polynomial may possess, alongside providing explicit expressions for these zeros.

We must refer that, since the algebra of coquaternions is isomorphic to the algebra of 2×2 real matrices, the problem at hand, albeit approached differently, can be regarded as a specific instance of the broader problem of solving matrix polynomial equations [6,16,29]. By employing a coquaternionic formalism, we were able to derive explicit results that offer simple and direct applicability, often bypassing the need for numerical methods.

Due to the manifest intricate product between coquaternions, we believe that the development of symbolic computation techniques in this context plays an important role. In particular, we have used the software system Mathematica to avoid all the tedious computations and the add-on `CoqPolynomial`, complementing the package `Coquaternions` [7] to run all the examples. The current versions of these tools are described and available for download at the webpage https://w3.math.uminho.pt/Coquaternions.

The paper is organized as follows: Sect. 2 contains the basic results on the coquaternion algebra; Sect. 3 gives a matrix reformulation of the problem under consideration, while the main results of the paper are presented in Sect. 4. The paper ends with several examples carefully chosen to draw attention to some aspects of the theory.

[1] A polynomial in one formal variable x whose coefficients are coquaternions located on the left (or right)-hand side of the powers of x.

2 Basic Results and Notation

Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$. The algebra of real coquaternions, which we denote by $\mathbb{H}_{\text{coq}}$, is generated by the product given according to the following rules

$$\mathbf{i}^2 = -1,\ \mathbf{j}^2 = \mathbf{k}^2 = 1,\ \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

The multiplication of two coquaternions $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is given by

$$\begin{aligned} pq = {} & p_0q_0 - p_1q_1 + p_2q_2 + p_3q_3 + (p_0q_1 + p_1q_0 - p_2q_3 + p_3q_2)\mathbf{i} \\ & + (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)\mathbf{j} + (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)\mathbf{k}. \end{aligned} \tag{1}$$

The expression (1) shows that the product pq can be computed using matrices as $L_p\mathbf{q}$, with

$$L_p = \begin{pmatrix} p_0 & -p_1 & p_2 & p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \tag{2}$$

and where, when performing the matrix multiplication, we identify q with the column vector $\mathbf{q} = (q_0\ q_1\ q_2\ q_3)^{\mathrm{T}}$.

The *conjugate* $\overline{q}$ of a coquaternion q is defined as $\overline{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$; the number q_0 is called the *real part* of q and denoted by $\operatorname{Re} q$ and the *vector part* of q, denoted by $\underline{q}$, is given by $\underline{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$.

We recall that every coquaternion q can be represented by the real matrix

$$\begin{pmatrix} q_0 + q_3 & q_1 + q_2 \\ q_2 - q_1 & q_0 - q_3 \end{pmatrix}$$

and for this reason the quantities $\operatorname{tr} q = q + \overline{q} = 2\operatorname{Re} q$ and $\det q = q\overline{q} = q_0^2 + q_1^2 - q_2^2 - q_3^2$ are called the *trace* and *determinant* of q, respectively. We endow $\mathbb{H}_{\text{coq}}$ with the *norm*[2] $\|q\| = \sqrt{|\det q|}$ and say that q is a *non-singular* coquaternion if $\|q\| \neq 0$ and call q a *unit coquaternion* if $\|q\| = 1$.

We identify the set of elements in $\mathbb{H}_{\text{coq}}$ whose vector part is zero with the set $\mathbb{R}$ of real numbers. The subspace of $\mathbb{H}_{\text{coq}}$ of coquaternions whose real part is zero (usually referred to as pure coquaternions) is identified with the $(2+1)$-Minkowski space $\mathbb{R}^{2+1}$. This is an inner product space consisting of the real vector space $\mathbb{R}^3$ equipped with the Lorentz inner product $\langle \underline{p}, \underline{q} \rangle_L = -p_1q_1 + p_2q_2 + p_3q_3$. We observe that $\langle \underline{q}, \underline{q} \rangle_L = -\det \underline{q}$.

The sign of $\det q$ determines the type of the coquaternion q; q is *space-like, light-like* or *time-like*, if $\det q < 0$, $\det q = 0$ or $\det q > 0$, respectively; the sets of such coquaternions will be denoted by $\mathbb{S}$, $\mathbb{L}$ and $\mathbb{T}$, respectively.

[2] Although this does not have all the properties of a true norm, it is still usually referred to as the norm of q.

We define now in $\mathbb{H}_{\text{coq}}$ the relation

$$p \sim q \iff \operatorname{Re} p = \operatorname{Re} q \text{ and } \det \underline{p} = \det \underline{q}. \tag{3}$$

This is an equivalence relation in $\mathbb{H}_{\text{coq}}$ and the corresponding equivalence class of q is the set

$$[q] := \left\{ x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_0 = q_0 \text{ and } x_1^2 - x_2^2 - x_3^2 = q_1^2 - q_2^2 - q_3^2 \right\}. \tag{4}$$

We can identify[3] $[q]$ with a hyperboloid in the hyperplane $x_0 = q_0$; this will be a hyperboloid of one sheet, a hyperboloid of two sheets or a degenerate hyperboloid (i.e., a cone), if $\underline{q} \in \mathbb{S}$, $\underline{q} \in \mathbb{T}$ or $\underline{q} \in \mathbb{L}$, respectively.

In a previous work [10] we presented a complete characterization of the zero-set $Z(P)$ of a unilateral coquaternionic polynomial P of degree n and proved that the zeros of P belong to, at most, $n(2n-1)$ equivalence classes. In the case of a quadratic equation, this result leads to the conclusion that such equation has at most six zeros. In this context, we have also showed that coquaternionic polynomials may have three different types of zeros, motivating us to introduce the following definition.

A zero z of a given coquaternionic polynomial P is said to be an *isolated zero* of P, if $[z]$ contains no other zeros of P; z is said to be a *hyperboloidal zero* of P, if $[z] \subseteq Z(P)$; z is said to be a *linear zero* of P, if z is neither an isolated zero nor a hyperboloidal zero of P.[4]

In what follows, we will treat all the zeros belonging to the same equivalence class as forming a single zero, i.e., we will refer to a whole hyperboloid of zeros or a line of zeros simply as a hyperboloidal zero or a linear zero, respectively.

