

Computational Procedures for Improving Extreme Value Estimation in Time Series Modelling

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Abstract. In the last decades, some work has been developed in parameter estimation of extreme values jointly with time series analysis. Those results show relevant asymptotic properties. However, for finite samples, limiting results provide approximations that can be poor. Some challenges have been developed by combining Extreme Value Theory and time series modelling to obtain more reliable extreme value parameter estimates. In classical time series modelling a key issue is to determine how many parameters must be included in the model. Special care must be given to extreme events in the series that need specific statistical procedures based on the behaviour of extremes. Resampling procedures such as the jackknife and the bootstrap have been used to improve parameters estimation in Extreme Value Theory combined with time series modelling. New approaches, based on bootstrap procedures are shown and are illustrated with a real data set using the R software.

Keywords: Extreme value theory · Parameter estimation · Resampling procedures · Time series

1 Introduction and a Few Details on EVT

Extreme Value Theory (EVT) aims to study and to predict the occurrence of extreme or even rare events, outside of the range of available data. These events are part of the real world but environmental extreme or rare events may have

This work is funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00006/2020 (CEAUL), UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

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O. Gervasi et al. (Eds.): ICCSA 2023 Workshops, LNCS 14105, pp. 84–96, 2023.
https://doi.org/10.1007/978-3-031-37108-0_6

a massive impact on everyday life and may have catastrophic consequences for human activities. Most environmental datasets have a time-dependent variation and short-term clustering are typical phenomena for extreme value data, and it is crucial that both are properly accounted for when making inferences.

Let us assume that we have a sample $(X_1, \ldots, X_n)$ of independent and identically distributed (iid) or possibly stationary, weakly dependent random variables (r.v.'s) from an unknown cumulative distribution function (cdf) F. The interest is focused on the distribution of the maxima, $M_n := \max(X_1, ..., X_n)$, for which we have

$$P\left(M_n \leq x\right) = P\left(X_1 \leq x\right) \ldots P\left(X_n \leq x\right) = F^n(x). \tag{1}$$

As n goes to ∞, the distribution F^n in (1) has a trivial limit: 0, if $F(x) < 1$ and 1, if $F(x) = 1$. So the idea for M_n was the same of *central limit theorem*: first, subtract a $n-$dependent constant, then rescale by a $n-$dependent factor. Assuming that there exist two sequences of real constants, $\{a_n\} \in R^+$ and $\{b_n\} \in R$ such that the maximum, linearly normalized, has a non-degenerate limiting distribution function, G,

$$\lim_{n\to\infty} P\left((M_n - b_n)/a_n \leq x\right) = G(x).$$

First results on the G distribution are due to [9,10,13,24]. But were [11] and [14] who gave conditions for the existence of those sequences $\{a_n\} \in R^+$ and $\{b_n\} \in R$ such that, when $n \to \infty$ and $\forall x \in R$,

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = \mathrm{EV}_\xi(\mathrm{x}), \tag{2}$$

where EV_ξ is a nondegenerate distribution function, denoted as the Extreme Value cdf, and given by

$$\mathrm{EV}_\xi(\mathrm{x}) = \begin{cases} \exp[-(1+\xi x)^{-1/\xi}], & 1+\xi x > 0 \text{ if } \xi \neq 0 \\ \exp[-\exp(-x)], & x \in R \qquad \text{if } \xi = 0. \end{cases} \tag{3}$$

When the above limit holds we say that F is in the domain of attraction (for maxima) of EV_ξ and write $F \in \mathcal{D}_\mathcal{M}(\mathrm{EV}_\xi)$.

The cdf in (3) can also incorporate location (λ) and scale ($\delta > 0$) parameters, and in this case, the EV_ξ cdf is given by,

$$\mathrm{EV}_\xi(x; \lambda, \delta) \equiv \mathrm{EV}_\xi((x-\lambda)/\delta). \tag{4}$$

2 Some Parameters of Interest in EVT

There is a large variety of parameters of extreme events, but the estimation of the *extreme value index* (EVI), the parameter denoted by ξ, in (3), is of primordial importance by itself and because it is the basis for the estimation of all other parameters of extreme events.

