Analytic Solutions of Fractional Differential Equation Associated with the k-Symbol Tremblay Differential Operator

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Abstract. A generalization of Tremblay differential operator is obtained in terms of the $k-$symbol fractional calculus in a complex domain. By using this operator, the analytic solutions of fractional differential equation are investigated. Our approach is based on a variety of ideas, spanning subordination and superordination as well as the use of Jack Lemma. For the normalized operator, some geometric characteristics are researched and discovered, including univalency and convexity. In order to find an analytical response to a class of fractional differential equations, the bounded turning characteristic will be investigated. Examples are illustrated in the sequel.

Keywords: Fractional calculus · Fractional differential equation · Fractional differential operator · Analytic function · Univalent function

1 Introduction

The Tremblay differential operator (TDO) is a special structure of fractional differential operator (usually uses the classic fractional differential operator) used

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in the study of special functions and mathematical physics. The TDO has applications in various fields such as quantum mechanics, signal processing, and image processing. It is also used in the study of special functions such as the Bessel function and the hypergeometric function and the Whittaker functions [1]. Ibrahim and Jahangiri studied TDO with a complex variable. Later, Esa et al. [2] considered a modification of the TDO of a complex variable. Recently, Irmak [3] presented different geometric studies including TDO. In this work, we aim to introduce a generalization of TDO in view of the $k-$symbol calculus.

The concept k-symbol fractional calculus is the overview of the gamma function to take account of the parameter k (see [4]). This simplification yields the recognized $k-$symbol fractional differential and integral operators. This postponement shields the fractional operators of complex variables. The k-symbol fractional calculus is a influential implement for analyzing and modeling complex schemes that display non-integer order dynamics (see [5–7]).

Global analytic solutions refer to mathematical solutions that provide a formula or expression for the function being studied over its entire domain. In other words, a global analytic solution provides a complete description of the behavior of a function, without any gaps or singularities. Examples of functions that have global analytic solutions include polynomials, trigonometric functions, exponential functions, and logarithmic functions. These functions can be expressed using finite combinations of elementary functions (see [8–10]). However, not all functions have global analytic solutions. For example, some transcendental functions such as the gamma function and the Riemann zeta function do not have closed-form expressions that can be evaluated for all inputs. In these cases, numerical methods may be used to approximate the function over its domain (see [11–13]).

The $k-$symbol TDO is formulated in the open unit disk. A global solution for a fractional differential equation involving it is formulated analytically. Our method is found by different concepts including the subordination and superordination together with the application of Jack Lemma. Some geometric properties, such as univalency and convexity, are investigated and found for the normalized operator. The bounded turning characteristic will be looked at in order to discover an analytical solution to a class of fractional differential equations. Instances are provided in the follow-up.

2 The $k-$Symbol Process

The concept of k-symbol calculus [4] is first proposed in order to alter the well-known fractional calculus based on the $k-$gamma function

$$\Gamma_k(t) = \int_0^\infty \exp(\frac{-t^k}{k}) t^{\eta-1} dt, \quad \Re(\eta) > 0,\ k \in \mathbb{R}^+.$$

Definition 1

$$\Gamma_k(\eta) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{\eta}{k}-1}}{(\eta)_{n,k}}, \tag{1}$$

where

$$(\eta)_{n,k} := \eta(\eta+k)(\eta+2k)...(\eta+(n-1)k) = \frac{\Gamma_k(\eta+nk)}{\Gamma_k(\eta)}.$$

Note that $\Gamma_k(\eta) \to \Gamma(\eta)$ *when* $k=1$, *and* $\Gamma_k(\eta+k) = \eta\Gamma_k(\eta), \quad \Gamma_k(k) = 1.$

By utilizing $\Gamma_k(\eta)$, the $k-$symbol Riemann-Liouville operators are formulated in [11,12], as follows:

Definition 2. *The following formula can be applied to the* $k-$*symbol Riemann-Liouville fractional integral*

$$\mathcal{R}^{\mu}_{a,k}u(\eta) = \frac{1}{k\Gamma_k(\mu)}\int_a^{\eta}(\eta-\xi)^{\mu/k-1}u(\xi)d\xi, \quad \mu > 0.$$

Note is that, for $u(\eta) = 1$, *then*

$$\mathcal{R}^{\mu}_{a,k}u(\eta) = \frac{(\eta-a)^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}, \quad \mu \in (0,1].$$

