The Weighted Vector Finite Element Method for Vector Wave Equation with Singularity

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Abstract. Boundary value problems for wave vector equations with singularity are used in mathematical models of electromagnetism problems in domains with a boundary containing reentrant corners. The solutions to such problems do not belong to the Sobolev space W_2^1 and the approximate solution by the classical finite element method has a low rate of convergence to the exact solution.

We define an R_ν-generalized solution to the boundary value problem for a wave vector equation with corner singularity in a set of the weighted Sobolev-Monk space. A weighted vector finite element method (WVFEM) is constructed to find an approximate R_ν-generalized problem. The WVFEM basis functions contain weighting functions to a degree depending on the sizes of the reentrant corners at the domain boundary. This allows us to weaken the influence of the singularity on the accuracy of finding the solution. The method has a convergence rate of $O(h)$ in the norm of the weighted space $L_{2,\alpha}(\Omega)^2$. Numerical calculations of model problems confirmed the theoretical estimate of the convergence rate.

Keywords: Vector wave equations with a singularity · R_ν-generalized solution · Weighted vector finite element method

1 Introduction

Over the past few decades, various numerical methods have been developed for solving boundary value problems of mathematical physics. One such method is the vector finite element method which was first proposed by J.C. Nédélec [8,9] and was further developed in works [2,4–7,26,27].

Vector FEM is used to solve various problems of electrodynamics, elasticity theory and other problems of mathematical physics (see, for example, [10,11]).

The main difference between the vector finite element method and the nodal method is that the degrees of freedom are associated with the sides of the finite elements, and not with the nodes of the finite element mesh. This makes it possible to accurately fulfill the solinoidal condition of the solution and to easily

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take into account the boundary conditions and behavior of the solution at the interface between the media. However, the convergence rate of the vector finite element method when solving a problem in a domain containing a reentrant corner at the boundary is reduced (see, for example, [3,25]). This circumstance requires the construction of a special numerical method for solving problems with corner singularity.

We proposed to define the R_ν-generalized solution for boundary value problems with the singularity [12]. The existence and uniqueness, coercive and differential properties of the R_ν-generalized solution for various problems with singularity in weighted Sobolev spaces and sets were studied in [15,18,21,24]. The introduction of the R_ν-generalized solution made it possible to create numerical methods in which the singularity does not affect the accuracy of finding the solution. We have created a weighted nodal finite element method for elliptic problems, problems in the theory of elasticity and hydrodynamics [13,14,19,20,22]. The proposed approach does not lead to ill-conditioned systems of linear equations and makes it possible significantly reduce the computation time to achieve a given accuracy [17,23].

In this paper, we introduce a special weight set from the weighted Sobolev-Monk space $H(\mathrm{rot}, \Omega)$. The R_ν-generalized solution of the boundary value problem for vector wave equation with corner singularity is defined in this set. The weighted vector finite element method is constructed to find an approximate R_ν-generalized solution to the problem under consideration. A special feature of this method is that a weight function is introduced into the definition of basis functions the degree of which depends on the asymptotic behavior of the singular component of the solution to the original boundary value problem. This makes it possible to better approximate solution near the singularity point and increase the rate of convergence of the approximate solution to the exact one in the norm of the weighted space $L_{2,\alpha}(\Omega)^2$. Numerical analysis of a series of model problems showed that the convergence rate of the constructed method is one and a half times higher than in the singular addition method and the regularization method.

2 Notations and Problem Setting

Let $\Omega = (-1,-1) \times (-1,-1) \setminus [-1,0] \times [-1,0]$ – L-shaped domain in $\mathbb{R}^2$, $\partial\Omega$ be a boundary of domain Ω and the point $O(0,0)$ be the vertex of the reentrant corner $3\pi/2$.

We denote the vector-function by $\mathbf{u}(x) = (u_1(x), u_2(x))$ with $x = (x_1, x_2)$, $\mathbf{n}$ and $\boldsymbol{\tau}$ are the unit outward normal and the tangent vector to point of the boundary $\partial\Omega$, $\{\mathbf{i}, \mathbf{j}\}$ is an orthonormal basis in $\mathbb{R}^2$.