3 Quadratic Polynomials in $\mathbb{H}_{\text{coq}}$

In what follows we consider the quadratic equation

$$x^2 + bx + c = 0, \tag{5}$$

where $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $b = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ are in $\mathbb{H}_{\text{coq}}$. As mentioned before, equations of this form have at most six zeros; what remains to be seen is how many zeros of each type they may have.

3.1 Reformulation of the Problem

Recalling the so-called left representation of a coquaternion (2) and using the fact that any coquaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is a zero of the equation[5]

$$x^2 - \operatorname{tr}(q)x + \det q = 0, \tag{6}$$

[3] Contrary to what happens with quaternions, the relation (3) does not coincide with the usual similarity relation; coquaternions satisfying (3) are called quasi-similar; see e.g. [10] for details.

[4] The term linear zero is justified by the fact that, in this case, the zeros in the class $[z]$ form a line in a given hyperplane.

[5] The polynomial on the left-hand side of (6) is called the characteristic polynomial of q.

one can write the Eq. (5) in the matrix form

$$(2x_0 I_4 + L_b)\mathbf{x} = \mathbf{c},$$

where

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix}^T \text{ and } \mathbf{c} = \begin{pmatrix} -c_0 + x_0^2 + x_1^2 - x_2^2 - x_3^2 & -c_1 & -c_2 & -c_3 \end{pmatrix}^T.$$

We follow now the approach of [17] adapted to the coquaternionic setting. Assume that Eq. (5) has a solution x such that $\operatorname{Re} x = t$, i.e., that the following nonlinear system is solvable,

$$\begin{pmatrix} 2t+b_0 & -b_1 & b_2 & b_3 \\ b_1 & 2t+b_0 & b_3 & -b_2 \\ b_2 & b_3 & 2t+b_0 & -b_1 \\ b_3 & -b_2 & b_1 & 2t+b_0 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -c_0 + t^2 + x_1^2 - x_2^2 - x_3^2 \\ -c_1 \\ -c_2 \\ -c_3 \end{pmatrix}.$$

Such a system combines the nonlinear equation

$$(x_1 + \tfrac{b_1}{2})^2 - (x_2 + \tfrac{b_2}{2})^2 - (x_3 + \tfrac{b_3}{2})^2 = t^2 + b_0 t + \tfrac{b_1^2 - b_2^2 - b_3^2}{4} + c_0 \qquad (7)$$

with the linear system

$$\mathcal{M}(t)\widetilde{\mathbf{x}}(t) = \widetilde{\mathbf{c}}(t), \qquad (8)$$

where

$$\mathcal{M}(t) := \begin{pmatrix} 2t+b_0 & b_3 & -b_2 \\ b_3 & 2t+b_0 & -b_1 \\ -b_2 & b_1 & 2t+b_0 \end{pmatrix}, \ \widetilde{\mathbf{x}}(t) := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ \widetilde{\mathbf{c}}(t) := \begin{pmatrix} -c_1 - b_1 t \\ -c_2 - b_2 t \\ -c_3 - b_3 t \end{pmatrix}. \qquad (9)$$

Observe that, for each value of t,

$$\mathcal{H}_t := \Big\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 = t, \\ (x_1 + \tfrac{b_1}{2})^2 - (x_2 + \tfrac{b_2}{2})^2 - (x_3 + \tfrac{b_3}{2})^2 = t^2 + b_0 t + \tfrac{b_1^2 - b_2^2 - b_3^2}{4} + c_0 \Big\} \qquad (10)$$

can be identified with a hyperboloid with center $(-\frac{b_1}{2}, -\frac{b_2}{2}, -\frac{b_3}{2})$ in the hyperplane $x_0 = t$; this will be a hyperboloid of two-sheets, one-sheet, or a cone, depending on whether $t^2 + b_0 t + \frac{1}{4}\det \underline{b} + c_0$ is negative, positive or zero, respectively. The solvability of the system (7)-(8) is linked to the rank of the matrix $\mathcal{M}(t)$, as indicated in the next subsection.

3.2 Rank of $\mathcal{M}(t)$

Simple calculations show that

$$\det \mathcal{M}(t) = (b_0 + 2t)(\det \underline{b} + (b_0 + 2t)^2).$$

The set $\mathcal{S}$ of real solutions of $\det \mathcal{M}(t) = 0$ depends on the type of the vector part of b, namely:

1. if $\underline{b}$ is space-like, then

$$\mathcal{S} = \left\{ -\tfrac{b_0}{2}, -\tfrac{b_0}{2} + \tfrac{\|\underline{b}\|}{2}, -\tfrac{b_0}{2} - \tfrac{\|\underline{b}\|}{2} \right\}$$

2. if $\underline{b}$ is time-like or light-like, then

$$\mathcal{S} = \left\{ -\tfrac{b_0}{2} \right\}. \tag{11}$$

From now on and for simplicity we adopt the following notation.

$$N_1 := -\|\underline{b}\|;\ N_2 := \|\underline{b}\|;\ s_0 := -\tfrac{b_0}{2};\ s_1 := s_0 + N_1;\ s_2 := s_0 + N_2. \tag{12}$$

Case 1: $\operatorname{rank} \mathcal{M}(t) = 0$

This is equivalent to saying that $b_1 = b_2 = b_3 = 0$ and $t = -\frac{b_0}{2}$. The system (8) is solvable iff $c_1 = c_2 = c_3 = 0$. In such case, the Eq. (5) reduces to the real quadratic equation

$$x^2 + b_0 x + c_0 = 0,\ x \in \mathbb{H}_{\text{coq}},\ b_0, c_0 \in \mathbb{R}$$

whose solution $-\frac{b_0}{2} + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ has to satisfy (7), i.e.,

$$x_1^2 - x_2^2 - x_3^2 = c_0 - \frac{b_0^2}{4}.$$

Case 2: $\operatorname{rank} \mathcal{M}(t) = 1$

If $\operatorname{rank} \mathcal{M}(t) < 3$, then $\det \mathcal{M}(t) = 0$, i.e., $t \in \mathcal{S}$. Consider first the case where $\underline{b}$ is not space-like, i.e., $t = s_0$ (cf. (11)-(12)). Then

$$\mathcal{M}(s_0) = \begin{pmatrix} 0 & b_3 & -b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}.$$

The case $b_1 = b_2 = b_3 = 0$ was already considered. If one of the b_i is non-zero, then it follows that $\operatorname{rank} \mathcal{M}(s_0) \neq 1$.