As a specific value of $a = 0$, *the integral becomes*

$$\mathcal{R}^{\mu}_{k}u(\eta) = \frac{1}{k\Gamma_k(\mu)}\int_0^{\eta}(\eta-\xi)^{\mu/k-1}u(\xi)d\xi.$$

The $k-$*symbol Riemann-Liouville fractional derivative has the structure*

$$D^{\mu}_{k}u(\eta) = \frac{d}{d\eta}\left(\mathcal{R}^{1-\mu}_{k}u(\eta)\right).$$

The formula for the Caputo derivative with the sign k *is*

$${}^{C}_{k}\Delta^{\mu}u(\eta) = \begin{cases} k^w\mathcal{R}^{wk-\mu}_{k}u^{(w)}(\eta) = \frac{k^{w-1}}{\Gamma_k(wk-\mu)}\int_0^{\eta}\frac{u^{(w)}(\xi)}{(\eta-\xi)^{\mu/k-w+1}}d\xi, & \mu \in ((w-1)k, wk] \\ \frac{d^w}{d\eta^v}u(\eta), & w\,k = \mu. \end{cases} \tag{2}$$

For example, when $k = 1$, *the classical Caputo derivative is obtained.*

Keep in mind that the function of the k-Riemann-Liouville singular kernel is expressed, as follows:

$$\Bbbk_{\mu,k}(\eta) = \frac{\eta^{\mu/k-1}}{k\Gamma_k(\mu)}, \quad \mu \in (0,1),\ k > 0.$$

Hence, the integral operator changes in terms of the convolution product

$$\mathcal{R}^{\mu}_{k}u(\eta) = \Bbbk_{\mu,k}(\eta) * u(\eta).$$

As an instance, when $u(\eta) = 1$ then

$$D^{\mu}_{k}u(\eta) = \frac{\eta^{\frac{1-\mu}{k}-1}}{\Gamma_k(1-\mu)}.$$

Generally, if $u(\eta) = \eta^m$ then

$$D_k^{\mu} u(\eta) = \left(\frac{\Gamma_k(k(m+1))}{\Gamma_k(1-\mu+km)} \right) \eta^{\frac{1-\mu}{k}+m-1}.$$

Clearly, for $k = 1$, the operator has the regular formula [14]

$$D_1^{\mu} u(\eta) = \left(\frac{\Gamma(m+1)}{\Gamma(1-\mu+m)} \right) \eta^{m-\mu}.$$

2.1 The $k-$Symbol TDO

By using D_k^{μ}, the $k-$symbol TDO is defined as follows:

$$ {}^{T}D_k^{\mu,\nu} := \frac{\Gamma_k(\nu)}{\Gamma_k(\mu)} \eta^{1-\nu} D_k^{\mu-\nu} \eta^{\mu-1} \tag{3}$$

$$\Big(\mu, \nu \in [0,1];\ \mu > \nu;\ \eta \in \mathbb{C}\Big).$$

Next, we discuss the upper bound of ${}^{T}D_k^{\mu,\nu}$ acting on the class of univalent analytic functions when $|\eta| < 1$.

Proposition 1. *For a complex variable η with $|\eta| < 1$, let $u(\eta)$ be univalent. If $0 < \rho := \mu - \nu < 1$, then*

$$|{}^{T}D_k^{\mu,\nu} u(\eta)| \leq \frac{\Gamma_k(\nu)}{\Gamma_k(\mu)\Gamma_k(2-\mu+\nu)} r\big(rF_k(2,1;2-\rho;r)\big)', \tag{4}$$

$$(r = |\eta| < 1,\ 0 < \rho < 1)$$

where F_k indicates the $k-$symbol hypergeometric function. The inequality is sharp for the Koebe function

$$B(\eta) := \frac{\eta}{(1-\eta)^2},\ |\eta| < 1.$$

Proof. Since u is univalent function then it has the expansion series

$$u(\eta) = \eta + \sum_{n=2}^{\infty} u_n \eta^n, \tag{5}$$

with $|u_n| < n$. A computation yields, when $u_1 = 1$

$$\begin{aligned} |{}^{T}D_k^{\mu,\nu} u(\eta)| &\leq \frac{\Gamma_k(\nu)}{\Gamma_k(\mu)} \sum_{n=1}^{\infty} \frac{\Gamma_k(n+1)}{\Gamma_k(n+1-\rho)} n r^n = \frac{r\Gamma_k(\nu)}{\Gamma_k(\mu)} \sum_{n=0}^{\infty} \frac{(n+1)\Gamma_k(n+2)}{\Gamma_k(n+2-\rho)} r^n \\ &= \frac{r\Gamma_k(\nu)}{\Gamma_k(\mu)\Gamma_k(2-\rho)} \sum_{n=0}^{\infty} \frac{(2)_{n,k}(1)_{n,k}}{(2-\rho)_{n,k}} \frac{(n+1)r^n}{\Gamma_k(n+1)} \\ &= \frac{r\Gamma_k(\nu)}{\Gamma_k(\mu)\Gamma_k(2-\mu+\nu)} \big(rF_k(2,1;2-\rho;r)\big)', \end{aligned}$$