Among the most relevant parameters of extreme events, and assuming that we are interested in large values, i.e. in the right tail of the underlying model F, we can mention a few parameters such as the *probability of exceedance*, the *return period* of a high level and the *right endpoint* of the underlying model F. In this paper, we consider the problem of estimating the *high quantile* of probability $1-p$ (p small), χ_{1-p}, defined by

$$\chi_{1-p} := \inf\{x : F(x) \geq 1-p\},$$
$$\chi_{1-p} := \lambda - \frac{\delta}{\xi}\left[1 - \{-\log(1-p)\}^{-\xi}\right], \quad \xi \neq 0.$$

Whenever independence is no longer valid, some important dependent sequences have been studied and the limit distributions of their order statistics under some dependence structures are known. Stationary sequences are examples of such sequences and are realistic for many real problems.

Suppose we have n observations from a stationary process $\{Y_n\}_{n\geq 1}$ with marginal distribution function F. For large n and a high level u_n, we have

$$F_n(u_n) = P\left(\max(Y_1, ..., Y_n) \leq u_n\right) \approx F^{n\theta}(u_n),$$

where $\theta \in [0,1]$ is a constant for the process and is known as the *extremal index*. This concept, even appearing in papers from [23,25,26], was only well characterized by [21]. It is still appropriate to model the distribution of the maximum, $\max(Y_1, ..., Y_n)$, using EV_ξ cdf, in (3), but location and scale parameters are different from those that would have been obtained in the independent setup. In this article, the parameter θ will not be considered.

EVT has been developed under two frameworks. The first one is the parametric framework that considers a class of models associated with the limiting behaviour of the maxima, given in (2). The main assumption behind the parametric approach is that estimators are calculated considering the data following approximately an exact EV probability distribution function defined by a number of parameters.

The other framework, the semi-parametric one, the only assumption made is that the limit in (2) holds, i.e., that the underlying distribution verifies the extreme value condition. In this framework, we do not need to fit a specific parametric model based on scale, shape and location parameters. Estimates are now usually based on the largest k order statistics in the sample, assuming only that the model F underlying the data is in $\mathcal{D}_{\mathcal{M}}(\mathrm{EV}_\xi)$.

The parameter ξ is estimated, on the basis of the k top statistics, with k intermediate, i.e. such that $k = k_n \to \infty$ and $k/n \to 0$, as $n \to \infty$. However, most estimators show a strong dependence on that value k. They usually present a small bias and a high variance for small values of k; bias increases and variance decreases when k increases. To overcome this difficulty, Jackknife and Bootstrap procedures are known as giving more stable estimates around the target value.

2.1 Some Semi-parametric Estimators of ξ

Suppose we have a sample $(X_1, X_2, \ldots, X_n)$ of iid random variables with a heavy regularly varying right tail.

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the associated non-decreasing order statistics from the sample of size n. The most popular semi-parametric EVI-estimator is the Hill estimator, H, [16]. This estimator can be defined as the average of the log-excesses, $V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}$, $1 \leq i \leq k < n$, above the high threshold $X_{n-k:n} > 0$,

$$\widehat{\xi}_n^H(k) \equiv H(k) := \frac{1}{k} \sum_{i=1}^{k} V_{ik}, \quad 1 \leq k < n. \tag{5}$$

There are several alternatives to the Hill estimator that are less sensitive to the choice of the level k. We shall consider the simplest class of reduced-bias EVI-estimators, the Corrected Hill (CH) estimator, [2], defined by

$$\hat{\xi}_n^{CH}(k) \equiv \hat{\xi}_{\hat{\beta},\hat{\rho}}^{CH}(k) \equiv \mathrm{CH}(k) := \mathrm{H}(k) \left(1 - \frac{\hat{\beta}(n/k)^{\hat{\rho}}}{1-\hat{\rho}}\right), \quad 1 \leq k < n, \tag{6}$$

with $(\hat{\beta}, \hat{\rho})$ an adequate estimator of the vector of second-order parameters (β, ρ). For the estimation of this vector of parameters, see [1,12].

