Trajectories in Rutherford Dispersion According to Lagrangian Dynamics

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Abstract. This study delves into the dynamics of physical systems using the Lagrangian formalism within polar coordinates, starting with the Lagrange function, $L = T - U$, where T denotes the kinetic energy and U is the potential energy. The kinetic term is reformulated regarding the radial distance and angular velocity by adapting Lagrange's equation to polar coordinates. In contrast, the possible term is inversely proportional to the square of the radial distance. By implementing the Euler-Lagrange equations, the Lagrange function is differentiated concerning the radial coordinate and its time derivative, leading to a differential equation regarding r and ϕ. A substitution to $u = 1/r$ simplifies and solves this equation, producing a solution correlating angular positions with time. Integrating the initial conditions identifies the constants of integration, culminating in a comprehensive description of the motion in polar terms, illustrating the relationship between inverse radial distance, angular position, and time, and providing a detailed understanding of the dynamic governed by an inversely quadratic central force. This approach reveals the dynamics of complex systems without direct analysis of forces, underscoring the usefulness of the Lagrangian perspective in fields such as celestial mechanics, particle physics, and field theory.

Keywords: Lagrangian Formalism · Polar Coordinates · Euler-Lagrange Equations

1 Introduction

This research explores the dynamics of physical systems by applying Lagrangian formalism in polar coordinates to deepen the understanding of phenomena beyond the scope of classical Newtonian mechanics, such as Rutherford dispersion. The limitation of Newton's inverse square law in the face of subatomic and nuclear phenomena highlights the need for more general theoretical tools to describe the complex interactions in these systems. The study transforms the equations of motion through Lagrangian formalism, revealing crucial symmetries and conservation laws and expanding the understanding of the fundamental forces at play. This approach opens the possibility of approaching particle

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dynamics from a perspective that integrates classical and modern physics, thus providing a more holistic interpretation of physical processes at various scales. The objective is to provide a comprehensive analysis of motion in polar terms, using the Euler-Lagrange equations to derive a detailed description of the dynamics under an inversely quadratic central force, which has significant applications in fields such as celestial mechanics, the physics of particles, and field theory, and also facilitates the transition towards a Lagrangian understanding of quantum mechanics.

1.1 Rutherford's Dispersion Challenge: Beyond Newton's Law

Rutherford's scattering experimentation challenged Newtonian understanding, suggesting that forces operating at the subatomic level do not necessarily align with classical laws of motion. The approach of Goldstein et al. [9] clarifies that Newton's inverse square law, applicable in planetary motion and ballistics, found limitations in the face of nuclear dispersion, which better fit the quantitative descriptions provided by particle physics (See Fig. 1). This divergence pointed to an inherent complexity in atomic forces that could not be effectively addressed by classical mechanics alone. Furthermore, Landau and Lifshitz [14] expand this perspective by examining the implications of theoretical mechanics in microscopic systems, highlighting the insufficiency of Newton's law to explain the interaction between particles with an atomic nucleus, which was more adequately described by the Quantum theory.

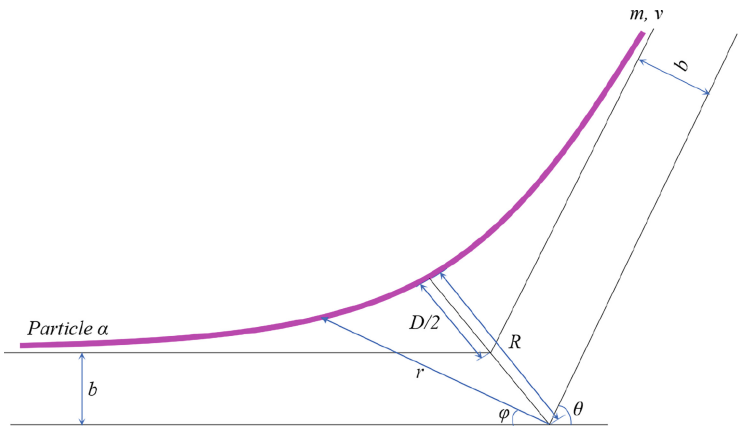


Fig. 1. Path of an Alpha particle scattered by a nucleus.

