

Modelling the Fuel Consumption of a NRP Ship Using a Kalman Filter Approach

M. Filomena Teodoro[1,2(✉)], Pedro Carvalho[2], and Ana Trindade[2]

[1] CEMAT, Center for Computational and Stochastic Mathematics, Instituto Superior Técnico, Lisbon University, 1048-001 Lisbon, Portugal
maria.teodoro@tecnico.ulisboa.pt

[2] CINAV, Center of Naval Research, Naval Academy, Military University, 2810-001 Almada, Portugal

Abstract. The Kalman filter can be applied in the most diverse areas of knowledge, for example, medicine, agriculture, social sciences, computing, etc. The Kalman filter is a recursive tool that can be used under the aim of Navigation and Integration Systems. We make a brief approach to the derivation of a Kalman filter dividing the work into two parts. By first, a Kalman filter is used to simulate different situations analyzing the "response" of the filter considering distinct cases for distinct states of the shop motor; at second, a specific Kalman filter is built to filter the fuel consumption data collected directly from the on-board records of a ship from Portuguese Republic.

Keywords: Kalman Filter · Fuel Consumption Modeling · Computational Algorithms

1 Introduction

In 1960, the engineer Rudolf Kalman published an article (Kalman, A new approach to linear filtering and prediction problems 1960), (Kalman, New methods in wiener filter theory 1963) in which he presented a new method of linear filtration. This method uses measurements of independent variables and the associated noise to filter the system signal and predict its next state using statistical techniques. This new method introduced by Kalman in early sixties decade came to be known as Kalman Filter (KF) and had its first use aboard the spacecraft navigation computers of the APOLLO project.

The KF is one of the most applied methods for tracking and estimation due to its simplicity, optimality, tractability and robustness (Uhlmann 1997).

Accordingly with the authors of (Sorenson 1985), (R. G. Brown 2012) the KF can be seen as a sequential minimum mean square error (MMSE) estimator of a signal with noise. This signal is described by a state (or dynamical) model. When the errors are Gaussian distributed, the KF conduces to an optimal MMSE estimator; if the errors are not Gaussian distributed, the estimator still is a linear MMSE estimator. In (G. Welch 2006) is illustrated in a simple way the basic concepts of FK.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
O. Gervasi et al. (Eds.): ICCSA 2023 Workshops, LNCS 14105, pp. 344–361, 2023.
https://doi.org/10.1007/978-3-031-37108-0_22

This work presents a brief approach to the derivation of a KF when the input is a scalar quantity. It was considered a KF to estimate a first order system. Several simulations were carried out with these models and the results of applying the filter in different situations were analyzed.

It ended with the analysis of a KF model, built specifically to filter NRP Douro consumption data.

The approach that was made in this work considered only the discrete KF, once, in practice, observations and controls were carried out in discrete case.

The KF was used in the form of a linear system to obtain the best estimate of the state vector conditional to past observations. The estimate was calculated using the reconstruction of the state vector using the previous state vector estimate, the known inputs and measured outputs. The observed consumption value was registered hourly in NRP Douro, with AV3 machine regime, using a one-dimensional KF considering white random noise in the measure equation. It was possible to verify that the applied KF was effective and conduced to good measures of estimation and prevision.

2 Preliminaries

In statistics, the Kalman filter is a mathematical method created by Rudolf Kalman, with the objective of using measurements of magnitudes carried out over time (contaminated with noise and other uncertainties) and generating results that tend to approximate the real values of the measured quantities and associated values.

In theoretical terms, the Kalman filter is an algorithm that produces estimates of the state of a system using noisy measurements observed over time. The filter operates recursively, that is, it only needs the measurement at the current instant and the calculation of the state of the previous instant. Furthermore, it is not always possible or desirable to measure some variables of interest for a given purpose (such as, for example, control systems) and the Kalman filter can be used to infer these variables through indirect measurements.

The easy implementation in digital computers due to its recursion, the possibility of being used without knowing exactly its "internal" functioning and the fact that it uses a representation in state space, are some of the reasons that make this filter so popular.

The Kalman filter can be applied in the most diverse areas of knowledge, including medicine, agriculture, social sciences, computing, etc. With numerous applications, the Kalman filter is one of the most recurrent topics in the academic world and serves as the theme of this work, developed within the curricular unit Navigation and Integration Systems, with the objective of applying the theory of signal filtering to a case practical, solidifying students' learning.

In this work we intend to make a brief approach to the derivation of a Kalman filter when the input is a scalar quantity. To meet the defined objectives, the work was divided into two parts. A first part, in which a Kalman filter is used, using MATLAB code to simulate different situations and analyze the "response" of the filter. In the second part, a specific Kalman filter is built to filter NRP Douro consumption data and analyze the results. The data used were collected directly from the on-board records and only a sample of these data was selected for filtering.

3 Kalman Filter

Kalman filters are based on discretized linear dynamical systems in the time domain. The state of the system is represented as a vector of real numbers. At each time step, a linear operator is applied to the state to generate the next state, with added noise and, optionally, some information about the control inputs if they are known. Later, another linear operator with more addition of noise generates the real state observations.

In order to use the Kalman filter to estimate the complete state of a process given only a sequence of noisy observations, it is necessary to model the process according to the structure of the Kalman filter. This means specifying the following matrices: a_j, the state transition model; h_j, the observation model; Q_j, the process noise covariance; R_j, the covariance of the observation noise; and, occasionally, b_j, the model of the control inputs, for each time step j, as described below.