2.2 A Semi-parametric Estimator of Extreme Quantiles

Let us consider again the sample $(X_1, X_2, \ldots, X_n)$ of iid random variables with a regularly varying right tail. For these heavy tailed models and for small values of p, we want to extrapolate beyond the sample, estimating not only the EVI, ξ, but also an extreme quantile χ_{1-p}, i.e., a value such that $F(\chi_{1-p}) = 1-p$, or equivalently,

$$\chi_{1-p} := F^{\leftarrow}(1-p) = U(1/p), \quad p = p_n \to 0, \quad \text{as } n \to \infty,$$

with the notation $F^{\leftarrow}(y) := \inf\{x : F(x) \geq y\}$ for the generalized inverse function of F and $U(t) := F^{\leftarrow}(1-1/t)$, $t \geq 1$, for the reciprocal quantile function. The classical semi-parametric extreme quantile estimator for heavy right-tails was proposed by [32] as

$$Q_{p,\hat{\xi}_n^{\bullet}}(k) = X_{n-k:n}\, r_n^{\hat{\xi}_n^{\bullet}(k)}, \quad 1 \leq k < n, \quad r_n \equiv r_n(k;p) = \frac{k}{np}, \tag{7}$$

where $X_{n-k:n}$ is the $(k+1)$-th upper order statistic and $\hat{\xi}_n^{\bullet}$ can be any consistent estimator of the EVI, ξ.

3 Some Notes on Time Series Analysis

Time series analysis significantly impacts various scientific, economic, and social applications and arises in numerous contexts. A time series is a sequence of observations indexed by time, denoted as $\{X_1, \cdots, X_T\}$, and are typically ordered at evenly spaced intervals with a correlation between the observations.

One of the important aspects of time series analysis is the selection of a model that could have generated them. In order to have a deep analysis of these time series, a statistical model should be considered to capture the stochastic movement of the time series. AutoRegressive Integrated Moving Average (ARIMA) and Exponential Smoothing are commonly used time series models. ARIMA models use past observations of the dependent variable to forecast its values based on a linear regression model. In contrast, Exponential Smoothing methods are more versatile approaches that continually update forecasts, emphasizing the most recent experience. This means recent observations are given more weight than older observations [18]. Exponential Smoothing models consider trend and seasonality patterns in the data, while ARIMA models aim to explain the autocorrelations in the data [18].

4 The Role of Bootstrap Methodology

Computer-intensive methods, which emerged when computers became more powerful, have been developed in the last decades. The most well-known are perhaps the Jackknife [27,28,31], and the Bootstrap [7,19] methodologies. Recently, these two methodologies have been used with success in EVT, revealing themselves as being promising in parameter estimation.

In their classical form, as first proposed by [7], nonparametric bootstrap methods are designed for being used in samples collected under an independent setup. The bootstrap method involves iteratively resampling a dataset with replacement. The nonparametric bootstrap method works without assuming the nature of the distribution underlying the sample. From the original sample, a new sample is resampled with replacement and on the basis of the estimator of interest, an estimate is calculated. This procedure is repeated a large number of times, let us say B times, and the average of the estimates calculated is referred to as "the bootstrap estimate".

In the context of dependent data, the situation is more complicated since the population is not characterized entirely by the one-dimensional marginal distribution F alone, but requires the knowledge of the joint distribution of the whole sequence $X_1, ..., X_T$. Singh [29] presented an example of the inadequacy of the classical bootstrap under dependence. Several attempts have been made to extend the bootstrap method to the dependent case. A breakthrough was achieved when resampling of single observations was replaced by block resampling, an idea that was put forward by [3,15,20,22], and others in various forms and in different inference problems. Several ways of blocking appeared.

Another idea to overcome that issue consists in performing the bootstrap methodology on the residuals after fitting an adequate model, see [4,5].