On the other hand, the formalization of a mathematical framework that overcomes these limits is manifested in the work of Arnold [2], who explains that Lagrangian and Hamiltonian mechanics provide the necessary tools to transcend the restrictions of Newtonian mechanics by using principles variations that are

congruent with the phenomena observed in Rutherford dispersion. Likewise, José and Saletan [13] show that classical dynamics, from a contemporary approach, can be complemented with more modern interpretations that allow integration between classical and contemporary physics, thus contributing to a more holistic understanding of the physical processes in different scales.

1.2 Fundamentals and Objectives of Lagrangian Analysis in Particle Scattering

Lagrangian analysis provides a robust theoretical framework for studying particle scattering by allowing the characterization of complex systems by simplifying the equations of motion. According to Hand and Finch [11], the Lagrangian formalization is especially useful in particle scattering since it transforms second-order differential equations into first-order equations, facilitating the identification of the conserved quantities of the system. Applying this formalism reveals the underlying symmetries and conservation laws essential for predicting the trajectory of particles after a collision. This approach expands the understanding of fundamental interactions and provides a methodology for calculating the scattering cross-section, a key goal for understanding the nature of forces between elementary particles.

Additionally, Greenwood [10] argues that the Lagrangian is a central element of variational principles that allow an elegant and versatile formulation that can be applied even when forces cannot be easily derived from a potential, as in specific dispersion interactions. On the other hand, Sussman and Wisdom [21] point out that Lagrangian mechanics not only simplifies calculations in systems with numerous particles but is also essential for the transition to quantum mechanics, providing a coherent connection between theories. Classic and modern. Consequently, Lagrangian analysis constitutes a critical foundation for advancing the study of particle interactions at a fundamental level.

1.3 Principles of Lagrangian Dynamics

The principles of Lagrangian dynamics are rooted in the variational principle, which stipulates that the actual motion of a system between two states is dictated by the stationarity of the action, an integral quantity defined as the time critical of the Lagrangian. Ván and Kovács [22] discuss how variational principles, applied to systems outside of thermodynamic equilibrium, can be extended to include non-equilibrium conditions, opening new horizons in understanding complex dynamical systems.

This approach aligns with the interpretation that equations of motion can be derived from a principle of most minor action. This simplifies the analysis of physical systems by providing a single scalar equation instead of multiple force vector equations [19].

Furthermore, the elegance of Lagrangian principles lies in their ability to directly incorporate the constraints of a physical system into the formulation through Lagrange multipliers, thus allowing a more direct and general treatment

of the constraints [15]. Rosenau [19] exposes how Lagrangian structures can be deployed even in systems of evolution equations that initially appear to have no variational structure. This demonstrates the versatility of the Lagrangian to describe a wide variety of physical phenomena, from classical mechanics to fluid dynamics and field theories.

The Lagrangian formulation provides a unified framework for mechanics that extends well beyond the conservative systems traditionally associated with classical mechanics, integrating the dissipative effects and driving forces that are fundamental in treating natural physical systems [20].

1.4 Historical Comparison: Newton's Law Versus Lagrange's Formalism

The historical comparison between Newton's law and Lagrange's formalism represents a fundamental transition in theoretical mechanics from the description of forces acting on particles to an energy-based approach. Newton's laws, formulated in the 17th century, constitute the basis of classical mechanics, providing a set of three equations that directly relate forces to the movement of a body. These equations are extremely powerful in predicting the motion of objects under the influence of conservative forces, and their application can be seen in contemporary studies of complex systems, such as flagellar synchronization in micro-swimmers [17].

In contrast, Lagrangian formalism, developed in the 18th century, offers a more generalized description of motion, not limiting itself to conservative systems. This formalism introduces the concept of Lagrangian, which is the difference between a system's kinetic and potential energy and the action, which must be extreme for the movement to be realizable [5].