We introduce the spaces $L_2(\Omega)$, $L_2(\Omega)^2$ and $H^1(\Omega)$, $H^1(\Omega)^2$ with norms

$$\|u\|_{L_2(\Omega)} = \sqrt{\int_\Omega u^2 dx}, \qquad \|\mathbf{u}\|_{L_2(\Omega)^2} = \sqrt{\|u_1\|_{L_2(\Omega)} + \|u_2\|_{L_2(\Omega)}},$$

$$\|u\|_{H^1(\Omega)} = \sqrt{\sum_{\|i\|\le 1} \int_\Omega |D^i u|^2 dx}, \qquad \|\mathbf{u}\|_{H^1(\Omega)^2} = \sqrt{\|u_1\|_{H^1(\Omega)} + \|u_2\|_{H^1(\Omega)}},$$

where $D^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \partial x_2^{i_2}}$; $|i| = i_1 + i_2$, i_1, i_2 are nonnegative integer numbers.

For functions $u(x)$ and vector-functions $\mathbf{u}(x)$ we define differential operators:

$$\operatorname{div} \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad \mathbf{rot}\ u = \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1} \right), \qquad \operatorname{rot} \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

Let $H(\operatorname{rot}\Omega) = \{\mathbf{u} \in L_2(\Omega)^2, \operatorname{rot} \mathbf{u} \in L_2(\Omega)\}$ be the space of functions with norm

$$\|\mathbf{u}\|^2_{H(\operatorname{rot},\Omega)} = \int_\Omega \left(u_1^2 + u_2^2 + (\operatorname{rot} \mathbf{u})^2 \right) dx$$

and $\mathring{H}(\operatorname{rot}, \Omega) = \{\mathbf{u} \in L_2(\Omega)^2, \operatorname{rot} \mathbf{u} \in H(\operatorname{rot}, \Omega), \mathbf{u} \cdot \boldsymbol{\tau} = 0$ almost everywhere on $\partial\Omega\}$.

Denote by $\Omega' = \left\{ x \in \overline{\Omega} : \left(x_1^2 + x_2^2\right)^{\frac{1}{2}} \le \delta \ll 1 \right\}$ the δ-neighborhood of the point O in Ω. Let us introduce a weight function $\rho(x)$ which satisfies the following conditions: $\rho(x) = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}$ for $x \in \overline{\Omega'}$, $\rho(x) = \delta$ for $x \in \overline{\Omega} \setminus \overline{\Omega'}$.

We define the weighted spaces $L_{2,\alpha}(\Omega)$, $L_{2,\alpha}(\Omega)^2$ and $H_\alpha(\operatorname{rot}, \Omega)$ of functions with norms:

$$\|u\|_{L_{2,\alpha}(\Omega)} = \sqrt{\int_\Omega \rho^{2\alpha} u^2 dx}, \qquad \|\mathbf{u}\|_{L_{2,\alpha}(\Omega)^2} = \sqrt{\|u_1\|_{L_{2,\alpha}(\Omega)} + \|u_2\|_{L_{2,\alpha}(\Omega)}},$$

$$\|\mathbf{u}\|^2_{H_\alpha(\operatorname{rot},\Omega)} = \|\mathbf{u}\|^2_{L_{2,\alpha}(\Omega)} + \|\operatorname{rot} \mathbf{u}\|^2_{L_{2,\alpha}(\Omega)} = \int_\Omega \rho^{2\alpha} \left(u_1^2 + u_2^2 + (\operatorname{rot} \mathbf{u})^2 \right) dx. \quad (1)$$

Let $\mathring{H}_\alpha(\operatorname{rot}, \Omega) = \{\mathbf{u} \in H_\alpha(\operatorname{rot}, \Omega), \mathbf{u} \cdot \boldsymbol{\tau} = 0$ almost everywhere on $\partial\Omega\}$. Here α is real number.