Discrete-time linear systems are often represented in a state variable format given by the following equation:

$$x_j = ax_{j-1} + bu_j \tag{1}$$

where, in this case, the state variable, x_j, and the input value, u_j, are scalars, a and b are constants, and j represents the time variable. Equation 1 shows that the current value of the state variable at time j (x_j) is equal to its value at the previous time (x_{j-1}) multiplied by a constant (a) plus the current input value (u_j) multiplied by another constant (b). In Fig. 1, diagram 1 schematically represents Eq. 1, where the block with T represents a time delay (input is x_j, output is x_{j-1}).

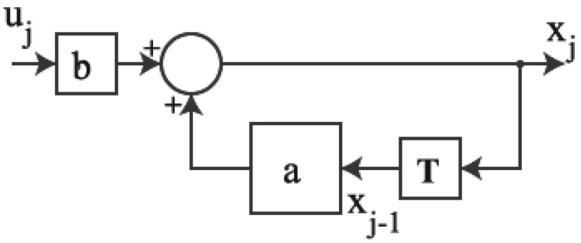

Fig. 1. Diagram 1.

If you add some noise to the process, you have the Eq. 2:

$$x_j = ax_{j-1} + bu_j + w_j. \tag{2}$$

The noise, w, is a source of white noise with zero mean and covariance Q and is uncorrelated with the input value. In Fig. 2, diagrama 2 schematically represents Eq. 2. In this situation, it is possible to filter the signal x so that the effects of noise w are minimized.

Assuming that it is not possible to directly measure the signal x, and instead, z is measured, which is x multiplied by a gain (h) with introduced noise (v) we consider Eq. 3.

$$z_j = hx_j + v_j. \tag{3}$$

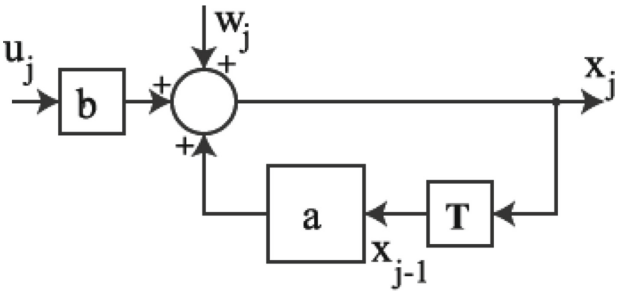

Fig. 2. Diagram 2.

The measured value, z, depends on the current value of x, as determined by the gain, h. Furthermore, the measurement has its own associated noise, v. The measurement noise, v, is a source of white noise with zero mean and covariance R and is uncorrelated with the input value or the noise w. The two noise sources are independent of each other and independent of the input value (Fig. 3).

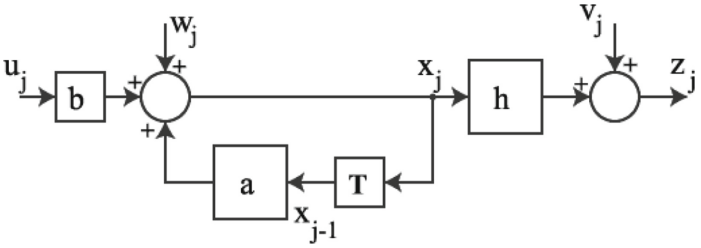

Fig. 3. Diagram 3.

Applying the Kalman filter to the previous system, it is possible to filter z and estimate the variable x while minimizing the effects of w and v. For this, it might seem reasonable to obtain an estimate of the state variable (and the output value) simply by reproducing the system architecture. This simple (and ultimately useless) way of getting an estimate of x_j (which we'll call $\hat{x}_j$) would be schematized as follows (see Diagram 4 displayed in Fig. 4). This model is very similar to the previous one. The first difference noted is that the original estimate of x_j is now designated $\hat{x}_j$; which is designated as a priori estimate:

$$\hat{x}_{\bar{j}} = a\hat{x}_{j-1} + bu_j. \tag{4}$$

This model is very similar to the previous one. The first difference noted is that the original estimate of xj is now designated $\hat{x}_j$; which is called a priori estimation (Eq. 4). However, this approach has two glaring weaknesses. The first is that there is no correction (update), if the exact quantities *a, b* or *h* (or the initial value x_0) are not known, the estimate will not allow to determine the exact value of x. And the second, does not compensate for the introduction of noise sources (w and v).

Diagram 5 (Fig. 5) presents a solution that allows overcoming the two problems presented.

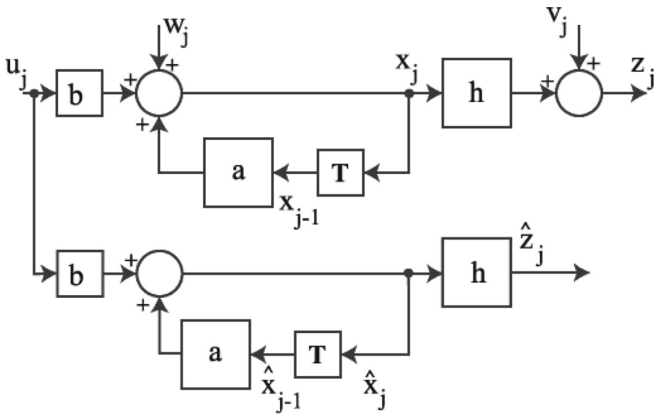

Fig. 4. Diagram 4.