where $F_k(.)$ presents the $k-$symbol hypergeometric function. The equality is indicated when $u(\eta)$ is replaced by the Koebe function.

Note that when $r \to 1$, Proposition 1 yields

$$|{}^{T}D_{k}^{\mu,\nu}u(\eta)| \leq \frac{\Gamma_k(\nu)}{\Gamma_k(\mu)\Gamma_k(2-\mu+\nu)}\big(F_k(2,1;2-\rho;1)\big)'.$$

Proposition 2. *The operator ${}^{T}D_{k}^{\mu,\nu}$ is normalized by the formula*

$$\mathfrak{T}_{k}^{\mu,\nu}u(\eta) := \frac{\nu}{\mu}\left({}^{T}D_{k}^{\mu,\nu}u(\eta)\right) = \eta + \sum_{n=2}^{\infty}\left(\frac{\Gamma_k(\nu+k)\Gamma_k(\mu+n)}{\Gamma_k(\mu+k)\Gamma_k(\nu+n)}\right)u_n\eta^n \tag{6}$$

$$:= \eta + \sum_{n=2}^{\infty}\hbar_n(\nu,\mu,k)u_n\eta^n, \quad \Big(0 < \nu \leq \mu \leq 1, k \geq 1\Big)$$

where u is given in (5).

Proposition 3. *For a complex variable η with $|\eta| < 1$, let $u(\eta)$ be convex. If $0 < \rho := \mu - \nu < 1$, then*

$$|{}^{T}D_{k}^{\mu,\nu}u(\eta)| \leq \frac{r\Gamma_k(\nu)}{\Gamma_k(\mu)\Gamma_k(2-\mu+\nu)}\big(F_k(2,1;2-(\mu-\nu);r)\big), \tag{7}$$

$$(r = |\eta| < 1,\ 0 < \rho < 1)$$

where F_k indicates the $k-$symbol hypergeometric function. The inequality is sharp for the Koebe function

$$J(\eta) := \frac{\eta}{(1-\eta)},\ |\eta| < 1.$$

Proof. Since u is convex function then it has the expansion series

$$u(\eta) = \eta + \sum_{n=2}^{\infty}u_n\eta^n, \tag{8}$$

with $|u_n| < 1$. A computation yields, for $u_1 = 1$

$$\begin{aligned}|{}^{T}D_{k}^{\mu,\nu}u(\eta)| &\leq \frac{\Gamma_k(\nu)}{\Gamma_k(\mu)}\sum_{n=1}^{\infty}\frac{\Gamma_k(n+1)}{\Gamma_k(n+1-\rho)}r^n = \frac{r\Gamma_k(\nu)}{\Gamma_k(\mu)}\sum_{n=0}^{\infty}\frac{\Gamma_k(n+2)}{\Gamma_k(n+2-\rho)}r^n\\ &= \frac{r\Gamma_k(\nu)}{\Gamma_k(\mu)\Gamma_k(2-\rho)}\sum_{n=0}^{\infty}\frac{(2)_{n,k}(1)_{n,k}}{(2-\rho)_{n,k}}\frac{r^n}{\Gamma_k(n+1)}\\ &= \frac{r\Gamma_k(\nu)}{\Gamma_k(\mu)\Gamma_k(2-\mu+\nu)}\big(F_k(2,1;2-(\mu-\nu);r)\big).\end{aligned}$$

The equality is indicated when $u(\eta)$ is replaced by the Koebe function.

Proposition 4. *Consider the operator* $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]$. *If it satisfies the inequality*

$$\sum_{n=2}^{\infty} n(n-(1-\sqrt{6}))|\hbar_n(\nu,\mu,k)||u_n| < \sqrt{6}$$

then $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]$ *is univalent in* Σ.