By $H_\alpha(\operatorname{rot}, \Omega, \delta)$ we denote a set of functions satisfying the following conditions:

(a) $|u_i(x)| \le C_1 \rho^{-\alpha}(x)$ $(i = 1, 2)$, $|\operatorname{rot} \mathbf{u}| \le C_1 \rho^{-\alpha-1}(x)$, $x \in \Omega'$, $C_1 > 0$;
(b) $\|\mathbf{u}\|_{L_{2,\alpha}(\Omega \setminus \Omega')^2} \ge C_2 > 0$

with norm (1).

We define the weighted sets
$\mathring{H}_\alpha(\operatorname{rot}, \Omega, \delta) = \{\mathbf{u} \in H_\alpha(\operatorname{rot}, \Omega, \delta), \mathbf{u} \cdot \boldsymbol{\tau} = 0$ almost everywhere on $\partial\Omega\}$,
$V(\Omega, \delta) = \{\mathbf{u} \in \mathring{H}_\alpha(\operatorname{rot}, \Omega, \delta), \operatorname{div} \mathbf{u} = 0\}$.

Let us consider the boundary value problem for vector wave equation

$$\operatorname{rot} \operatorname{rot} \mathbf{E} - k^2 \mathbf{E} = \mathbf{f}, \qquad \text{in } \Omega, \quad (2)$$

$$\operatorname{div} \mathbf{E} = 0, \qquad \text{in } \Omega, \quad (3)$$

$$\mathbf{E} \cdot \boldsymbol{\tau} = 0, \qquad \text{on } \partial\Omega. \quad (4)$$

Comment. The solution of the problem (2)–(4) belongs to the space $H(\mathrm{rot}, \Omega)$ (see, for example, [1]) and does not belong to $H^1(\Omega)^2$. A solution from this space does not allow to construct a numerical method of high degree of accuracy on the basis of the principle of consistent estimates (see, for example, [3,25]). We will define an R_ν-generalized solution in the weighted set V to construct a weighted vector finite element method with a high rate of convergence of the approximate solution to the exact one.

Denote by

$$a(\mathbf{u}, \mathbf{v}) = (\mathrm{rot}\ \mathbf{u}, \mathrm{rot}(\rho^{2\nu}\mathbf{v})) - k^2(\mathbf{u}, \rho^{2\nu}\mathbf{v}) =$$

$$\int_\Omega \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\left[\rho^{2\nu}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) - \left(\frac{\partial \rho^{2\nu}}{\partial x_2}v_1 - \frac{\partial \rho^{2\nu}}{\partial x_1}v_2\right)\right] dx - k^2 \int_\Omega \rho^{2\nu}\,(u_1 v_1 + u_2 v_2)\, dx,$$

$$b(\mathbf{v}) = (\mathbf{f}, \rho^{2\nu}\mathbf{v}) = \int_\Omega \rho^{2\nu}\,(f_1 v_1 + f_2 v_2)\, dx$$

the bilinear and linear forms, respectively.

Definition 1. A function $\mathbf{E}_{\boldsymbol{\nu}}$ in the weighted set $V(\Omega, \delta)$ is called a R_ν-generalized solution of the boundary value problem (2)–(4) if for any $\mathbf{v} \in V(\Omega, \delta)$ the identity

$$a(\mathbf{E}_{\boldsymbol{\nu}}, \mathbf{v}) = b(\mathbf{v}) \tag{5}$$

holds.

3 Construction of the Weighted Vector Finite Element Method

The scheme of the weighted vector finite element method for determination of an R_ν-generalized solution was constructed in [16]. Here we briefly describe construction of the WVFEM for the boundary value problem of vector wave Eq. (2)–(4).

We perform a quasi-uniform triangulation of the domain Ω. Let mesh step $h = \frac{2}{N}$, N is a positive even integer. We divide Ω by straight lines $x_1 = -1 + jh$, $x_2 = -1 + lh$, $j, l = 0, \ldots, N$, into a set of finite elements $\{K\} = \{K_i\}_{i=1}^{N_h}$, $N_h = \frac{3N^2}{4}$. Let S_i be the side of the finite element. Let S be a subset of $\{K\}$:

$$S = \{S_i, i = 1, \ldots, N_S, S_i \not\subset \partial\Omega, N_S = \frac{3N^2}{2} - 2N\}.$$

Let us introduce notation for a separate finite element K_i (Fig. 1): $S_j^{K_i}$ $\left(j = \overline{1,4}\right)$ are its sides K_i, $M_j^{K_i}\left(m_{1j}^{K_i}, m_{2j}^{K_i}\right)$ are the centers of $S_j^{K_i}$, $O^{K_i}\left(o_1^{K_i}, o_2^{K_i}\right)$ is its center K_i. The rule for local numbering of sides K_i is shown in Fig. 1.