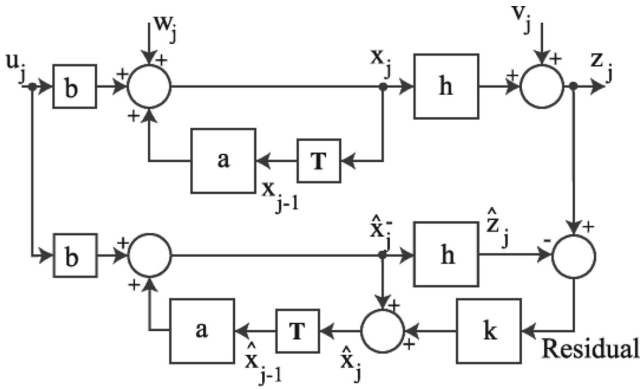

Fig. 5. Diagram 5.

This *a priori* estimate is used to predict an estimate for the output value, $\hat{z}_j$. The difference between this estimated value and the actual value is called the residual, or innovation (Eq. 5).

$$Residual = z_j - \hat{z}_{\bar{j}} = z_j - h\hat{x}_{\bar{j}} \tag{5}$$

If this residual difference is small, it usually means you have a good estimate; if it is large, the estimate is not so good. We can use this information to refine the initial guess of x_j; this new estimate is called the *a posteriori* estimate, $\hat{x}_j$. If the residual value is small, so will the correction of the estimate. As the residual value grows, so does the correction. A relevant equation will be Eq. 6:

$$\hat{x}_j = \hat{x}_{\bar{j}} + k(Residual) = \hat{x}_{\bar{j}} + k\left(z_{\bar{j}} - h\hat{x}_{\bar{j}}\right). \tag{6}$$

Nesta fase, o filtro de *Kalman* é utilizado para determinar a quantidade k (designada por Ganho de *Kalman)* que é usada para refinar a estimativa inicial. E é este processo que está no coração da filtragem de *Kalman.*

Note: The aim is to try to find an ideal estimator and, so far, we have only been able to optimize the gain value, k. However, assuming that a copy of the original system (that is, the gains a, b and h were defined as shown) should be used to form the estimator. It can then be assumed that the estimator, shown above, is the ideal (optimal) linear estimator.

3.1 Kalman Gain

To determine the *Kalman gain*, k, we define the errors of initial estimate: the *a priori* error $e_{\bar{j}}$, and the *a posteriori error*, e_j. Each one is defined as the difference between the real value x_j and its estimate (respectively *a priori* and a posteriori) as we can find in Eq. 7:

$$\begin{aligned} e_{\bar{j}} &= x_j - \hat{x}_{\bar{j}} \\ e_j &= x_j - \hat{x}_j \end{aligned}. \tag{7}$$

The variance of the errors defined by Eq. 7 are given by Eq. 8

$$\begin{aligned} p_{\bar{j}} &= \mathrm{E}\left\{ \left(e_{\bar{j}}\right)^2 \right\} \\ p_j &= \mathrm{E}\left\{ \left(e_j\right)^2 \right\} \end{aligned}, \tag{8}$$

where the *E{}* operator represents the expected or average value. These definitions will be used in the calculation of the quantity k.

A Kalman filter minimizes the *a posteriori* variance, p_j, by appropriately choosing the value of k. Substituting Eq. 7 in Eq. 8 and then substituting in Eq. 6, we obtain:

$$\begin{aligned} p_j &= \mathrm{E}\left\{ \left(x_j - \hat{x}_j\right)^2 \right\} \\ p_j &= E\left\{ \left(x_j - \hat{x}_{\bar{j}} - k\left(z_j - h\hat{x}_{\bar{j}}\right)\right)^2 \right\}. \end{aligned} \tag{9}$$

To determine the value of k that minimizes the variance, we solve the equation $\frac{\partial p_j}{\partial k} = 0$.

$$\begin{aligned} \frac{\partial p_j}{\partial k} &= 0 = \frac{\partial E\left\{ \left(x_j - \hat{x}_{\bar{j}} - k\left(z_j - h\hat{x}_{\bar{j}}\right)\right)^2 \right\}}{\partial k} \\ &= 2E\left\{ \left(x_j - \hat{x}_{\bar{j}} - k\left(z_j - h\hat{x}_{\bar{j}}\right)\right)\left(z_j - h\hat{x}_{\bar{j}}\right) \right\} \\ &= 2E\left\{ x_j z_j - \hat{x}_{\bar{j}} z_j - k z_j^2 + kh\hat{x}_{\bar{j}} z_j - h x_j \hat{x}_{\bar{j}} + h\left(\hat{x}_{\bar{j}}\right)^2 + kh z_j \hat{x}_{\bar{j}} - kh^2\left(\hat{x}_{\bar{j}}\right)^2 \right\} \\ &= 2E\left\{ x_j z_j - \hat{x}_{\bar{j}} z_j - h x_j \hat{x}_{\bar{j}} + h\left(\hat{x}_{\bar{j}}\right)^2 \right\} - 2kE\left\{ z_j^2 - 2h\hat{x}_{\bar{j}} z_j + h^2\left(\hat{x}_{\bar{j}}\right)^2 \right\} \end{aligned} \tag{10}$$

Solving equality (10) we obtain the Kalman gain expression given by Eq. 11.

$$k = \frac{E\left\{ x_j z_j - \hat{x}_{\bar{j}} z_j - h x_j \hat{x}_{\bar{j}} + h\left(\hat{x}_{\bar{j}}\right)^2 \right\}}{E\left\{ z_j^2 - 2h\hat{x}_{\bar{j}} z_j + h^2\left(\hat{x}_{\bar{j}}\right)^2 \right\}}. \tag{11}$$

Rewriting expression (11) we obtain a simplified formua for the Kalman gain, given by expression (12)

$$k = \frac{h p_{\bar{j}}}{h^2 p_{\bar{j}} + R}. \tag{12}$$

The expression (12) needs a value for the *a priori* covariance which, in turn, requires knowledge of the state variable x_j. Therefore, it will be necessary to determine an estimate for the *a priori* covariance. Before that, it is important to analyze this equation in detail. First note, the "constant", k (gain), changes with each iteration. For that reason, it should actually be written with an index (ie k_j). Next, and more significantly, one can analyze what happens when each of the three terms of Eq. (12) is changed.