Proof. A computation yields

$$\left|\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}\right| = \left|\frac{\sum_{n=2}^{\infty} n(n-1)\hbar_n(\nu,\mu,k)u_n\zeta^{n-1}}{1+\sum_{n=2}^{\infty} n\hbar_n(\nu,\mu,k)u_n\eta^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)|\hbar_n(\nu,\mu,k)||u_n|}{1-\sum_{n=2}^{\infty} n|\hbar_n(\nu,\mu,k)||\psi_n|}.$$

To prove that $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]$ is univalent, by the condition inequality of the result yields that

$$\left|\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}\right| < \sqrt{6}.$$

Then in virtue of the Umezawa lemma [17], we obtain univalency of the operator $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]$.

Due to the severe nature of the Koebe function as a univalency function, we will analyze some key characteristics of a special example of $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]$ when u is the Koebe function.

Proposition 5. *Consider the operator* $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]$. *If it satisfies the inequality*

$$\frac{\eta^2[\mathfrak{T}_k^{\mu,\nu}u(\eta)]^{(3)}}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'} - \left(\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}\right)^2 + \frac{(2\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'')}{([\mathfrak{T}_k^{\mu,\nu}u(\eta)]')} + 1$$

$$\prec \chi + (1-\chi)\left(\frac{1+\eta}{1-\eta}\right)^{3/2}, \quad \chi \in [0,1)$$

Then $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]$ *is convex of order* χ.

Proof. Let

$$\Phi(\eta) = 1 + \left(\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}\right), \quad \Psi(\eta) = \frac{\Phi(\eta)-\chi}{1-\chi}.$$

Obviously, $\Psi(0) = 1$ and

$$\eta\Psi'(\eta) = \frac{\eta\Phi'(\eta)}{1-\chi}.$$

Then, in view of Miller and Mocanu lemma [16], we obtain

$$\Psi(\eta) + \eta\Psi'(\eta) \prec \left(\frac{1+\eta}{1-\eta}\right)^{3/2}$$

where $\Psi(\eta) \prec \dfrac{1+\eta}{1-\eta}$. But

$$\Phi(\eta)+\eta\Phi'(\eta)=\frac{\eta^2[\mathfrak{T}_k^{\mu,\nu}u(\eta)]^{(3)}}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}-\left(\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}\right)^2+\frac{(2\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'')}{([\mathfrak{T}_k^{\mu,\nu}u(\eta)]')}+1$$

$$\prec \chi+(1-\chi)\left(\frac{1+\eta}{1-\eta}\right)^{3/2},\quad \chi\in[0,1)$$

Then $\Phi(\eta) \prec \dfrac{1+(1-2\chi)\eta}{1-\eta}$ or $\Re(\Phi(\eta)) > \chi$; which yields that $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]$ is convex of order χ.

2.2 Fractional Differential Equation (FDE)

In this part, we have study the generalized FDE in terms of the suggested $k-$symbol TDO, as follows:

$$\mathcal{V}(\eta)=\left(\frac{\eta}{\mathfrak{T}_k^{\mu,\nu}u(\eta)}\right)^2[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'-1,\quad \eta\in\Sigma:=\{\eta\in\mathbb{C}:|\eta|<1\},\tag{9}$$

where $\mathcal{V}(\eta)$ is analytic in Σ with $|\mathcal{V}(\eta)|<1$.

Example 1. Let $\mathcal{V}(\eta)=\eta/(1-\eta)$. Then the solution of Eq. (9) is given by the formula (see Fig. 1)

$$\begin{aligned}\mathfrak{T}_k^{\mu,\nu}u(\eta)&=-\frac{\eta}{(c\eta-\eta\log(1-\eta)+\eta\log(\eta)-1)}\\&=\eta+\eta^2(c+\log(\eta))+\eta^3(2c\log(\eta)+c^2+\log^2(\eta)+1)\\&+\eta^4(c(3\log^2(\eta)+2)+3c^2\log(\eta)+c^3+\log^3(\eta)+2\log(\eta)+1/2)\\&+\eta^5(c(4\log^3(\eta)+6\log(\eta)+1)+c^2(6\log^2(\eta)+3)+4c^3\log(\eta)\\&+c^4+\log^4(\eta)+3\log^2(\eta)+\log(\eta)+4/3)+O(\eta^6),\end{aligned}$$

where c is a constant.