Assume now that $\underline{b}$ is space-like and consider the matrices

$$\mathcal{M}(s_k) = \begin{pmatrix} \frac{N_k}{2} & b_3 & -b_2 \\ b_3 & \frac{N_k}{2} & -b_1 \\ -b_2 & b_1 & \frac{N_k}{2} \end{pmatrix};\ k = 1, 2.$$

Since $N_k \neq 0$, it is easy to see that the systems

$$\big(N_k, b_3, -b_2\big) = \alpha\big(b_3, N_k, -b_1\big) = \beta\big(-b_2, b_1, N_k\big)$$

have no solution for $\alpha, \beta \in \mathbb{R}$, which in turn implies that $\operatorname{rank} \mathcal{M}(s_k) \neq 1$. In conclusion, the case $\operatorname{rank} \mathcal{M}(t) = 1$ never occurs.

Case 3: $\operatorname{rank} \mathcal{M}(t) = 2$

In this situation the zero-set of the system (8) is an affine line L_t, provided that $\operatorname{rank}([\mathcal{M}(t)\ \widetilde{\mathbf{c}}(t)]) = 2$; the discussion of the problem under consideration reduces to the discussion of the intersection of

$$\mathcal{L}_t = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 = t, (x_1, x_2, x_3) \in L_t\} \tag{13}$$

with the hyperboloid $\mathcal{H}_t$ in (10), which, as it is well known, can be (in general): an empty set, one point, two points or $\mathcal{L}_t$ itself. We refer in this context the paper [27], where the intersection of a double cone and a line is studied in the coquaternionic setting.

Case 4: $\operatorname{rank}\mathcal{M}(t) = 3$

In this case, $t \notin \mathcal{S}$ and for each t, the system (8) has a unique solution $(x_1, x_2, x_3) = (x_1(t), x_2(t), x_3(t))$ which can be obtained by using any computer algebra tools. Using the function `LinearSolve` of Mathematica, we end up in (7) with a sixth degree polynomial of the form

$$\begin{aligned} p_6(t) = &-16t^6 - 48b_0t^5 - 8(7b_0^2 + 2c_0 + \alpha)t^4 - 8(b_0(4(b_0^2 + c_0) + \alpha) + 2\beta)t^3 \\ &+ (-9b_0^4 - 2(12c_0 + \alpha)b_0^2 - 20\beta b_0 + 3\alpha^2 + 4c_0\alpha - 4\gamma)t^2 \\ &+ (-b_0^5 - 8c_0b_0^3 - 8\beta b_0^2 + (\alpha^2 + 4c_0\alpha - 4\gamma)b_0 + 4\alpha\beta)t \\ &+ \beta^2 - b_0^4c_0 - b_0^3\beta + b_0\alpha\beta + b_0^2(c_0\alpha - \gamma), \end{aligned} \tag{14}$$

where

$$\alpha := \langle \underline{b}, \underline{b} \rangle_L, \quad \beta := \langle \underline{b}, \underline{c} \rangle_L, \quad \gamma := \langle \underline{c}, \underline{c} \rangle_L. \tag{15}$$

This leads to the conclusion that there are at most six non-isolated solutions of the problem (5), which can be obtained, for each real solution t of (14), from

$$\begin{cases} x_0 = & t \\ x_1 = & -\dfrac{(b_3c_2 - b_2c_3)(b_0 + 2t) - (b_0 + 2t)^2(c_1 + b_1t) + b_1t\alpha + b_1\beta}{\det \mathcal{M}(t)} \\ x_2 = & -\dfrac{(b_3c_1 - b_1c_3)(b_0 + 2t) - (b_0 + 2t)^2(c_2 + b_2t) + b_2t\alpha + b_2\beta}{\det \mathcal{M}(t)} \\ x_3 = & \dfrac{(-b_2c_1 + b_1c_2)(b_0 + 2t) - (b_0 + 2t)^2(c_3 + b_3t) + b_3t\alpha + b_3\beta}{\det \mathcal{M}(t)} \end{cases} \tag{16}$$

4 Main Results

Based on the conclusions of the previous section, the approach to obtain and prove the results concerning the identification and computation of the linear zeros of a monic quadratic coquaternionic equation can be described in three steps:

Step 1: For each value of s, such that $\det \mathcal{M}(s) = 0$, determine conditions on b and c in order to guarantee that the linear system

$$\mathcal{M}(s)\widetilde{\mathbf{x}}(s) = \widetilde{\mathbf{c}}(s),$$

has infinitely many solutions; see (9);

Step 2: Determine the set L_s of solutions of such system and obtain the corresponding line $\mathcal{L}_s$ given by (13);

Step 3: Determine additional conditions, if any, on b and c in order to guarantee that $\mathcal{L}_s \subset \mathcal{H}_s$, i.e., that all the elements in $\mathcal{L}_s$ are also solutions of (7).

All the steps are based on computer algebra tools. In particular, we have used the software system Mathematica to avoid all the tedious computations. Details about such computations can be obtained in the notebooks available at https://w3.math.uminho.pt/QuadraticsTools.

The nature and number of non-isolated solutions of the quadratic equation (5) depend on the form of the coquaternions b and c. Next results give the explicit expressions of the linear (respectively, hyperboloidal) zeros that Eq. (5) may have and show that the co-existence of isolated and linear (respectively, hyperboloidal) zeros is possible, but the number of isolated zeros in such cases is at most two.