If the *a priori* error is very small, the corresponding k will be very small, so the correction (update) will also be very small. In other words, you can ignore the current measurement and simply use the previous estimates to form the new estimate. This would be expected: if the first (*a priori*) estimate is good (that is, with a small error), there is very little need to correct it.

If the *a priori* error is very large (so that the measurement noise term, R, in the denominator is not important), then $k = 1/h$. In effect, this shows that one can disregard the *a priori* estimate and use the current (measured) output value to estimate the value of the state variable. This is clear from its substitution in Eq. (6). Again, this would be expected: if the *a priori* error is large, one should disregard the *a priori* estimate and instead use the current measurement of the output value to form the new estimate of the state variable.

$$\hat{x}_j = \hat{x}_{\bar{j}} + k(Residual) = \hat{x}_{\bar{j}} + k\left(z_j - h\hat{x}_{\bar{j}}\right) = \hat{x}_{\bar{j}} + \frac{1}{h}\left(z_j - h\hat{x}_{\bar{j}}\right) = \frac{z_j}{h} \tag{13}$$

If the measurement noise, R, is too large, k will again be too small, and the current measurement is again neglected in forming the new estimate. This would be expected: if the measurement noise is large, there is little confidence in the measurement and the new estimate will depend more on the previous estimates.

3.2 A Priori Covariance

To determine the a *priori* variance we consider its definition

$$\begin{aligned} p_{\bar{j}} &= \mathrm{E}\left\{ \left(x_j - \hat{x}_{\bar{j}}\right)^2 \right\} \\ &= \mathrm{E}\left\{ \left(ax_{j-1} + bu_j - w_j - \left(a\hat{x}_{j-1} + bu_j\right)\right)^2 \right\} \\ &= \mathrm{E}\left\{ \left(a\left(x_{j-1} - \hat{x}_{j-1}\right) + w_j\right)^2 \right\} \\ &= \mathrm{E}\left\{ a^2\left(x_{j-1} - \hat{x}_{j-1}\right)^2 + 2aw_j\left(x_{j-1} - \hat{x}_{j-1}\right) + w_j^2 \right\} \end{aligned} \tag{14}$$

and the Eq. (15)

$$\mathrm{E}\left\{ 2aw_j\left(x_{j-1} - \hat{x}_{j-1}\right)\right\} = 2aE\left\{ w_j e_{j-1}\right\} = 0 \tag{15}$$

obtaining a simpler expression of $p_{\bar{j}}$:

$$p_{\bar{j}} = \mathrm{a}^2\mathrm{E}\left\{ \left(x_{j-1} - \hat{x}_{j-1}\right)^2 \right\} + E\left\{ w_j^2 \right\} = \mathrm{a}^2 p_{j-1} + Q. \tag{16}$$

3.3 A Posteriori Covariance

Similarly to the computation of the variance *a priori*, the computation of the variance *a posteriori* start with its definition following some simplifications.

$$\begin{aligned} p_j &= \mathrm{E}\left\{ \left(x_j - \hat{x}_j\right)^2 \right\} \\ &= \mathrm{E}\left\{ \left(x_j - \left(\hat{x}_{\bar{j}} - hk\hat{x}_{\bar{j}} + kz_j\right)\right)^2 \right\} \\ &= \mathrm{E}\left\{ \left(x_j - \left(\hat{x}_{\bar{j}} - hk\hat{x}_{\bar{j}} + k\left(hx_j + v_j\right)\right)\right)^2 \right\} = \mathrm{E}\left\{ \left(\left(x_j - \hat{x}_j\right)(1 - hk) - kv_j\right)^2 \right\} \\ &= \mathrm{E}\left\{ \left(x_j - \hat{x}_j\right)^2(1 - hk)^2 - 2kv_j\left(x_j - \hat{x}_j\right)(1 - hk) + k^2 v_j^2 \right\} = \ldots \\ &= \mathrm{E}\left\{ \left(x_j - \hat{x}_j\right)^2 \right\} + k^2\mathrm{E}\left\{ v_j^2 \right\} = (1 - hk)^2 p_{\bar{j}} + k^2 R \end{aligned} \tag{17}$$

Reorganizing formula (17) obtain (18):

$$R = \frac{p_{\bar{j}}\left(h - h^2 k\right)}{k}. \tag{18}$$

Including formula (18) in expression (17) we get

$$p_j = (1 - hk)^2 p_{\bar{j}} + k^2 \frac{p_{\bar{j}}\left(h - h^2 k\right)}{k} = p_{\bar{j}}\left(1 - 2hk + h^2 k^2 + kh - h^2 k^2\right) = p_{\bar{j}}(1 - hk). \tag{19}$$

3.4 Iterative Process Summary

The *Kalman* filter process starts with a system descrition in terms of a, *b* e *h*. The state variable is *x*, *u* and *z* are respectively the input and output of the system. The index of time is *j*. The equations of transition and the measure rule all system:

$$x_j = ax_{j-1} + bu_j + w_j$$
$$z_j = hx_j + v_j.$$

The process comprises two steps, a prediction step (which calculates the next state estimate based only on past measurements of the output values) and a correction step (which uses the current estimate value to refine the result provided by the output).