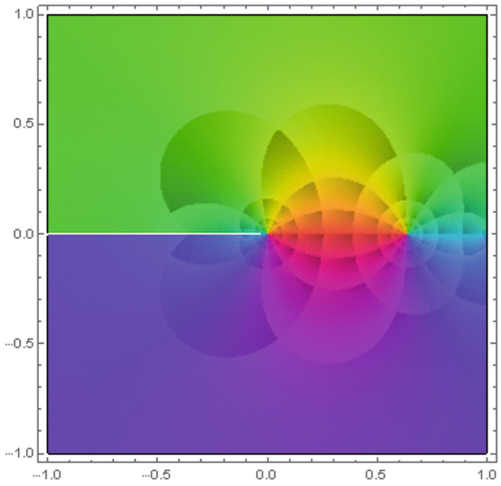

Fig. 1. Plot of the solution in Example 1

Example 2. Let $\mathcal{V}(\eta) = \eta/(1-\eta)^2$. Then the solution of Eq. (9) is given by the formula (see Fig. 2)

$$
\begin{aligned}
\mathfrak{T}_k^{\mu,\nu} u(\eta) &= \frac{(\eta - \eta^2)}{(c\eta^2 - c\eta - 2\eta - (\eta-1)\eta\log(1-\eta) + (\eta-1)\eta\log(\eta) + 1)} \\
&= \eta + \eta^2(c + \log(\eta) + 1) \\
&+ \eta^3(2c(\log(\eta)+1) + c^2 + \log^2(\eta) + 2\log(\eta) + 3) \\
&+ \eta^4(c(3\log^2(\eta) + 6\log(\eta) + \eta) + 3c^2(\log(\eta)+1) + c^3 + \log^3(\eta) + 3\log^2(\eta) \\
&+ 7\log(\eta) + 13/2) \\
&+ \eta^5(6c^2(\log^2(\eta) + 2\log(\eta) + 2) + c(4\log^3(\eta) + 12\log^2(\eta) + 24\log(\eta) + 19) \\
&+ 4c^3(\log(\eta)+1) + c^4 + \log^4(\eta) + 4\log^3(\eta) + 12log^2(\eta) + 19\log(\eta) + 46/3) + O(\eta^6)
\end{aligned}
$$

where c is a constant.

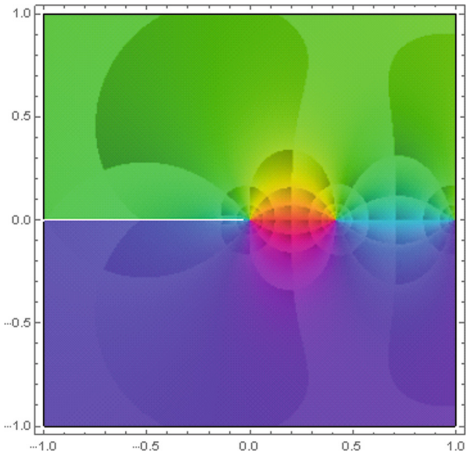

Fig. 2. Plot of the solution in Example 2

A univalent solution of a differential equation is a solution that is unique for a given set of initial or boundary conditions. In other words, there is only one solution that satisfies the differential equation and the given conditions. Univalent solutions are important in many applications of differential equations, particularly in physics and engineering, where unique solutions are necessary to make predictions and design systems. In the next section, we deal with the univalent solution of Eq. (9) in Σ. Our method is to present sufficient conditions to get a bounded turning behavior of $k-$symbol TDO. For this process, we have use the following information.

3 Information

3.1 Bounded Turning

A bounded turning analytic function is a complex function that has a property of bounded turning. A function is said to have a bounded turning at a point if the argument of the function remains within a bounded range as the function approaches that point [14,15]. More formally, a function $u(\eta)$ is said to have a bounded turning at a point η_0 if there exists a positive constant M such that $|\arg[u(\eta)] - \arg[u(\eta_0)]| < M$ for all z in some neighborhood of η_0. In simpler terms, the function does not oscillate rapidly as it approaches the point η_0. This property is often used in the study of complex analysis, where it is useful in understanding the behavior of functions near singularities and poles.

Examples of bounded turning analytic functions include the exponential function, the trigonometric functions, and any rational function that has simple poles. These functions have a smooth behavior near their singularities, and the argument of the function remains bounded as it approaches these points. In general, a bounded turning analytic function is a well-behaved function that does not exhibit any wild or erratic behavior near its singularities. This property is important in many areas of mathematics, including complex analysis, differential equations, and harmonic analysis.