In this article the parametric bootstrap approach is proposed. This approach was studied by Efron [8], among other authors. An adequate model is fitted to the data and the model parameters are estimated. When it can be applied, this type of bootstrap method can give better results since it considers the nature of the distribution underlying the data instead of being used the empirical distribution function as is done in the nonparametric bootstrap.

The parametric bootstrap will be applied to the residuals of the time series obtained after an adequate modelling procedure. See the detailed description below.

Step 1: Fit a model to the time series using AIC criterion;
Step 2: Select the best model using accuracy measures;
Step 3: Fit a GEV distribution to the residuals of the best model, $e_1, e_2, \cdots e_T$;
Step 4: For each b $\in B$ a large number
a) Generate a $e_1^b, e_2^b, \cdots e_T^b$ from the distribution estimated in Step 3;
b) Combine $e_1^b, e_2^b, \cdots e_T^b$ with the fitted values (from Step 2). A reconstructed sample path of the original time series is obtained;
c) Repeat Step 4;
Step 5: Obtain the mean of the B values simulated.

This algorithm is an adapted version of a first one developed for forecasting in time series and fully described in [4] and successfully applied in [5]. Here a GEV distribution is fitted to the residuals, and the parametric bootstrap is performed. Computational time depends on the sample size, for moderately sized samples it takes a few minutes. Several accuracy measures were considered in previous applications, such as the *Mean Absolute Error* (MAE), the *Root Mean Squared Error* (RMSE), the *Mean Absolute Percentage Error* (MAPE). They have been also used here.

A very preliminary version of the approach here described joining EVT and time series in order to improve the estimation of extreme quantities of interest was submitted for publication in the Proceedings of the 9th International Conference on Risk Analysis (ICRA9) [6].

The following packages in R software are used: `forecast` [17], `ets()`-EXPOS methods and `auto.arima()`-ARIMA models; and the function `fgev()` - Extreme value distributions, from package `evd` [30].

5 Estimation of Extreme Quantiles for Real-Data Sets

The high quantiles estimation of heavy tailed has many important applications. Studying and modelling river flow discharge rates is required for river management, including water resources planning, pollution prevention, and flood control. This study considers the daily mean flow discharge rate values (m^3/s) of two Portuguese rivers, Tejo and Guadiana. The time horizon for Tejo and Guadiana are January 1974 until June 2022 and from January 2002 until June

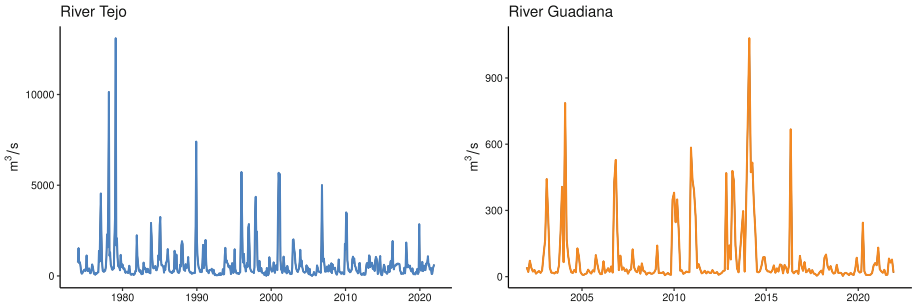

Fig. 1. Monthly maximum flow discharge for river Tejo and river Guadiana.

2022, respectively[1]. As we are interested in studying the maximum values of the series, the maximum of each month is taken (see Fig. 1).

As seen in Fig. 2 and Table 1, the descriptive study of these data revealed a tail heavier than that of the normal.