Prediction Step
The a priori state estimate is formed based on the previous state estimate and the current value of the input.

$$\hat{x}_{\bar{j}} = a\hat{x}_{j-1} + bu_j$$

To calculate the a priori covariance:

$$p_{\bar{j}} = \mathrm{a}^2 p_{j-1} + Q$$

It is possible to observe that these two equations use previous values of the estimate and posterior state covariance. Therefore, the first iteration of a Kalman filter requires estimates (which are usually just guesses) of these two variables. The exact estimate is usually not important as the values converge to the correct value over time; an ill-defined initial guess just takes more iterations to converge.

Correction Step (Update)
To correct (update) the prior estimate, we need the Kalman filter gain, *K*.

$$K = \frac{hp_{\bar{j}}}{h^2 p_{\bar{j}} + R}$$

This gain is used to correct (update) the prior estimate and thus obtain the posterior estimates.

$$\hat{x}_j = \hat{x}_{\bar{j}} + K\left(z_{\bar{j}} - h\hat{x}_{\bar{j}}\right)$$

And finally calculate the covariance *a posteriori.*

$$p_j = p_{\bar{j}}\left(1 - hK_j\right)$$

Notation
The used notation is displayed in Table 1.

The notation $\hat{x}(k|k)$ can be red as the estimate the *x* knowing the information in time *k−1*, meaning a priori estimate. $\hat{x}(k|k)$ is red as as the estimate the *x* knowing the information in time *k*, meaning a posteriori estimate.

Table 1. Notation meaning

Variable	Notation
time	j
state	x_j
System gains	a, b, h
input	u_j
result	z_j
gain	k_j
a priori estimate	$\hat{x}_{\bar{j}}$
a posteriori estimate	$\hat{x}_j$
a priori covariance	$p_{\bar{j}} = p(j\|j-1)$
a posteriori covariance	$p_j = p(j\|j)$

4 Empirical Approach

The next examples of the Kalman filter application, include two that are based on a model that uses the Kalman filter to estimate a constant and, with some few changes, and a a first order system, both in simulated environment. By last it was considered a model constructed with the specific purpose of estimating the consumption of NRP Douro.

4.1 Example: Estimating a Constant

The use of a Kalman filter is analyzed to estimate the value of a system with a gain a K = 1. If there is no noise in the system (w), the system output is a constant. This is shown in Fig. 6. The output gain, h, is set to 3; h = 3. The measurement noise covariance (v), R, is set to 1; R = 1. The initial estimate of the state is 1.5 and the initial estimate of the posterior covariance is 1. In Fig. 6 the upper left graph shows the actual state (red), the prior estimate (blue - the prediction step) and the a posteriori estimation (green - the corrected (updated) step). The lower left graph shows the actual error between the estimates and the actual state. The graph in the upper right corner shows the calculated covariances and the graph in the lower right corner shows the gain of the Kalman filter.

As expected, the posterior estimate is closer to the exact value at each step than the prior estimate. Error covariances start out large but decrease quickly. The Kalman filter gain decreases as confidence in the prior estimate increases. The graphs in Fig. 7 show when process noise is introduced (Q = 0.01) so the state does not remain constant. The remaining parameters are kept the same. Again, the a posteriori estimate is closer to the exact value at each step than the a priori estimate. Error covariances start out large but decrease quickly. Again, the a posteriori estimate is closer to the exact value at each step than the a priori estimate. Error covariances start out large but decrease quickly. The Kalman filter gain decreases as the confidence in the prior estimate increases, but it does not decrease as sharply as in the first case (without process noise).

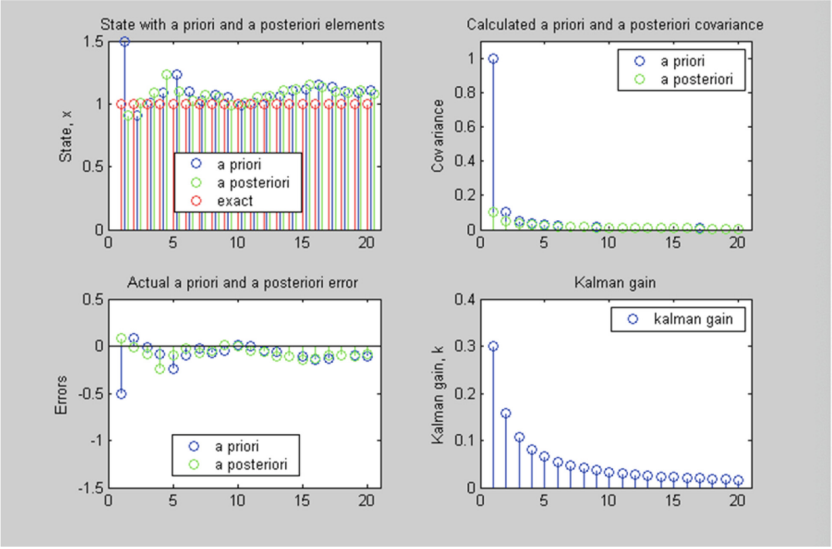

Fig. 6. Parameters: a = 1, h = 3, Q = 0, R = 1; x0 = 1,5, p0 = 1 (Color figure online)

This indicates that a larger correction is needed at each step – which is to be expected since noise is being introduced to the state itself at each step.