Definition 3. *An analytic function $u(\eta)$ of the normalized form $u(0) = 0$ and $u'(0) = 1$ is called bounded turning in Σ if and only if* $\Re\left(u'(\eta)\right) > 0, \quad \eta \in \Sigma.$

3.2 Subordination

Differential subordination [16] is a technique used in complex analysis to compare two analytic functions. Given two analytic functions $u(\eta)$ and $v(\eta)$, we say that $u(\eta)$ is differentially subordinate to $v(\eta)$ if there exists an analytic function $w(\eta)$ satisfying the conditions: $w(\eta)$ maps the unit disk $|\eta| < 1$ onto itself, i.e., $w(\eta)$ is a conformal mapping of the unit disk onto itself. $w(0) = 0, w'(0) > 0$ such that $u(\eta) = v(w(\eta))$ for all η in the unit disk, denoting by $u \prec v$. Intuitively, differential subordination means that $u(\eta)$ is "controlled" by $v(\eta)$ in a certain sense. Specifically, the function $w(\eta)$ provides a change of variable that transforms the unit disk into itself, and maps $u(\eta)$ to $v(\eta)$ in such a way that

$\Re w'(\eta) > 0$ for all $\eta \in \Sigma$. This condition ensures that the conformal mapping preserves the orientation of the unit disk, so that the angle between any two curves is preserved under the mapping.

Differential subordination has many applications in complex analysis, particularly in the study of univalent and starlike functions. By comparing two analytic functions using differential subordination, one can often derive useful information about their geometric properties, such as their radii of convexity or starlikeness.

3.3 Jack Lemma

The Jack Lemma states that if a function $u(\eta)$ is analytic Σ in the complex plane, and if $u(\eta)$ is not identically zero on Σ, then the zeros of $u(\eta)$ are isolated. In other words, if there is a point $\eta_0 \in \Sigma$ such that $u(\eta_0) = 0$, then there exists a small disk centered at η_0 in which $u(\eta)$ is also zero only at η_0. This result is important because it shows that the zeros of an analytic function are always isolated, which is not true for general continuous functions. The proof of the Jack Lemma relies on the fact that analytic functions can be locally represented by power series expansions, and that the zeros of a power series are also isolated. The Jack Lemma has many applications in complex analysis, such as in the study of the zeros of entire functions, the factorization of meromorphic functions, and the proof of the fundamental theorem of algebra.

We need the following result.

Lemma 1. *[16]- p.19*
For $\eta_0 \in \Sigma$ and $\rho_0 = |\eta_0|$, we assume that $u(\eta) = u_n\eta^n + u_{n+1}\eta^{n+1} + \cdots$ is analytic on $\Sigma_{\rho_0} \cup \{\rho_0\}$ such that $u \not\equiv 0$ and $n \geq 1$. If $|u(\eta_0)| = \max\{|u(\eta)| : \eta \in \Sigma_{\rho_0}\}$, then there is an $m \geq n$ with

- $\Re\left(\dfrac{\eta_0 u'(\eta_0)}{u(\eta_0)}\right) = m$, *and*
- $\Re\left(\dfrac{\eta_0 u''(\eta_0)}{u'(\eta_0)} + 1\right) \geq m$.

3.4 Janowski Starlike Functions (JSFs)

JSFs are a family of holomorphic functions in the unit disk that are important in complex analysis. They were introduced by Stefan Janowski in 1929. A function $u(\eta)$ is said to be JSF in the unit disk if it has the form (13), where u_n are complex coefficients satisfying the condition $|u_2|2$. A function is JSF if it is analytic in the unit disk, maps it onto a domain that is starlike with respect to the origin, and satisfies a certain inequality involving the second coefficient of its power series expansion. JSFs are a generalization of the class of starlike functions introduced by Bieberbach. They have many interesting properties, including sharp distortion theorems and connections to geometric function theory. One important subclass of JSFs are the Janowski close-to-convex functions, which are functions that

satisfy the condition $|(\eta u'(\eta))/u(\eta)|1/2$ in the unit disk. These functions have a particularly simple geometric interpretation: they map the unit disk onto a domain that is contained in a convex domain with diameter at most 2.