Although the Lagrangian approach is traditionally considered more suitable for conservative systems, recent research has extended its application to non-conservative systems, providing methods for including dissipative forces within the Lagrangian formalism [4]. This progress allows using the Lagrangian formalism in a broader range of physical situations, including systems with friction or forces that do not derive from a potential, as in specific orbital dynamics and control problems.

Furthermore, significant interest has developed in the identification of nonlinear dynamical systems using sparse data, where the Lagrangian formalism provides a systematic approach to modeling the equations of motion, even when forces are not directly accessible or when they are subject to geometric constraints [18]. In conclusion, while Newtonian mechanics provides a direct and concrete description of force interactions, Lagrangian mechanics offers a more abstract but robust framework for analyzing the dynamics of a system under a broader range of conditions [16].

2 Methodology

2.1 Elaboration of the Lagrangian Model for Rutherford Scattering

To model Rutherford scattering via Lagrangian dynamics, we consider an alpha particle approaching a stationary atomic nucleus. In this approach, the Lagrangian function, $L = T - U$, is constructed from the kinetic energy T and potential energy U of the system. The kinetic energy in polar coordinates is:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

while the potential energy due to the electrostatic interaction is:

$$U = \frac{1}{4\pi\epsilon_0}\frac{zZe^2}{r}$$

where m is the mass of the alpha particle, z and Z are the charges of the particle and the nucleus, respectively, and ϵ_0 is the permittivity of the empty.

2.2 Differential Equations in Particle Dynamics

The equations of motion are derived by applying the Euler-Lagrange equations to the Lagrangian function. For the radial coordinate r, the equation of motion is obtained as:

The Euler-Lagrange equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

For our Lagrangian function, we have:

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\phi}^2 + \frac{zZe^2}{4\pi\epsilon_0 r^2}$$

Which leads to the following second order differential equation in r:

$$m\ddot{r} - mr\dot{\phi}^2 - \frac{zZe^2}{4\pi\epsilon_0 r^2} = 0$$

Or alternatively:

$$m\frac{d^2r}{dt^2} - mr\dot{\phi}^2 - \frac{zZe^2}{4\pi\epsilon_0 r^2} = 0$$

This is a second-order differential equation in r.

2.3 Analysis of Initial Conditions and Physical Variables Involved

The initial conditions of the alpha particle are considered, which define its position and speed at the beginning of dispersion, and the constants of movement are established, such as angular momentum:

$$M = mr^2\dot{\phi} = m \vee b = \text{constante}$$

where m is the mass of the alpha particle, r is the radial distance, $\dot{\phi}$ is the angular velocity, v is the linear velocity and b is the impact parameter.

2.4 Derivation of the Lagrangian Differential Equation

We begin the analysis with the inverse relationship between u and r, where $r = \frac{1}{u}$. This transformation facilitates the description of the trajectory in terms of the dependent variable u and the independent variable ϕ. By differentiating r with respect to ϕ, we obtain the following relations:

$$\frac{dr}{dt} = \frac{dr}{du}\frac{dr}{d\phi}\frac{d\phi}{dt} = -\frac{dr}{u^2}\frac{dr}{d\phi}\frac{M}{mr^2} = -\frac{du}{d\phi}\frac{M}{m}u^2 \tag{1}$$

$$\frac{dr}{dt} = -\frac{M}{m}\frac{du}{d\phi}$$

$$\frac{d^2r}{dt^2} = \frac{d}{d\phi}\left(\frac{dr}{dt}\right)\frac{d\phi}{dt} = \frac{d}{d\phi}\left(-\frac{M}{m}\frac{du}{d\phi}\right)\frac{M}{m}u^2$$

$$\frac{d^2r}{dt^2} = -\frac{M^2}{m^2}u^2\frac{d^2u}{d\phi^2} \tag{2}$$

By substituting these relationships into the differential equation of particle dynamics, we arrive at:

$$m\left(-\frac{M^2u^2}{m^2}\right) - \frac{M^2u^3}{m} - \frac{zZe^2u^2}{4\pi\varepsilon_0} = 0$$
$$-\frac{M^2}{m}\frac{d^2u}{d\phi^2} + u = \frac{zZe^2}{4\pi\epsilon_0}$$
$$\frac{d^2u}{d\phi^2} + u = -\frac{mzZe^2}{4\pi\varepsilon_0 M^2}$$