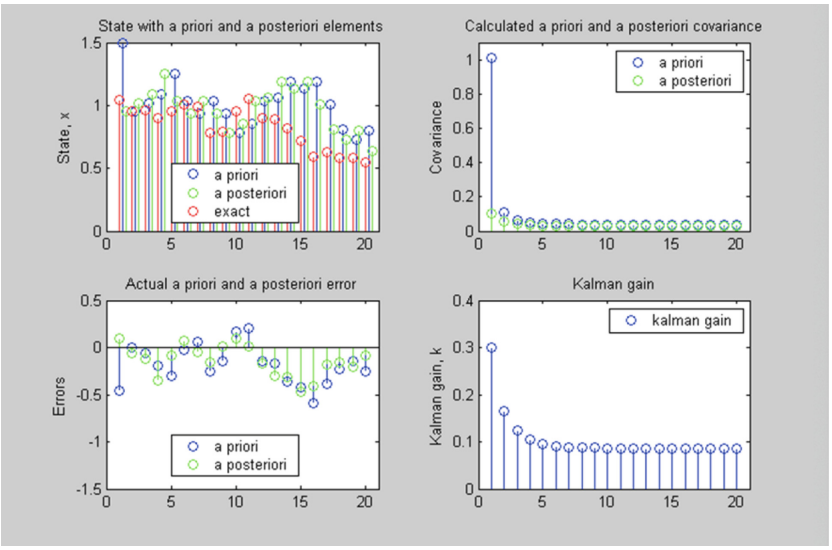

Fig. 7. Parameters: a = 1, h = 3, Q = 0,01, R = 1; x0 = 1,5, p0 = 1.

The Fig. 8 present graphs that are similar to previous case (above), only the process noise has increased *(Q = 0.1)*. Again, at each iterate, the posterior estimate is closer

to the exact value than the prior estimate. Error covariances start out large but decrease quickly. The Kalman filter gain decreases as the a priori estimate becomes more reliable, however it does not reach values as low as in the first case (without process noise) or in the second case *(Q = 0.1)*. This indicates that a larger correction is needed at each step – which is again expected, since the increase in the noise value of the measurement at each step makes the a priori estimate less good.

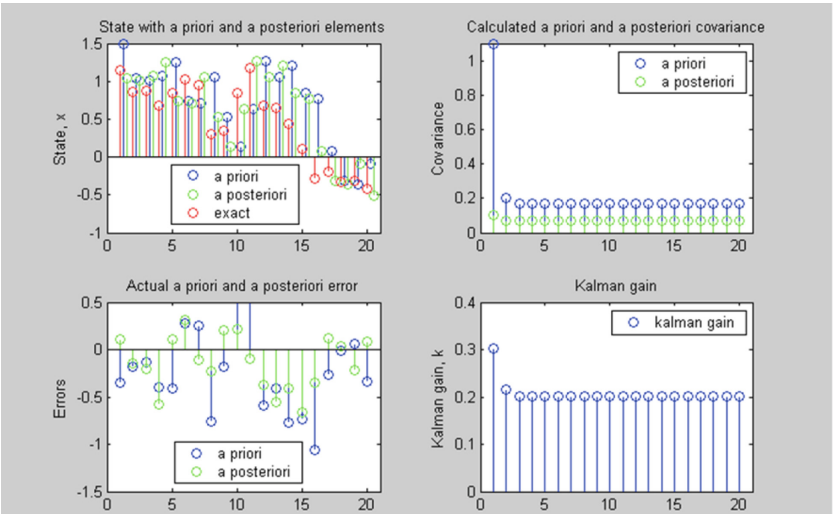

Fig. 8. Parameters: a = 1, h = 3, Q = 0,1, R = 1; x0 = 1,5, p0 = 1.

The graphs in Fig. 9 are similar to the ones presented in Fig. 7. Here only the measure error is increased *(R = 2)*.

Again, the a posteriori estimate is closer to the exact value at each step than the a priori estimate. Error covariances start out large but decrease quickly. The Kalman filter gain decreases as confidence in the prior estimate increases. Although it does not decrease as much as in the first case (without process noise), it reaches lower values than in the second case *(Q = 0.1)*. This means that, in relation to these two cases, one can rely more on the a priori estimate and less on the measurement, as the measurement noise is higher. Similar work was developed considering first order systems. By a question of space in article, it is not described in the manuscript. Follows a model constructed with the specific purpose of estimating the consumption of NRP Douro.

4.2 Estimating the Consumption of NRP Douro

This model is considered to estimate the consumption of the NRP Douro, with an AV3 machine regime (Fig. 10).

Assuming that the consumption at a stable machine speed is constant, that is, if the machine speed is not changed, the hourly consumption value is constant. However, from