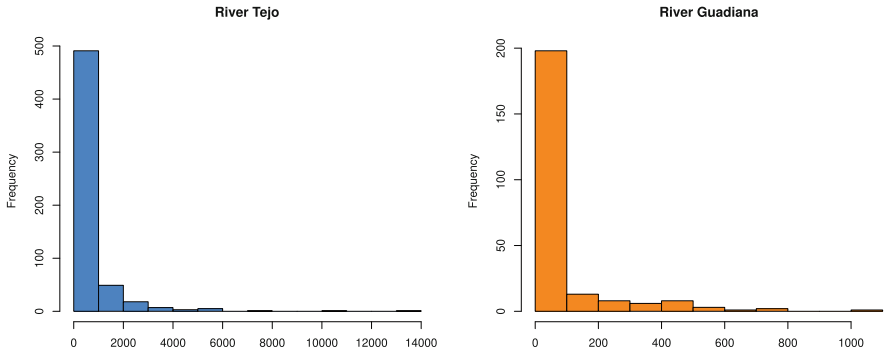

Fig. 2. The histograms for data: Tejo and Guadiana.

For quantiles estimation, it is crucial the estimation of EVI parameter. Fig. 3 shows the sample paths of the EVI estimates obtained when using the aforementioned semiparametric estimators (Sect. 2.1) over the original data.

Considering the time series of monthly maximum values (see Fig. 1) the ARIMA and the Exponential Smoothing models were fitted, and some accuracy measures are obtained and presented in Table 2. According to this table, Exponential Smoothing is the best model because it has the lowest values of the accuracy measures. In fact for the Tejo river, MAE is a little smaller in the `auto.arima` model, but the RMSE and MAPE measures reveal a better performance of the `ets` model, so we continued the work with `ets` The EV distribution is fitted to the residuals of the best model. The graphical representation of these

[1] Download at "http://snirh.apambiente.pt".

Table 1. Descriptive statistics measures for both rivers.

River	n	min	max	skewness	kurtosis
Tejo	576	0.99	13102.55	5.7495	47.373
Guadiana	240	3.35	1080.1	3.2483	12.38033

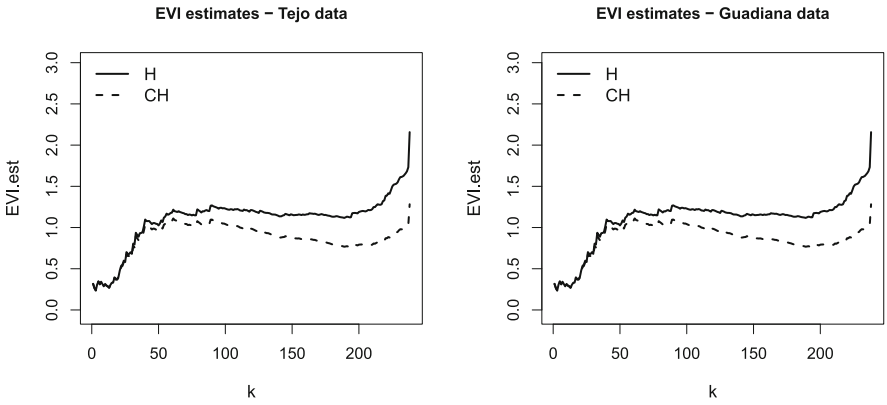

Fig. 3. Sample paths of the EVI-estimates considered.

residuals in Fig. 4 and the positive values for the skewness as well as the high kurtosis, given in Table 3, suggest a right heavy tail.

Table 2. Results of the accuracy measures.

	Tejo			Guadiana		
R function	RMSE	MAE	MAPE	RMSE	MAE	MAPE
`ets`	869.76	432.96	98.93	115.34	63.68	145.80
`auto.arima`	922.19	426.81	195.99	123.58	67.28	176.53

Using EV parameters estimates, the parametric bootstrap was performed and the reconstructed time series was obtained. This procedure was repeated $B = 1,000$ times, and mean values for each month were calculated. In Fig. 5 are represented three simulated time series and the mean of all simulated time series, whereas in Fig. 6 are represented the original and the reconstructed time series.

For both rivers can be seen that the reconstructed time series captures the extreme values of the original time series, see Fig. 6.