Theorem 1. *If $\underline{b} = 0$, the quadratic equation has non-isolated zeros iff $\underline{c} = 0$. In such conditions, the equation has* **one hyperboloidal zero** *given by*

$$\mathcal{H}_{-\frac{b_0}{2}} = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 = -\tfrac{b_0}{2}, x_1^2 - x_2^2 - x_3^2 = c_0 - \tfrac{b_0^2}{4}\right\}. \tag{17}$$

In addition, if $\underline{c} = 0$ and $b_0^2 - 4c_0 > 0$, there are also **two isolated real zeros**, *given by*

$$z_1 = \frac{-b_0 + \sqrt{b_0^2 - 4c_0}}{2} \qquad \text{and} \qquad z_2 = \frac{-b_0 - \sqrt{b_0^2 - 4c_0}}{2}. \tag{18}$$

Proof. If $\underline{b} = \underline{c} = 0$, the result (17) follows at once from Case 1 of Sect. 3.2. In addition and under the assumption that $\underline{c} = 0$ and $b_0^2 - 4c_0 > 0$, we can conclude from (14) and (15) that the real part of the isolated zeros (different from $-\frac{b_0}{2}$) are the solutions of the equation $c_0 + t(b_0 + t) = 0$, and the result (18) follows. On the other hand, if $\underline{c} \neq 0$, then (14) reduces to

$$-(b_0 + 2t)^2 \left(\gamma + b_0^2 c_0 + (b_0^3 + 4b_0 c_0)t + (5b_0^2 + 4c_0)t^2 + 8b_0 t^3 + 4t^4\right),$$

and the result is proved, since $b_0 + 2t \neq 0$. □

Theorem 2. *If $\underline{b}$ is a time-like coquaternion, then the quadratic equation has no non-isolated zeros.*

Proof. If $\det \underline{b} > 0$, then $b_1 \neq 0$ and the unique value of s such that $\det \mathcal{M}(s) = 0$ is $s = s_0 = -\frac{b_0}{2}$ (cf. (11)). For $t = s_0$, the linear system (8) can be written in the equivalent form (cf. (12))

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & -\frac{\alpha b_0 - 2\beta}{2b_1} \\ b_3 & 0 & -b_1 & \frac{b_0 b_2}{2} - c_2 \\ -b_2 & b_1 & 0 & \frac{b_0 b_3}{2} - c_3 \end{array}\right).$$

This shows that the system is consistent iff $\alpha b_0 - 2\beta = 0$, i.e.,

$$c_1 = \frac{-\alpha b_0 + 2b_2c_2 + 2b_3c_3}{2b_1}. \tag{19}$$

The set L_{s_0} of solutions of the above system is given by

$$L_{s_0} = \left\{ \left(\lambda, \frac{b_0b_3 - 2c_3}{2b_1} + \frac{b_2}{b_1}\lambda, \frac{2c_2 - b_0b_2}{2b_1} + \frac{b_3}{b_1}\lambda \right) : \lambda \in \mathbb{R} \right\}. \tag{20}$$

The intersection of the corresponding line $\mathcal{L}_{s_0}$ with the hyperboloid $\mathcal{H}_{s_0}$ in (10) is the set of points such that

$$\alpha b_0^2 + 4b_1(b_1c_0 - b_2c_3 + b_3c_2) - 4b_0(b_3c_3 + b_2c_2) + 4c_2^2 + 4c_3^2 + (8b_3c_2 - 8b_2c_3 + 4b_1\alpha)\lambda + 4\alpha\lambda^2 = 0. \tag{21}$$

Since $\underline{b} \in \mathbb{T}$, it follows that $\alpha = -\det \underline{b} \neq 0$, i.e., this last equation has at most two real solutions which lead to isolated zeros of (5) with real part s_0. □

Theorem 3. *If $\underline{b} \neq 0$ is a light-like coquaternion, then the quadratic equation has non-isolated zeros iff*

$$c = mb - m^2, \quad \text{for some } m \in \mathbb{R}. \tag{22}$$

In such conditions, the equation has **one linear zero**, *given by*

$$\mathcal{L}_{s_0} = \left\{ s_0 + \frac{s_0 + m}{b_1}(b_2\mathbf{k} - b_3\mathbf{j}) + \lambda\underline{b} : \lambda \in \mathbb{R} \right\}. \tag{23}$$

In addition to the assumption (22), if $m \neq -s_0$, there are also **two distinct isolated zeros**, *given by*

$$z_1 = -m \qquad \text{and} \qquad z_2 = m - b. \tag{24}$$

Proof. If $\det \underline{b} = 0$ and $\underline{b} \neq 0$, relations (19)-(21) remain true; in particular, (21) reads as

$$b_1(b_0c_1 - b_1c_0 + b_2c_3 - b_3c_2) - c_2^2 - c_3^2 - -2(b_3c_2 - b_2c_3)\lambda = 0,$$

which means that $\mathcal{L}_{s_0} \subset \mathcal{H}_{s_0}$, iff

$$b_1(b_0c_1 - b_1c_0 + b_2c_3 - b_3c_2) - c_2^2 - c_3^2 = b_3c_2 - b_2c_3 = 0. \tag{25}$$

Combining (19) and (25) we obtain, after some manipulation,

$$c_0 = \frac{c_1(b_0b_1 - c_1)}{b_1^2}, \quad c_2 = \frac{b_2c_1}{b_1}, \quad c_3 = \frac{b_3c_1}{b_1}$$

and the first part of the result is proved. Using these relations together with (20), we obtain

$$\mathcal{L}_{s_0} = \left\{ -\frac{b_0}{2} + \lambda\mathbf{i} + \frac{b_0b_1b_3 - 2b_3c_1 + 2b_1b_2\lambda}{2b_1^2}\mathbf{j} + \frac{-b_0b_1b_2 + 2b_2c_1 + 2b_1b_3\lambda}{2b_1^2}\mathbf{k} : \lambda \in \mathbb{R} \right\}$$

and result (23) follows.