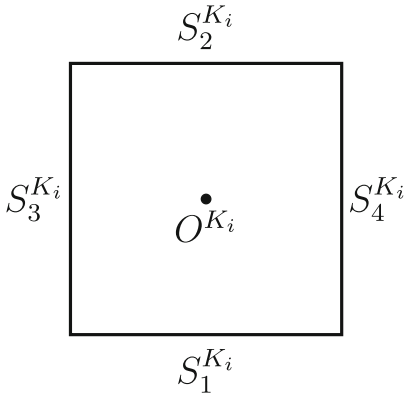


Fig. 1. [16] The rule for the local numeration $S_j^{K_i}$ $\left(j = \overline{1,4}\right)$ of the finite element K_i

For every finite element K_i we define form functions associated with its sides $S_j^{K_i}$ $\left(j = \overline{1,4}\right)$:

$$\psi_1^{K_i} = \frac{1}{h}\rho^{-\gamma}\left(m_{11}^{K_i}, x_2\right)\left(\frac{h}{2} + o_2^{K_i} - x_2\right)\mathbf{i}, \quad \psi_2^{K_i} = \frac{1}{h}\rho^{-\gamma}\left(m_{12}^{K_i}, x_2\right)\left(\frac{h}{2} - o_2^{K_i} + x_2\right)\mathbf{i},$$

$$\psi_3^{K_i} = \frac{1}{h}\rho^{-\gamma}\left(x_1, m_{23}^{K_i}\right)\left(\frac{h}{2} + o_1^{K_i} - x_1\right)\mathbf{j}, \quad \psi_4^{K_i} = \frac{1}{h}\rho^{-\gamma}\left(x_1, m_{24}^{K_i}\right)\left(\frac{h}{2} - o_1^{K_i} + x_1\right)\mathbf{j}.$$

Here γ is a real number.

Remark 1. The weight function is introduced into the form functions in such a way as to preserve the property of their solenoidality.

Now we define the basis functions ψ_i, $i = \overline{1, N_S}$. Let S_i be common side for elements K_m and K_n and $\psi_j^{K_m}$,$\psi_l^{K_n}$ be the form functions associated with S_i. For side S_i we assign the basis function

$$\psi_i = \begin{cases} \psi_j^{K_m}, & x \in K_m, \\ \psi_l^{K_n}, & x \in K_n, \\ 0, & x \in \overline{\Omega} \setminus (K_m \cup K_n). \end{cases}$$

Let us denote by $\tilde{V}^h$ the linear span $\{\psi_i\}_{i=1}^{N_S}$. Obviously, $\tilde{V}^h \in \mathring{H}_\nu(\mathrm{rot}, \Omega)$. By virtue of the conditions of basis functions, we have $\mathrm{div}\, \mathbf{v}_h = 0$ for any $\mathbf{v}_h \in \tilde{V}^h$. We define the weighted sets $V_\nu^h(\Omega, \delta) = \tilde{V}^h \cap H_\nu(\mathrm{rot}, \Omega, \delta)$.

Definition 2. A function $\mathbf{E}_\nu^h \in V_\nu^h(\Omega, \delta)$ we call an approximate R_ν-generalized solution of problem (2)–(4) by weighted vector finite element method if for any $\mathbf{v}_h \in V_\nu^h(\Omega, \delta)$ the following integral identity

$$a\left(\mathbf{E}_{\nu}^{h}, \mathbf{v}_{h}\right) = b\left(\mathbf{v}_{h}\right)$$

holds.