The quantile estimates are calculated when considering the original time series and the reconstructed time series. For both shape parameter estimators,

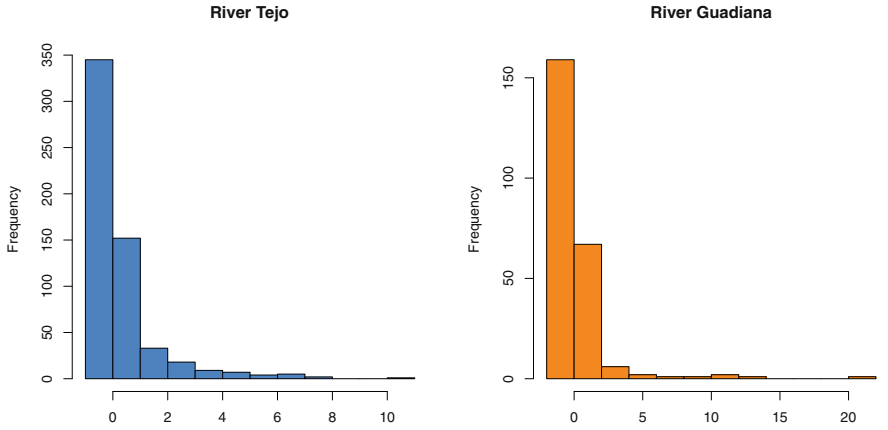

Fig. 4. The histograms for residuals.

Table 3. Descriptive statistics measures for residuals.

River	n	min	max	skewness	kurtosis
Tejo	576	–0.9695	10.4015	3.194409	13.04744
Guadiana	240	–0.9422	21.1081	5.802513	41.72199

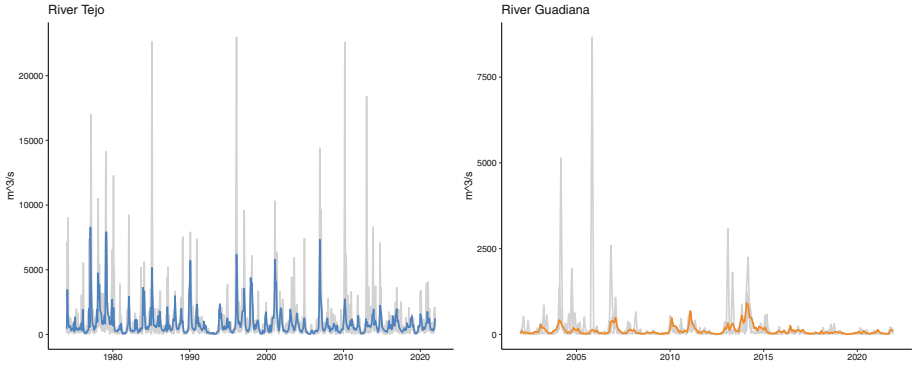

Fig. 5. Simulated time series (gray) and the mean of all.

the sample path of the quantile estimates reveals an underestimation relatively to the sample path obtained over the reconstructed time series, as seen in Fig. 7 and Fig. 8.

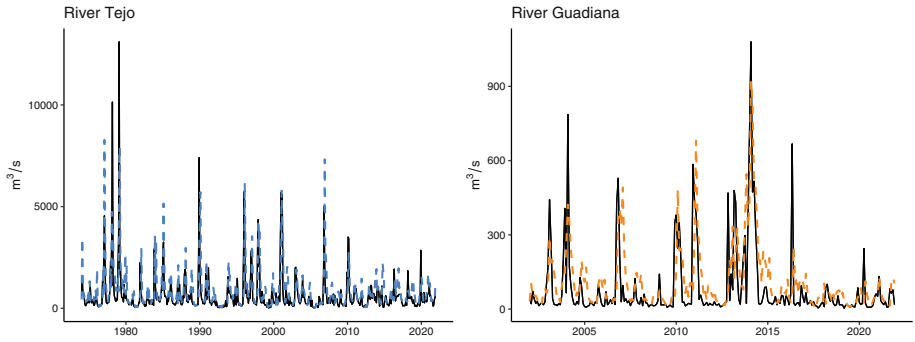

Fig. 6. Time series and the time series reconstructed (dashed).