With the relation $M = mvb$, the equation simplifies to:

$$\frac{d^2u}{d\phi^2} + u = -\frac{mzZe^2}{4\pi\varepsilon_0(m^2v^2b^2)} = \frac{zZe^2}{4\pi\varepsilon_0\frac{mv^2}{2}2b^2}$$

Defining the constant D as:

$$D = \frac{zZe^2}{4\pi\varepsilon_0\frac{mv^2}{2}}$$

we can rewrite the differential equation in its final form:

$$\frac{d^2u}{d\phi^2} + u = -\frac{D}{b^2}$$

This equation is identical to the equation obtained through Lagrange dynamics, without resorting to Newton's second law.

2.5 Transformations and Solutions of the Equation of Motion

The general solution of the differential equation for u is expressed as a combination of trigonometric functions and the constant derived from the initial conditions:

$$u = A\cos\phi + B\sin\phi - \frac{D}{2b^2}$$

Imposing the boundary conditions, where $\phi \to 0$ when $r \to \infty$ and that the radial velocity $\frac{dr}{dt} \to -v$ when $r \to \infty$, we determine that:

$$\frac{1}{r} = 0 = A\cos\theta + B\sin\theta - \frac{D}{2b^2}$$

$$A = \frac{D}{2b^2}$$

$$\frac{dr}{dt} = -\frac{M}{m}\frac{du}{dt} = -v = -\frac{M}{m}(-A\cos\theta + B\sin\theta)$$

$$B = \frac{mv}{M} = \frac{mv}{mvb} = \frac{1}{b}$$

These constants allow us to write the complete solution for u as:

$$u = -\frac{D}{2b^2}\cos\phi + \frac{1}{b}\sin\phi - \frac{D}{2b^2}$$

2.6 Physical Implications of the Lagrangian Solution

Finally, the relationship between u and ϕ translates into a description of the trajectory of the alpha particle in terms of the distance r and the angle ϕ:

$$\frac{1}{r} = \frac{1}{b}\sin\phi + \frac{D}{2b^2}(\cos\phi - 1)$$

This expression provides a framework to analyze how the alpha particle disperses when interacting with the nucleus, allowing to predict the qualitative and quantitative behavior of the Rutherford scattering phenomenon.

3 Results

3.1 Comparison of Trajectories: Newtonian Approach vs. Lagrangian

The differential equation obtained through Lagrange dynamics,

$$\frac{d^2u}{d\phi^2} + u = -\frac{D}{2b^2} \tag{3}$$

where $u = \frac{1}{r}$, is compared with the analogous equation derived from the Newtonian approach. Direct comparison reveals that both formulations are consistent and produce identical equations for the trajectory of the alpha particle in the electrostatic field of the nucleus. This validates the equivalence of the Lagrangian and Newtonian approaches for this specific scattering problem.

The solution to the differential equation provides a function $u(\phi)$ that describes how the inverse distance of the alpha particle from the nucleus varies with the polar angle ϕ. The calculated trajectory is consistent with the expected and previously known results of Rutherford scattering, demonstrating that the Lagrangian method is a viable tool to study this phenomenon.

3.2 Physical Interpretation of the Lagrangian Solution

The Lagrangian solution of the differential equation, expressed as

$$\frac{1}{r} = \frac{1}{b}\sin\phi + \frac{D}{2b^2}(\cos\phi - 1) \tag{4}$$

provides a deep physical interpretation of Rutherford scattering. The angle dependence ϕ reflects how the alpha particle is deflected from its original trajectory as it passes near the nucleus. The constant D, which incorporates the charge of the alpha particle and the nucleus, as well as the initial kinetic energy of the alpha particle, is essential in determining the magnitude of the scattering.