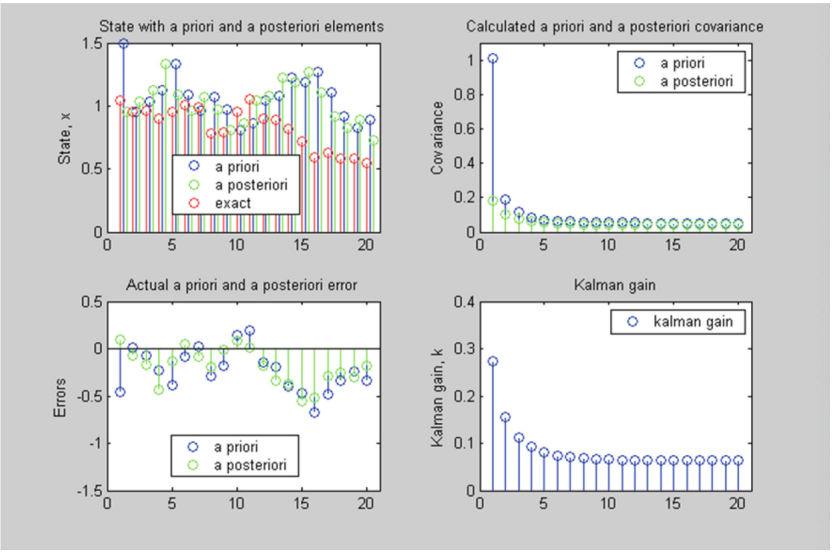

Fig. 9. Parameters: a = 1, h = 3, Q = 0,01, R = 2; x0 = 1,5, p0 = 1

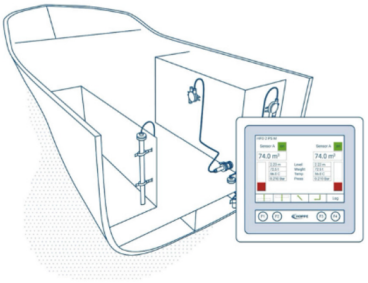

Fig. 10. Example of a fuel tank with an electronic sensor.

experience it is known that the true value of consumption suffers from fluctuations. Thus, one can describe the dynamics of the system through the following equation:

$$x_j = c + w_j \tag{20}$$

where c is the constant consumption rate and w_j is isa random noise value, which corresponds to a covariance Q.

The following assumptions were considered:

1. The consumption of the NRP Douro, with AV3 machines, is 200 L/hour.
2. Assuming the model is accurate, the noise variance value w, Q, is 0.5.
3. The range of error associated with the measurements is 64 L.
4. The range of error associated with the initial estimate is 400 L.
5. Measurements are taken hourly.
6. The measured consumption values correspond to a sample of 150 measurements.

7. The actual consumption values corresponding to each measurement with a random noise w introduced.

4.3 Kalman Filter Construction

ITERATION ZERO

Before first iteration, we need to initilize the Kalmen filter and estimate the next state.

INICIALIZATION

No body knows the real consumption of NRP Douro. It was assumed that is 200 L per hour (lts/h).

$$\hat{x}_{0,0} = 200 l/h$$

Considering that this assumption is very inaccurate, we define the initialization estimation error σ as 20 l/h. The estimated initialization uncertainty is the error variance (σ2):

$$p_{0,0} = 20^2 = 400$$

This variation is very high. If initialized with a more significant value, faster convergence of the Kalman filter is obtained.

PREDICTION

It is possible to predict the next state based on initialization values. As this dynamic model is a constant, the posterior estimate is equal to the prior estimate:

$$\hat{x}_{1,0} = 200\, l/h$$

The extrapolated estimated uncertainty (variance):

$$p_{1,0} = p_{0,0} + Q = 400 + 0{,}25 = 400{,}25$$

FIRST ITERATE

STAGE 1-MEASUREMENT

The measured value:

$$z_1 = 200\, l/h$$

The measurent error is 8 (σ), tthe variance ir given by (σ2) é 64, so:

$R_1 = 64$.

STAGE 2 - CORRECTION (UPDATE)

Computing the Kalman gain:

$$k_1 = p_{1,0} \div \left(p_{1,0} + R_1\right) = 400{,}25 \div (400{,}25 + 64) = 0{,}86214324$$

The Kalman gain is almost 1, which shows that the estimation error is much larger than the measurement error.

Estimating the actual state:

$$\hat{x}_{1,1} = \hat{x}_{1,0} + k_1\left(z_1 - \hat{x}_{1,0}\right) = 200 + 0{,}86214324 \times (200 - 200) = 200$$

Evaluating the variance of the actual state:

$$p_{1,1} = (1 - k_1) \times p_{1,0} = (1 - 0{,}86214324) \times 400{,}25 = 55{,}1771682$$

STAGE 3- PREDICTION

As the Dynamic Model of the system in use is constant, that is, the consumption value per hour does not change, then:

$$\hat{x}_{2,1} = \hat{x}_{1,1} = 200$$

The predict variance is:

$$p_{2,1} = p_{1,1} + Q = 55{,}537188 + 20 = 75{,}537188$$

REMAINING ITERATES

The remaining iterative process is just the repetition of the scheme.

5 Results and Discussion

In Fig. 11 is displayed actual value (green), measured value (red) and estimated value (blue) of consumption hourly rate. The analysis of the graph, it is verified that the estimated value converges to the real value. Graph 2 shows the uncertainty in the estimate.

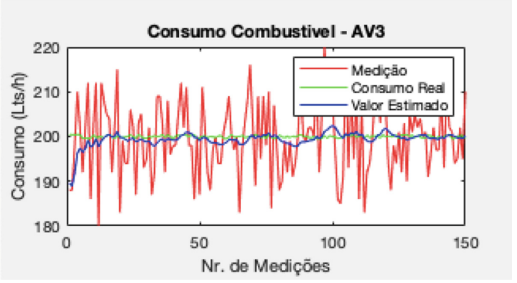

Fig. 11. KF simulation.

Figure 12 shows the uncertainty in the estimate. The value of the estimate uncertainty decreases rapidly. After 150 measurements, the value of the uncertainty in the estimate is 1.403, that is, the standard deviation of the uncertainty in the estimate is 1.969 lts/h. It can thus be said that the estimated consumption is 200.167 ± 1969 lts/h.

Figure 13 displays the Kalman gain.

From the analysis of Fig. 13, it is verified that the Kalman gain value decreases rapidly. The Kalman filter gain decreases as confidence in the prior estimate increases, making the measurement weight smaller and smaller.

Figure 14 shows the actual error between the estimates and the actual state. From the analysis of the graph, it can be seen that the error value starts out large, but decreases quickly. The error decreases as the estimated value approaches the actual value.