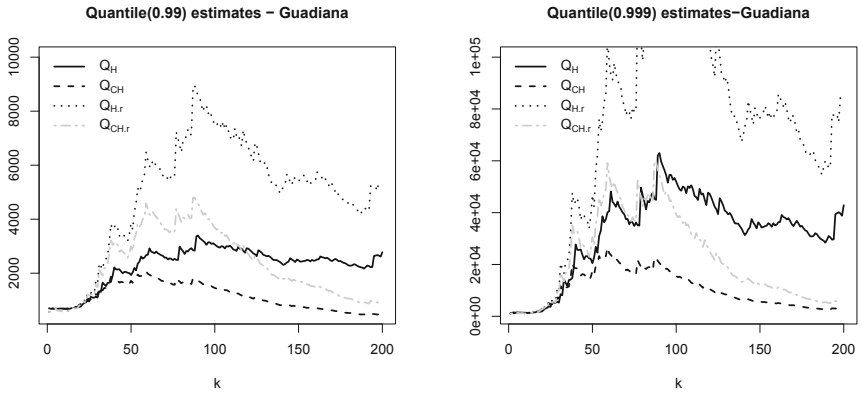

Fig. 7. Sample path of the quantile estimates based on the original time series and on the reconstructed associated for river Guadiana. These are denoted with subscript ".r".

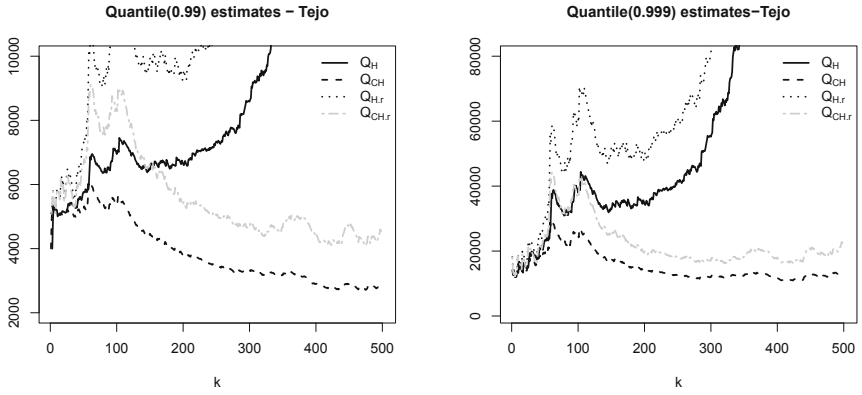

Fig. 8. Sample path of the quantile estimates based on the original time series and on the reconstructed associated for river Tejo. These are denoted with subscript ".r".

6 Concluding Remarks

Several attempts have been made in recent literature to improve the quality of extreme quantile estimators. Estimation of very extreme quantiles is complicated by the lack of observed values. The accuracy of the estimates depends on the estimation of key parameters in EVT, as well as the quantile estimators themselves. Research into both types of estimators is ongoing.

Modeling and forecasting in time series analysis, together with EVT methods, have proven to be valuable procedures for estimating extreme values when capturing time series features. Resampling methods have recently been applied in EVT, but they require special procedures because the resampling of a given sample does not allow obtaining values greater than the observed maximum.

Some procedures to overcome that difficulty were proposed and in this work, we present another one performing the parametric bootstrap on the residuals.

The method is illustrated considering two time series, but other time series were studied, verifying a good performance of the approach presented here.

In future work, an extensive application will be carried out, involving various sets of time series of similar features to those here considered (note that we intend to capture the highest values of a time series, as these may be responsible for catastrophic situations).

Acknowledgments. The authors thank the three referees for their constructive comments and valuable suggestions, which led to substantial improvements to this work. Clara Cordeiro and Manuela Neves are partially financed by national funds through FCT - Fundação para a Ciência e a Tecnologia under the project UIDB/00006/2020 (CEAUL). This work is funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

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