Concerning isolated zeros, if $t \neq s_0$ and $s_0 \neq m$, the polynomial (14) reduces, under the assumptions (22), to

$$-\frac{(b_0+2t)^4}{b_1^2}(c_1+b_1t)(-c_1+b_1(b_0+t))$$

whose real solutions are $t=-\frac{c_1}{b_1}$ and $t=\frac{-b_0b_1+c_1}{b_1}$, and the isolated zeros (24) follow from relation (16). Finally, in the general case, where the assumptions (22) are not valid and $t \neq s_0$, we end up with at most six isolated zeros of the quadratic equation (5), which can be obtained from (14) and (16). □

Theorem 4. *If $\underline{b}$ is a space-like coquaternion, then the quadratic equation has non-isolated zeros iff*

$$c = mb - m^2, \qquad \text{for some } m \in \mathbb{R}. \tag{26}$$

In such conditions, the quadratic equation has **two linear zeros** $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *whose expressions are as follows:*

1. *If* $b_1^2 \neq b_2^2$

$$\mathcal{L}_k = \Big\{ s_k - \tfrac{s_k+m}{b_1^2-b_2^2}\big((b_2b_3-b_1N_k)\mathbf{i}+(b_1b_3-b_2N_k)\mathbf{j}\big)+ \\ \lambda\big((b_1b_3-b_2N_k)\mathbf{i}+(b_2b_3-b_1N_k)\mathbf{j}+(b_1^2-b_2^2)\mathbf{k}\big) : \lambda \in \mathbb{R}\Big\};\ k=1,2.$$

2. *If* $b_1 = (-1)^s b_2 \neq 0$ $(s=1,2)$

$$\mathcal{L}_1 = \Big\{ \tfrac{-b_0+(-1)^s b_3}{2} + \tfrac{b_0-(-1)^s b_3-2m}{4b_2b_3}\Big((b_2^2-b_3^2)\mathbf{i}+(-1)^s(b_2^2+b_3^2)\mathbf{j}\Big)+ \\ \lambda\Big((b_2^2+b_3^2)\mathbf{i}+(-1)^s(b_2^2-b_3^2)\mathbf{j}+2(-1)^s b_2b_3\mathbf{k}\Big) : \lambda \in \mathbb{R}\Big\},$$

$$\mathcal{L}_2 = \Big\{ \tfrac{-b_0-(-1)^s b_3}{2} - (-1)^s \tfrac{b_0+(-1)^s b_3-2m}{2}\mathbf{k} + \lambda\big((-1)^s\mathbf{i}+\mathbf{j}\big) : \lambda \in \mathbb{R}\Big\}.$$

3. *If* $b_1 = b_2 = 0$

$$\mathcal{L}_k = \Big\{ s_k - \tfrac{s_k+m}{N_k}\underline{b} + \lambda(-b_3\mathbf{i}+N_k\mathbf{j}) : \lambda \in \mathbb{R}\Big\};\ k=1,2.$$

In addition to the assumption (26), if $m \neq -s_k$, there are also **two distinct isolated zeros**, *given by*

$$z_1 = -m \qquad \text{and} \qquad z_2 = m-b. \tag{27}$$

Proof. We first prove that when $\underline{b} \in \mathbb{S}$, there are no linear zeros with real part s_0. In fact, for $t=s_0$ and $b_3 \neq 0$, the linear system (8) can be written in the equivalent form

$$\left(\begin{array}{ccc|c} 0 & b_3 & -b_2 & \frac{b_0b_1}{2}-c_1 \\ b_3 & 0 & -b_1 & \frac{b_0b_2}{2}-c_2 \\ 0 & 0 & 0 & -\frac{\alpha b_0-2\beta}{2b_3} \end{array}\right).$$

This shows that the system is solvable iff $c_3 = \frac{\alpha b_0 + 2b_1c_1 - 2b_2c_2}{2b_3}$. The set L_{s_0} of solutions of the above system is given by (cf. (20))

$$L_{s_0} = \Big\{\Big(\frac{b_0b_2 - 2c_2}{2b_3} + \frac{b_1}{b_3}\lambda, \frac{-2c_1 + b_0b_1}{2b_3} + \frac{b_2}{b_3}\lambda, \lambda\Big) : \lambda \in \mathbb{R}\Big\}.$$

The intersection of the corresponding line $\mathcal{L}_{s_0}$ with the hyperboloid $\mathcal{H}_{s_0}$ in (10) is the set of points such that

$$-\alpha b_0^2 + 4b_3(b_3c_0 - b_2c_1 + b_1c_2) - 4b_0(b_1c_1 + b_2c_2) + 4c_1^2 - 4c_2^2 + 4(2b_3c_2 - 2b_2c_3 + b_1\alpha)\lambda + 4\alpha\lambda^2 = 0.$$

Since $\underline{b} \in \mathbb{S}$, it follows that $\alpha = -\det \underline{b} \neq 0$, i.e., this last equation has at most two real solutions which lead to isolated zeros of (5), with real part s_0. When $b_3 = 0$, the corresponding equation reads as

$$-\alpha b_0^2 + 4b_2(b_2c_0 - b_1c_3) - 4b_0b_1c_1 + 4c_1^2 - 4c_3^2 + (-8b_1c_3 + 4b_2\alpha)\lambda + 4\alpha\lambda^2 = 0$$

and the conclusion is the same.

The computation of the linear zeros with real part s_k requires the study of three distinct cases; observe that when $b_1^2 = b_2^2$ (Cases 2. and 3.), then $b_3 \neq 0$. We just give the proof of the first identity, for $k = 1$, using once more Mathematica capabilities for algebraic computation; the other identities can be proved in the same way (cf. notebook). For $t = s_1$ and under the assumption that $b_1^2 \neq b_2^2$, the linear system (8) can be written in the equivalent form

$$\left(\begin{array}{ccc|c} 2N_1 & 2b_3 & -2b_2 & b_0b_1 - b_1N_1 - 2c_1 \\ 0 & 2(b_1^2 - b_2^2) & -2b_2b_3 & b_3(b_0b_1 - 2c_1) + (2c_2 - b_1b_3 - b_0b_2)N_1 + \alpha b_2 \\ 0 & 0 & 0 & b_3(b_1c_1 - b_2c_2) + (b_2c_1 - b_1c_2)N_1 - c_3(b_1^2 - b_2^2) \end{array}\right),$$

which shows that the system is consistent iff

$$c_3 = \frac{b_3(b_1c_1 - b_2c_2) + (b_2c_1 - b_1c_2)N_1}{b_1^2 - b_2^2}. \tag{28}$$