An approximate solution $\mathbf{E}_{\nu}^{h}$ will be found in the form

$$\mathbf{E}_{\nu}^{h} = \sum_{i=1}^{N_S} d_i \psi_i,$$

where $d_i = \rho^{\nu}\left(M_i\right) q_i$, $q_i = \text{const}$.

The coefficients d_i are defined from the system of equations

$$a\left(\mathbf{E}_{\nu}^{h}, \psi_i\right) = b\left(\psi_i\right), \quad i = \overline{1, N_S}.$$

4 Numerical Results

We have carried out a set of numerical tests for boundary value problem (2)–(4) using a weighted vector FEM. The first problem contains only a singular component in the solution (problem A), and the second problem contains a singular component and a regular function from the space $H^2\left(\Omega\right)$ (problem B). Calculations were carried out on meshes with different numbers of partitions $N = 16, 32, 64, 128$. We have established the optimal parameters ν, δ, γ for the calculations (see [23]). For the found approximate R_ν-generalized solutions $\mathbf{E}_{\nu}^{h}$ we calculated the value of the relative error in the norm of the space $L_{2,\alpha}\left(\Omega\right)^2$: $\eta = \frac{\|\mathbf{E}_{\nu} - \mathbf{E}_{\nu}^{h}\|_{L_{2,\alpha}(\Omega)^2}}{\|\mathbf{E}_{\nu}\|_{L_{2,\alpha}(\Omega)^2}}$. The values of η for various numbers of partitions N in Tables 1 and 3 for problems (A) and (B) are presented. In addition, these tables show the ratios λ between error norms obtained on meshes with number of segments N increasing twice. Convergence rate graphs for problems (A) and (B) are presented in Figs. 2 and 4. At each point M_i the absolute error $\delta_{ij} = |E_{i\nu}(M_i) - E_{i\nu}^{h}(M_i)|$, $i = 1, 2$ was calculated. We also determined the number and coordinates of points M_i with an absolute error δ_{ij} less than the specified limit error $\overline{\Delta} = 10^{-3}$. The results of numerical experiments are shown in Tables 2, 4 and Figs. 3, 5.

Model Problem (A). We introduce an auxiliary function

$$\varphi = x_1 x_2 \sin\left(\pi x_1\right) \cos\left(0.5\pi x_1\right) \sin\left(\pi x_2\right) \cos\left(0.5\pi x_2\right) \left(\sqrt{x_1^2 + x_2^2}\right)^{-3.3}.$$

We define the solution to problem (A) as $\mathbf{E} = \mathbf{rot}\ \varphi$ and calculate the right side of Eq. (2) for two values of the coefficient $k_1 = 300$ and $k_2 = 220$ (frequency 14.314 and 10.497 GHz, respectively).

Table 1. Dependence of the relative errors η_ν for problem (A) from number of segments N. The parameters of weighted vector FEM are $\nu = 2.0$, $\delta = 0.063$ $(k = k_1)$; $\nu = 1.9$, $\delta = 0.055$ $(k = k_2)$

N	16	λ	32	λ	64	λ	128
$\eta, k = k_1$	0.537303	1.84	0.291715	2.44	0.119754	2.24	0.053544
$\eta, k = k_2$	0.543997	1.77	0.307029	2.29	0.134271	2.08	0.064547

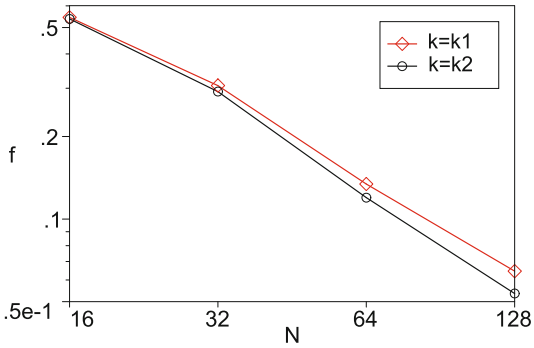


Fig. 2. Convergence rate of the relative error η_ν of the approximate R_ν-generalized solution of the problem (A) on the number of segments N.