The term $\sin(\phi)$ dominates when ϕ is small, indicating minimal deflection, consistent with the small-angle scattering observed experimentally. As ϕ increases, the term proportional to $\cos(\phi)$ becomes more significant, reflecting greater deflection of the particle, which is consistent with scattering at larger angles.

This interpretation of the solution not only confirms the validity of the Lagrangian model but also reinforces our understanding of the physical process underlying Rutherford scattering. The results obtained are consistent with experimental observations and demonstrate the usefulness of the Lagrangian formalism in the description of systems governed by central forces.

4 Discussion

4.1 Critical Analysis of the Applicability of Lagrange Formalism in Dispersion

The use of the Lagrangian formalism to describe Rutherford scattering has proven to be a robust theoretical tool. According to Arnold et al. [3], the underlying symplectic geometry provides a general framework for dynamical systems that is particularly suitable for studying conservation and symmetry properties. The application of this formalism to the specific problem of dispersion allows for elegant manipulation of the system variables and equations of motion. Adame-Carrillo et al. [1] highlight the importance of second-order approaches for k-presymplectic Lagrangian field theories, which could imply broader applications of Rutherford scattering in fields such as quantum gravity.

However, the applicability of Lagrangian formalism is not without criticism. Some studies, such as those by Frolov A. V. and Frolov V. P. [8], suggest that in systems with inequality constraints, conventional Lagrangian formulations may require modifications. Although Rutherford sparseness imposes no such restrictions, the possibility that Lagrange's formalism needs adaptations in more general contexts is a topic of continued research.

4.2 Advantages and Limitations of the Lagrangian Approach

The advantages of the Lagrangian approach are multiple. As Cortizo [6] highlights, Lagrange's equations offer a more general description than Newton's second law, especially useful in systems where conservative forces are predominant. Additionally, the Lagrangian formulation often simplifies the analysis by working with scalar quantities instead of vectors, which can simplify the calculation of the equations of motion.

However, there are limitations inherent to the Lagrangian formalism. Wagner and Guthrie [23] discuss how the Lagrangian formulation may be less intuitive than Newtonian methods for those with little experience in the subject. On the other hand, in systems where non-conservative forces play a significant role, or where non-holonomic constraints are prominent, the Lagrangian formulation may be less effective or require non-trivial adaptations, as discussed in the works of Jarab'ah [12] and Fan et al. [7].

Thus, while the Lagrangian approach offers a powerful alternative to Newtonian methods, it is important to recognize its limitations and apply it carefully, especially in complex systems or under conditions that challenge its fundamental assumptions.

5 Conclusions and Recommendations

5.1 Synthesis of Findings and Contributions to the Field of Particle Physics

This study has demonstrated the validity of the Lagrangian formalism in the analysis of Rutherford dispersion. The differential equation obtained and its

corresponding solution are consistent with the results that would be derived from Newtonian mechanics, confirming that Lagrangian mechanics is a powerful tool to address dispersion problems in particle physics. The congruence between the Lagrangian and Newtonian approaches strengthens the understanding of Rutherford scattering and offers a different perspective for addressing similar problems in physics. Furthermore, this work highlights the importance of symmetries and conservation laws in particle physics, and how the Lagrangian formalism naturally incorporates these features into its mathematical structure. These findings contribute to the field by providing an alternative framework that may be particularly useful in systems where forces are conservative and symmetries play a central role.

5.2 Recommendations for Future Research and Practical Applications

In light of the results obtained, it is recommended that future research explore the applicability of the Lagrangian formalism to more complex problems in particle physics, such as those involving non-central interactions or quantum mechanical effects. The adaptation of the formalism to systems with non-conservative forces or non-holonomic constraints also deserves special attention, which could open new paths in the simulation of particle systems and in the understanding of complex phenomena. Regarding practical applications, the use of the Lagrangian formulation is suggested in the design of dispersion experiments and in the development of simulation software that can benefit from its simplified structure and its focus on scalar quantities. Additionally, formalism can be a valuable resource in physics education, providing students with a deeper understanding of the underlying principles of classical and modern mechanics.

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