Definition 4. *The class of JSFs is defined as follows.*
$\mathbb{J}[\aleph, \flat]$ *is the class of all functions* $\wp : \Sigma \to \mathbb{C}$ *with* $\wp(0) = 1$ *fulfilling*

$$\wp(\eta) \prec \frac{1 + \aleph\eta}{1 + \flat\eta}, \quad -1 \leq \flat < \aleph \leq 1.$$

Lemma 2. *[18] Suppose that* $\wp$ *satisfies* $\wp(0) = 1$ *and* $|\wp_n| \leq 2$. *If* $(\wp(\eta), \eta\wp'(\eta)) \in \Delta$, *where* Δ *is a complex domain; and* $\Re\top(\wp(\eta), \eta\wp'(\eta)) > 0$, *then* $\Re\wp(\eta) > 0$ *for* $\eta \in \Sigma$.

4 Solvability

Our aim is to present the sufficient conditions to obtain $\Re\left([\mathfrak{T}_k^{\mu,\nu}u(\eta)]'\right) > 0$

Theorem 1. *Assume the following inequality*

$$\mathrm{Re}\left(1 + \frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}\right) < 1 + \frac{\kappa}{2}, \quad \eta \in \Sigma,\ \kappa > 0. \tag{10}$$

(i) If $2 \geq 1 + \kappa$, *and* $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]/\eta \neq 0$, *then* $\mathrm{Re}\,[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' > 0$, $\eta \in \Sigma$.
(ii) If $2 \geq 1 + \kappa$, *and* $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]/\eta \neq 0$, *then*

$$\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]} \prec (1-\eta)/(1-k\,\eta),\ \eta \in \Sigma, \qquad k = 1/(1+\kappa).$$

(iii) If $2 \geq 1 + \kappa$ *and* $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]/\eta \neq 0$, *then* $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]$ *is univalent.*

Proof. (i) We aim to show that $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' \neq 0$ for all $\eta \in \Sigma$ and $\eta \neq 0$ (since $[\mathfrak{T}_k^{\mu,\nu}u(0)]' = 1$ in Proposition 2). If there is η_1, $0 < |\eta_1| < 1$ and $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' = (\eta - \eta_1)^{m_1}\psi(\eta)$, where $m_1 \geq 1$ and ψ is analytic in Σ and $\psi(\eta_1) \neq 0$, then

$$1 + \frac{\eta\,[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'} = 1 + \frac{m_1\eta}{\eta - \eta_1} + \frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{\psi(\eta)}.$$

Therefore, $1 + \dfrac{\eta\,[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'} \to \infty$ when $\eta \to \eta_1$, which contradicts (26).

Let

$$[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' = (1 - w(\eta))^{\frac{\kappa}{n-1}}, \tag{11}$$

where $0 < \kappa/(n-1) \leq 1$. Since $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' \neq 0$ for all $\eta \in \Sigma$, then $w(\eta) = w_{n-1}\eta^{n-1} + \cdots$ is analytic in Σ with $w(0) = 0$. In addition, we have

$$\left(1 + \frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}\right) = 1 - \frac{\kappa\eta w'(\eta)}{(n-1)(1-w(\eta))}.$$

Consider the equality $\max_{|\eta|\leq|\eta_0|}|w(\eta)| = |w(\eta_0)| = 1$. By Lemma 1, we get

$$\eta_0 w'(\eta_0) = kw(\eta_0); \quad (k \geq n-1; w(\eta_0) = e^{i\theta}; \theta \in \mathbb{R}).$$

Thus,

$$\mathrm{Re}\left(1 + \frac{\eta_0[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]'}\right) = 1 - \frac{\kappa}{n-1}\mathrm{Re}\left(\frac{k\,e^{i\theta}}{1-e^{i\,\theta}}\right).$$

Hence, the following fact is obtained

$$\mathrm{Re}\left(1 + \frac{\eta_0[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]'}\right) = 1 + \frac{k\kappa}{2} \geq 1 + \frac{\kappa}{2},$$

which contradicts (26). That is $|w(\eta)| < 1, \eta \in \Sigma$, and by (11), we have

$$|\arg([\mathfrak{T}_k^{\mu,\nu}u(\eta)]')| = \kappa|\arg((1-w(\eta))| < \frac{\kappa}{(n-1)}\frac{\pi}{2} \leq \frac{\pi}{2},$$

where $2 \geq 1 + \kappa$, which leads to $\mathrm{Re}\,[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' > 0$, $\eta \in \Sigma$,

(ii) Formulate a function w as follows:

$$\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]} = \frac{1-w(\eta)}{1-kw(\eta)}, \tag{12}$$

where $k = 1/(1+\kappa)$, then $w(\eta) = c_1\eta + \cdots$ is analytic in Σ. Now, let $\eta_0 \in \Sigma$ with $\max_{|\eta|\leq|\eta_0|}|w(\eta)| = |w(\eta_0)| = 1$. According to Lemma 1, we have

$$\eta_0 w'(\eta_0) = kw(\eta_0); \quad (k \geq 1; w(\eta_0) = e^{i\theta}; \theta \in \mathbb{R}).$$