Table 2. The percentage of points from the total number of points M_i in which the absolute error δ_{ij} is less than the specified limit error $\overline{\Delta} = 10^{-3}$

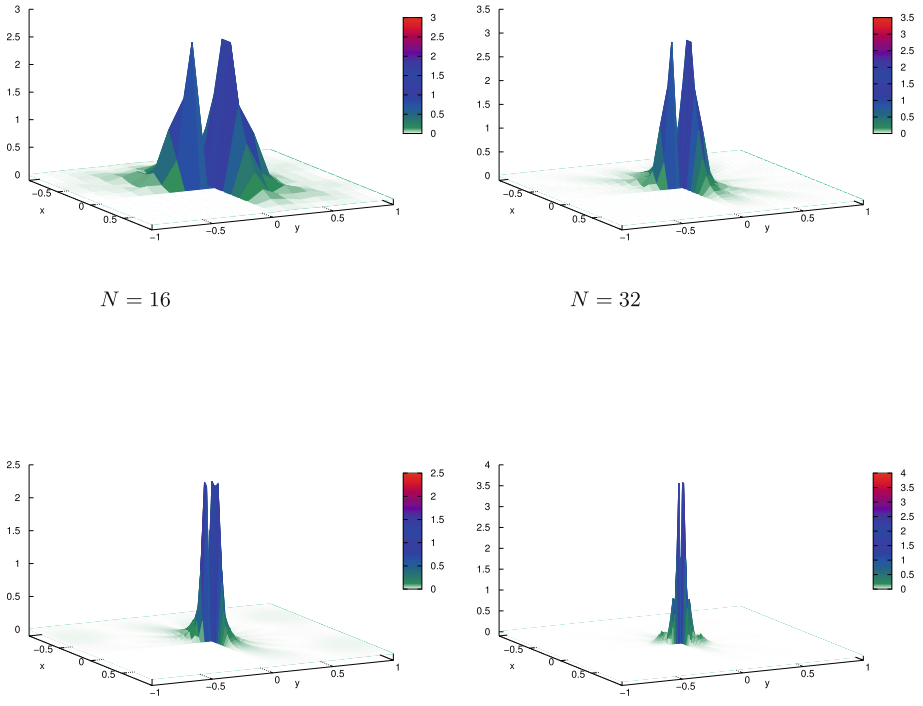
N	16	32	64	128
$\eta, k = k_1$	4.55%	13.59%	42.05%	73.56%
$\eta, k = k_2$	4.55%	13.18%	42.05%	73.58%

Remark 2. For the function φ the exponent was chosen equal to -3.3 so that the asymptotic of the solution $\mathbf{E}$ in the vicinity of the singularity point would be close to the theoretical $\rho^{-1/3}$.

Model Problem (B). For model problem (B) we introduce an auxiliary function

$$\varphi = x_1 x_2 \sin(\pi x_1) \cos(0.5\pi x_1) \sin(\pi x_2) \cos(0.5\pi x_2) \left(\left(\sqrt{x_1^2 + x_2^2} \right)^{-3.3} + 10 \right).$$

We present results for model problem (B).



$N = 16$ $N = 32$

$N = 64$ $N = 128$

Fig. 3. Graph of the absolute error δ_{1j} for the component $E_{1\nu}^h$ of the approximate R_ν-generalized solution of the problem (A) using the weighted vector finite element method on meshes with the number of partition segments N.

Table 3. Dependence of the relative errors η_ν for the problem (B) from number of segments N. The parameters of weighted vector FEM are $\nu = 2.0$, $\delta = 0.063$ $(k = k_1)$; $\nu = 1.9$, $\delta = 0.055$ $(k = k_2)$

N	16	λ	32	λ	64	λ	128
$\eta, k = k_1$	0.310778	1.87	0.166349	2.41	0.069088	2.18	0.031701
$\eta, k = k_2$	0.308519	1.92	0.160973	2.41	0.066790	2.17	0.030406

Table 4. The percentage of points from the total number of points M_i in which the absolute error δ_{ij} is less than the specified limit error $\overline{\Delta} = 10^{-3}$