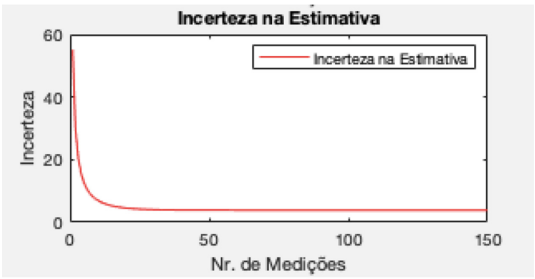

Fig. 12. Uncertainty of the estimate.

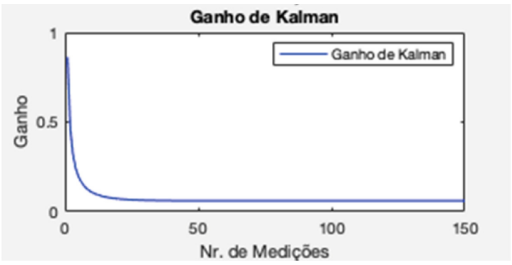

Fig. 13. Kalman gain.

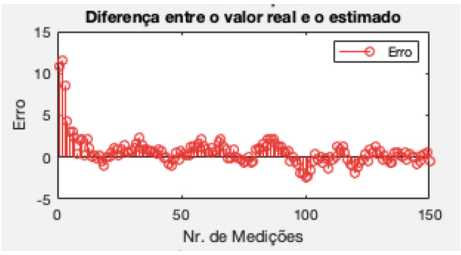

Fig. 14. Error between estimates and actual state.

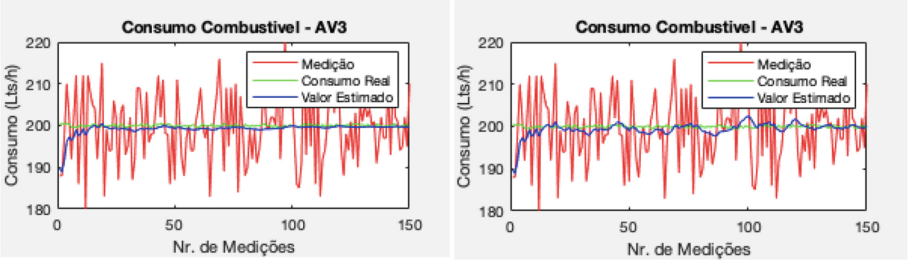

Fig. 15. KF Simulation. On right $Q = (5)^2$. On left $Q = (0.5)^2$.

The following simulations show how the Kalman filter performance is influenced by the measurement noise parameter, when, $Q = (0,5)^2$ and $Q = (5)^2$ (Fig. 15).

In this case, the constant, c, consumption, was observed 150 times, where the observations were corrupted by white noise with zero mean and variance 0.52. The Kalman filter estimated the value of the constant across the 150 noisy measurements. In the filter simulations, in which the variance value is changed, it is verified that the best performance among the simulations performed is the one that uses the real value of the variance of the measurement error. When using a value higher than the real one, it is observed that the filter responds more slowly to the measures, which results in a reduced estimated variance, but a slower convergence. When using a value lower than the real one, the filter responds more quickly to the measurements, which increases the estimated variance.

6 Conclusions

This work presents a brief approach to the derivation of a Kalman filter when the input is a scalar quantity. This was followed by the analysis of a Kalman filter model to estimate a constant and, of a similar model, to estimate a first-order system. Several simulations were carried out with these models and the results of applying the filter in different situations were analyzed. It ended with the analysis of a Kalman filter model, built specifically to filter NRP Douro consumption data.

The approach that was taken to the Kalman filter in this work was dedicated only to the discrete case, since in practical situations observations and controls are carried out in discrete time.

The Kalman filter was used in a linear system to obtain the best estimate of the state vector conditioned to the observations. This estimate was calculated by reconstructing the state vector using the previous state vector estimate, the known inputs, and the measured outputs.

In the consumption model presented, the hourly consumption value of the NRP Douro was estimated, with AV3 machine regime, using a one-dimensional Kalman filter. Despite having used a random noise value for the measurements, it was possible to verify that the applied Kalman filter was effective and presented a good estimate.

Acknowledgements. This work was supported by Portuguese funds through the Center of Naval Research (CINAV), Portuguese Naval Academy, Portugal and *The Portuguese Foundation for Science and Technology* (FCT), through the Center for *Computational and Stochastic Mathematics* (CEMAT), University of Lisbon, Portugal, project UID/Multi/04621/2019.

References

Brown, R.G., Hwang, P.Y.: Introduction to Random Signals and Applied Kalman Filtering: with MATLAB Exercises, 4th edn. Wiley, New Jersey (2012)

Julier, S.J., Uhlmann, J.K.: New extension of the Kalman filter to nonlinear systems. In: Signal Processing, Sensor Fusion, and Target Recognition VI, vol. 3068, pp. 182–193. SPIE (1997)

Kalman, R.E.: A new approach to linear filtering and prediction problems. J. Basic Eng. **82**(1), 35–45 (1960)

Kalman, R.E.: New methods in wiener filter theory. In: Bogdanoff, J.L., Kozin, F. (eds.) Proceedings of the First Symposium on Engineering Application of Random Function Theory and Probability, Wiley, New York (1963)

Sorenson, H.W.: Kalman Filtering: Theory and Application. IEEE Press, New Jersey (1985)

Welch, G., Bishop, G.: An introduction to Kalman filter (2006). Technical Report TR 95-041