The set L_{s_1} of solutions of the above system is given by

$$L_{s_1} = \Big\{\Big(\frac{-b_3(c_2 + b_2s_1) + N_1(c_1 + b_1s_1)}{b_1^2 - b_2^2} + \frac{b_1b_3 - b_2N_1}{b_1^2 - b_2^2}\lambda, \frac{-b_3(c_1 + b_1s_1) + N_1(c_2 + b_2s_1)}{b_1^2 - b_2^2} + \frac{b_1b_3 - b_2N_1}{b_1^2 - b_2^2}\lambda, \lambda\Big) : \lambda \in \mathbb{R}\Big\}$$

The intersection of the corresponding line $\mathcal{L}_{s_1}$ with the hyperboloid $\mathcal{H}_{s_1}$ in (10) is the set of points such that

$$(-b_1^2 + b_2^2)c_0 + (b_0b_1 + b_2b_3)c_1 - (b_0b_2 + b_1b_3)c_2 + c_2^2 - c_1^2 + 2(b_2c_1 - b_1c_2)\lambda = 0,$$

which means that $\mathcal{L}_{s_1} \subset \mathcal{H}_{s_1}$ iff

$$b_2c_1 - b_1c_2 = (-b_1^2 + b_2^2)c_0 + (b_0b_1 + b_2b_3)c_1 - (b_0b_2 + b_1b_3)c_2 + c_2^2 - c_1^2 = 0. \tag{29}$$

Combining (28) and (29) we obtain, after some manipulation,

$$c_0 = (b_0 - m)m, \quad c_1 = b_1 m, \quad c_2 = b_2 m, \quad c_3 = b_3 m$$

which leads to

$$\mathcal{L}_{s_1} = \left\{ s_1 + \frac{N_1(b_1 m + b_1 s_1 - b_2 \lambda)}{b_1^2 - b_2^2}\mathbf{i} + \frac{N_1(b_2 m + b_2 s_1 - b_1 \lambda)}{b_1^2 - b_2^2}\mathbf{j} + \lambda \mathbf{k} : \lambda \in \mathbb{R} \right\},$$

and the result in Case 1. (for $k = 1$) is proved.

Concerning now the coexistence of linear zeros and isolated zeros, we observe that, if $t \neq s_k$ and $s_k \neq m$, the polynomial (14) reduces, under the assumptions (26), to

$$-(b_0 - m + t)(m + t)(b_1^2 - b_2^2 - b_3^2 + (b_0 + 2t)^2)^2$$

whose real solutions are $t = -m$ and $t = -b_0 + m$, and the isolated zeros (27) follow from relation (16). Finally, in the general case, where the assumptions (26) are not valid and $t \neq s_k$; $k = 0, 1, 2$, we end up with at most six isolated zeros of the quadratic equation (5), which can be obtained from (14) and (16). □

As an immediate consequence of Theorems 1-4 we have the following results.

Corollary 1. *The zeros, if any, of the equation $x^2 = c$ (i.e., the square roots of c) are always either isolated or hyperboloidal.*

Corollary 2. *The equation $x^2 + bx + c = 0$ has linear zeros if and only if $\underline{b} \neq 0$ is a coquaternion in $\mathbb{L} \cup \mathbb{S}$ and $c = mb - m^2$, for some $m \in \mathbb{R}$. In particular, under these conditions, the equation $x^2 + bx = 0$ always has linear zeros.*

Corollary 3. *Concerning the number of non-isolated zeros, a quadratic equation of the form (5) can have at most:*

- *one hyperboloidal zero (and up to 2 isolated zeros);*
- *two linear zeros (and up to 2 isolated zeros).*

The results of Corollaries 1 and 3 align with the ones obtained in [9,23] and [13], respectively.

5 Examples

Recently we have developed a collection of Mathematica functions, `CoqPolynomial`, complementing the package `Coquaternions` whose current versions are described and available at http://w3.math.uminho.pt/Coquaternions for download. Details about the use of these tools together with several examples illustrating their performance can be found in [7].

With the help of `CoqPolynomial` we have developed a Mathematica m-file `QuadraticsAlgorithm.m` to implement the results of Theorems 1-4. In particular the function `ZerosQuadratic[b,c]` gives a list {`Izeros,Lzeros,Hzeros`} of the isolated, linear and hyperboloidal zeros of the polynomial $x^2 +$

$bx + c$, where b and c are coquaternions; these tools are available at https://w3.math.uminho.pt/QuadraticsTools.

We give now some examples illustrating the results of the previous section. For the first example we present all the details, including the use of the aforementioned Mathematica tools.

Example 1. Consider the polynomial

$$P(x) = x^2 + (3 + 5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})x + 2 + 10\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}.$$

Since $\underline{b} = 5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \in \mathbb{L}$ and $c = 2b - 4$, the polynomial P fulfills assumption (22) of Theorem 3 with $m = 2$. In this case, the matrix $\mathcal{M}(t)$ in (9) is

$$\mathcal{M}(t) := \begin{pmatrix} 2t+3 & 4 & -3 \\ 4 & 2t+3 & -5 \\ -3 & 5 & 2t+3 \end{pmatrix}$$

whose determinant is $(2t + 3)^3$. This means that $s_0 = -\frac{3}{2} = -\frac{b_0}{2}$ (cf. (11)-(12)) and the quadratic equation has one linear zero given by (23), i.e.,

$$\mathcal{L}_{-\frac{3}{2}} = \left\{ -\tfrac{3}{2} + \tfrac{1}{10}(3\mathbf{k} - 4\mathbf{j}) + \lambda(5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) : \lambda \in \mathbb{R} \right\}.$$

In addition, since $m \neq s_0$, the equation also has the two distinct isolated zeros (24) -2 and $-1 - 5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$.