N	16	32	64	128
$\eta, k = k_1$	4.55%	12.17%	36.85%	69.03%
$\eta, k = k_2$	4.55%	12.78%	36.05%	68.18%

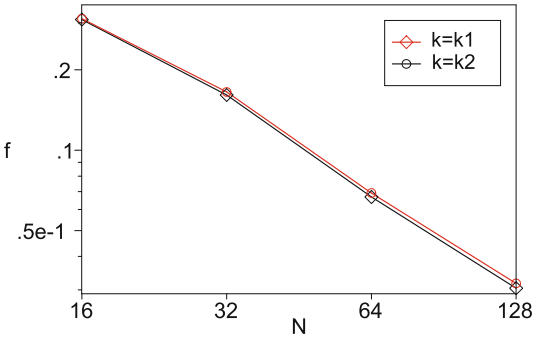


Fig. 4. Convergence rate of the relative error η_ν of the approximate R_ν-generalized solution of the problem (B) on the number of segments N.

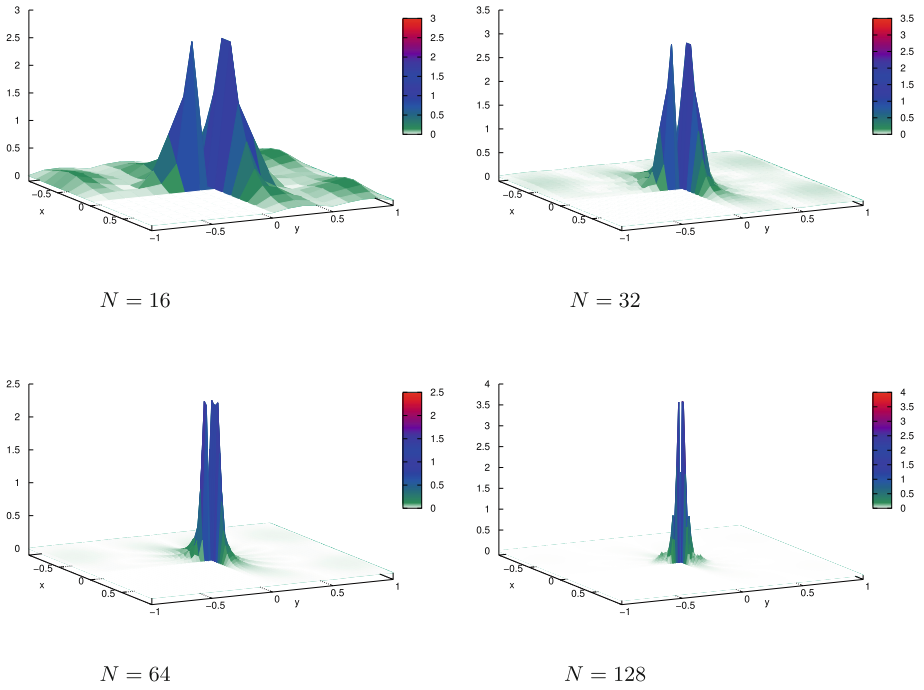


Fig. 5. Graph of the absolute error δ_{1j} for the $E^h_{1\nu}$ of the approximate R_ν-generalized solution of the problem (B) using the weighted vector finite element method on meshes with the number of partition segments N.

5 Conclusions

In this paper, a weighted vector finite element method for solving vector wave boundary value problems with singularity is constructed. We proposed to define the solution to the problem as an R_ν-generalized solution in a special weighted set $V(\Omega, \delta)$. The basis functions of the constructed method include a weight function to a degree depending on the asymptotic behavior of the solution near the singularity point.

Numerical experiments were carried out for model problems of two types: with a solution containing only a singular component, and with a solution containing both singular and regular components. The results of a series of calculations showed that when the mesh is refined, the approximate R_ν-generalized solution converges to the exact solution with a rate $O(h)$ in the $L_{2,\alpha}(\Omega)^2$ norm. This convergence rate is more than one and a half times higher in order of degree h than in the singular complement method and the regularization method. Also noteworthy is the algorithmic simplicity of WVFEM and the naturalness of the solution definition.

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