All these calculations can be easily performed using the Mathematica add-on `CoqPolynomial` and the function `ZerosQuadratic` which implements the results of this paper.

```
In[1]:= <<QuadraticsAlgorithm`
In[2]:= b=Coquaternion[3,5,3,4]; c=Coquaternion[2,10,6,8]; Nature[Vec[b]]
        (* checks the nature of b *)
Out[2]= Lightlike
In[3]:= Solve[List@@(c-mb+m^2)=={0,0,0,0},m] (* checks the assumptions *)
Out[3]= {{m → 2}}
```

The zeros of P can be obtained as follows:

```
In[4]:= (zeros=ZerosQuadratic[b,c])// TraditionalForm
Out[4]= {{-2, -1 - 5i - 3j - 4k}, {-3/2 + 5iλ + j(-2/5 + 3λ) + k(3/10 + 4λ)}, {}}
```

We can check the solution by constructing and evaluating the polynomial P at each computed zero.

```
In[5]:= P=Polynomial[1,b,c]; Eval[P,Flatten[zeros]]
Out[5]= {0, 0, 0}
```

Table 1. The number and nature (LZ-linear zeros; IZ-isolated zeros; HZ-hyperboloidal zeros) of the zeros of the polynomials of Example 2

Pol.	Coefficients	Type $\underline{b}$	Result	m	s_k	Number of zeros
P_1	$b = 2 + 4\mathbf{k}$ $c = 1 + 4\mathbf{k}$	$\underline{b} \in \mathbb{S}$	Th. 4 (3)	1	$s_1 = -3$ $s_2 = 1$	2 LZ 2 IZ
P_2	$b = 3 + 5\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ $c = 0$	$\underline{b} \in \mathbb{S}$	Th. 4 (1)	0	$s_1 = -3$ $s_2 = 0$	2 LZ
P_3	$b = 4 + 2\mathbf{i} - 2\mathbf{j}$ $c = 4 + 4\mathbf{i} - 4\mathbf{j}$	$\underline{b} \in \mathbb{L}$	Th. 3	2	$s_0 = -2$	1 LZ
P_4	$b = 4$ $c = 3$	$\underline{b} \in \mathbb{L}$	Th. 1	–	$s_0 = -2$	1 HZ 2 IZ

Example 2. We consider now other four polynomials $P_i; i = 1, \ldots, 4$, with non-isolated zeros. The polynomials and some details about the structure of their zeros are listed in Table 1.

The result of using function `ZerosQuadratic` for each polynomial P_i, gives the following zeros:

$$\texttt{P}_1 : \{\{-1, -1 - 4\mathbf{k}\}, \{-3 - 4\lambda\mathbf{i} - 4\lambda\mathbf{j} - 2\mathbf{k}, 1 - 4\lambda\mathbf{i} + 4\lambda\mathbf{j} - 2\mathbf{k}\}, \{\,\}\}$$

$$\texttt{P}_2 : \{\{\,\}, -3 + \left(\tfrac{45}{8} + 34\lambda\right)\mathbf{i} + \left(\tfrac{51}{8} + 30\lambda\right)\mathbf{j} + 16\lambda\mathbf{k}, 16\lambda\mathbf{i} + 16\lambda\mathbf{k}\}, \{\,\}\}$$

$$\texttt{P}_3 : \{\{\,\}, \{-2 + 2\lambda\mathbf{i} - 2\lambda\mathbf{j}\}, \{\,\}\}$$

$$\texttt{P}_4 : \{\{-1, -3\}, \{\,\}, \{-2 + \mathbf{j}\}\}$$

We remark that in the case of a hyperboloidal zero, the output is one representative of the equivalence class (4).

Example 3. Finally we consider four polynomials $Q_i; i = 1, \ldots, 4$, which have only isolated zeros. The polynomials and their zeros are listed in Table 2. We point out that in this situation, since the number of zeros of Q_i is at most 6, the nonlinear system corresponding to the quadratic equation (5), i.e.,

$$\begin{cases} c_0 + b_0 x_0 + x_0^2 - b_1 x_1 - x_1^2 + b_2 x_2 + x_2^2 + b_3 x_3 + x_3^2 = 0 \\ c_1 + b_1 x_0 + b_0 x_1 + 2x_0 x_1 + b_3 x_2 - b_2 x_3 = 0 \\ c_2 + b_2 x_0 + b_3 x_1 + b_0 x_2 + 2x_0 x_2 - b_1 x_3 = 0 \\ c_3 + b_3 x_0 - b_2 x_1 + b_1 x_2 + b_0 x_3 + 2x_0 x_3 = 0 \end{cases} \tag{30}$$

can be easily solved (numerically, if necessary) by using the Mathematica function `Solve`.

Table 2. The zeros of the polynomials of Example 3

Pol.	Coefficients	Type of $\underline{b}$	Zeros
Q_1	$b = 2 - \mathbf{i} - \mathbf{j}$ $c = 3 - \mathbf{i} - \mathbf{j} - \mathbf{k}$	$\underline{b} \in \mathbb{L}$	$z_1 = -1 + 2\mathbf{i} + \mathbf{j}$
Q_2	$b = -1 + 2\mathbf{i} - \mathbf{j}$ $c = \mathbf{j} - 2\mathbf{k}$	$\underline{b} \in \mathbb{T}$	$z_1 = \mathbf{j}$ $z_2 = 1 - 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
Q_3	$b = 2 - \mathbf{i} + 2\mathbf{j}$ $c = -1 + 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$	$\underline{b} \in \mathbb{S}$	$z_1 = \frac{1}{2}(-5 + 5\mathbf{i} - 5\mathbf{j} - 3\mathbf{k})$ $z_2 = -2 + 9\mathbf{i} - 6\mathbf{j} - 7\mathbf{k}$ $z_3 = \frac{1}{2}(-3 - 3\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$ $z_4 = \frac{1}{2}(-1 - 3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ $z_5 = \mathbf{k}$ $z_6 = \frac{1}{6}(3 - \mathbf{i} - \mathbf{j} + 3\mathbf{k})$
Q_4	$b = -2 - 2\mathbf{i} - 2\mathbf{j}$ $c = 3 + 2\mathbf{i} + 2\mathbf{j}$	$\underline{b} \in \mathbb{L}$	no zeros

6 Final Remarks

Solving the quadratic equation (5) in the variable x is equivalent to solving the system (30) of four nonlinear equations in the variables x_0, x_1, x_2, x_3. When the number of solutions is finite (i.e., ≤ 6), one can use a computer algebra system to obtain the isolated zeros (eventually with finite precision), but this turns out to be a very demanding approach when there are infinite solutions.

Nothing prevents the approach followed in this paper from being used also to determine isolated zeros. In this situation, the solution to the problem may involve solving 6th or 4th degree (real) equations, whose computational effort would be analogous to that of solving the system (30).

The results of Sect. 4 provide a very simple and elegant way to construct monic quadratic polynomials in $\mathbb{H}_{\text{coq}}$ with prescribed zeros, which can be a valuable help when performing numerical experiments such as those usually carried out in the study of the dynamics of the quadratic map.

Ackowledgment. Research at CMAT was partially financed by Portuguese funds through FCT - Fundação para a Ciência e a Tecnologia, within the Projects UIDB/00013/2020 and UIDP/00013/2020.

References

1. Ata, E., Yayli, Y.: Split quaternions and semi-euclidean projective spaces. Chaos, Solitons Fractals **41**(4), 1910–1915 (2009)
2. Bekar, M., Yaylı, Y.: Involutions of complexified quaternions and split quaternions. Adv. Appl. Clifford Algebras **23**(2), 283–299 (2013)
3. Brody, D.C., Graefe, E.M.: On complexified mechanics and coquaternions. J. Phys. Math. Theor. **44**(7), 072001 (2011)

4. Cao, W.: Quadratic equation in split quaternions. Axioms **11**(5), p.188 (2022)
5. Cockle, J.: On systems of algebra involving more than one imaginary; and on equations of the fifth degree. Phil. Mag. **35**(3), 434–435 (1849)
6. Dennis, J.E., Traub, J.F., Weber, R.P.: The algebraic theory of matrix polynomials. SIAM J. Numer. Anal. **13**(6), 831–845 (1976)
7. Falcão, Maria Irene, Miranda, Fernando, Severino, Ricardo, Soares, Maria Joana: Mathematica tools for coquaternions. In: Gervasi, O., et al. (eds.) ICCSA 2021. LNCS, vol. 12952, pp. 449–464. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-86973-1_32
8. Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: Basins of attraction for a quadratic coquaternionic map. Chaos, Solitons Fractals **104**, 716–724 (2017)
9. Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: On the roots of coquaternions. Advances in Applied Clifford Algebras **28**(5) (2018)
10. Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: The number of zeros of unilateral polynomials over coquaternions revisited. Linear Multilinear Algebra **67**(6), 1231–1249 (2019)
11. Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: Iteration of quadratic maps on coquaternions. Int. J. Bifurcat. Chaos **27**(12), 1730039 (2017)
12. Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: Dynamics of the coquaternionic maps x^2+bx. Rend. Circ. Mat. Palermo, II. Ser. **72**, 959–975 (2023). https://doi.org/10.1007/s12215-021-00715-6
13. Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: Remarks on the zeros of quadratic coquaternionic polynomials. In: Computational Science and Its Applications – ICCSA 2022, vol. 13377 LNCS, pp. 466–477. Springer International Publishing (2022)
14. Gao, C., Chen, X., Shen, Y.G.: Quintessence and phantom emerging from the split-complex field and the split-quaternion field. Gen. Relativ. Gravit. **48**(11), 1–23 (2016)
15. Gogberashvili, M.: Split quaternions and particles in (2+1)-space. Eur. Phys. J. **74**(12) (2014)
16. Higham, N.J., Kim, H.M.: Numerical analysis of a quadratic matrix equation. IMA J. Numer. Anal. **20**(4), 499–519 (2000)
17. Jia, Z., Cheng, X., Zhao, M.: A new method for roots of monic quaternionic quadratic polynomial. Comput. Math. Appl. **58**(9), 1852–1858 (2009)
18. Jiang, T., Zhang, Z., Jiang, Z.: Algebraic techniques for Schrödinger equations in split quaternionic mechanics. Comput. Math. Appl. **75**(7), 2217–2222 (2018)
19. Kula, L., Yayli, Y.: Split quaternions and rotations in semi Euclidean space E_2^4. J. Korean Math. Soc. **44**(6), 1313–1327 (2007)
20. Marrani, A., Romano, L.: Orbits in nonsupersymmetric magic theories. Int. J. Mod. Phys. A **34**(32), 1950190 (2019)
21. Ni, Q.Y., Ding, J.K., Cheng, X.H., Jiao, Y.N.: 2×2 matrix representation forms and inner relationships of split quaternions. Advances in Applied Clifford Algebras **29**(2) (2019)
22. Niven, I.: Equations in quaternions. Amer. Math. Monthly **48**, 654–661 (1941)
23. Özdemir, M.: The roots of a split quaternion. Appl. Math. Lett. **22**(2), 258–263 (2009)
24. Özdemir, M., Ergin, A.: Some geometric applications of split quaternions. In: Proceedings 16th International. Conference Jangjeon Mathematics Society. 16, pp. 108–115 (2005)
25. Özdemir, M., Ergin, A.: Rotations with unit timelike quaternions in Minkowski 3-space. J. Geom. Phys. **56**(2), 322–336 (2006)

26. Scharler, D.F., Siegele, J., Schröcker, H.P.: Quadratic split quaternion polynomials: Factorization and geometry. Adv. Appl. Clifford Algebras **30**(1) (2020)
27. Serôdio, R., Beites, P.D., Vitória, J.: Intersection of a double cone and a line in the split-quaternions context. Adv. Appl. Clifford Algebras **27**(3), 2795–2803 (2017)
28. Simsek, H., Özdemir, M.: Rotations on a lightcone in Minkowski 3-space. Adv. Appl. Clifford Algebras **27**(3), 2841–2853 (2017)
29. Slusky, M.: Zeros of 2×2 matrix polynomials. Comm. Algebra **38**(11), 4212–4223 (2010)