In view of (12), a logarithmic differentiation implies

$$\begin{aligned}\mathrm{Re}\left(1 + \frac{\eta_0[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]'}\right) &= \mathrm{Re}\left(a\frac{\eta_0 w'(\eta_0)}{1-kw(\eta_0)} - \frac{\eta_0[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]'}{1-w(\eta_0)} + \frac{1-w(\eta_0)}{1-kw(\eta_0)}\right)\\ &= \mathrm{Re}\left(k\frac{k\,e^{i\theta}}{1-k\,e^{i\theta}} - \frac{k\,e^{i\theta}}{1-e^{i\theta}} + \frac{1-e^{i\theta}}{1-k\,e^{i\theta}}\right)\\ &= \left(\frac{1+k}{2}\right)f(t), \quad t = cos\theta\end{aligned}$$

where

$$f(t) = \frac{k(1-k)+2(1-t)}{1-2kt+k^2}, \ -1 \leq t \leq 1.$$

Since $\kappa > 0$, $0 < k = 1/(1+\kappa) < 1$ and $2 \geq 1+\kappa$, then $2k-1 \geq 0$. Hence,

$$f'(t) = 2(1-k)\frac{(k+1)k-1}{(1-2kt+k^2)^2} \geq 2(1-k)\frac{2k-1}{(1-2kt+k^2)^2} \geq 0,$$

which gives that f is a non-decreasing function and

$$f(t) \geq f(-1) = \frac{(n-1)(1-k)+4}{(1+k)^2}.$$

Hence, the previous relation shows that

$$\mathrm{Re}\left(1+\frac{\eta_0[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]'}\right) \geq \frac{(1-k)+4}{2(1+k)} \geq 1+\frac{\kappa}{2},$$

which contradicts (26). In virtue of $|w(\eta)| < 1, \eta \in \Sigma$ then (12) shows that the argument of the theorem is occurred.

(iii) Assume that

$$\left(\frac{\eta}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]}\right)^2 [\mathfrak{T}_k^{\mu,\nu}u(\eta)]' - 1 = V(\eta). \tag{13}$$

Then $V(\eta) = d_1\eta + \cdots$ is analytic in Σ. A logarithmic differentiation implies

$$1+\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'} = \frac{\eta V'(\eta)}{1+V(\eta)} + 2\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]} - 1. \tag{14}$$

In view of (14), we obtain

$$\mathrm{Re}\left(\frac{\eta V'(\eta)}{1+V(\eta)} + 2\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]} - 1\right) < 1+\frac{\kappa}{2}. \tag{15}$$

Consequently, we have

$$\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]} \prec \frac{1-\eta}{1-k\eta},$$

where $0 < k = 1/(1+\kappa) < 1, \kappa > 0$ which yields $\mathrm{Re}\left(\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]}\right) > 0$. Thus, (15) presents

$$\mathrm{Re}\left(\frac{\eta V'(\eta)}{1+V(\eta)} - 1\right) < 1+\frac{\kappa}{2} - 2\mathrm{Re}\left(\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]}\right) < 1+\frac{\kappa}{2}. \tag{16}$$

For $\eta_0 \in \Sigma$ with $|V(\zeta_0)| = 1$. According to Lemma 1, we get

$$\eta_0 V'(\eta_0) = kV(\eta_0); \quad (k \geq 1; V(\eta_0) = e^{i\theta}; \theta \in \mathbb{R}).$$

Thus,

$$\mathrm{Re}\left(\frac{\eta_0 V'(\eta_0)}{1+V(\eta_0)} - 1\right) = \mathrm{Re}\left(\frac{ke^{i\theta}}{1+e^{i\theta}}\right) - 1 = \frac{k}{2} - 1.$$

Hence, $\mathrm{Re}\left(\frac{\eta_0 V'(\eta_0)}{V(\eta_0)}-1\right) \geq 1+\frac{\kappa}{2}$, where $k \geq 1$, which contradicts the assumption. That is $|V(\eta)|<1, \eta \in \Sigma$. In addition, (13) shows that

$$\left|\left(\frac{\eta}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]}\right)^2 [\mathfrak{T}_k^{\mu,\nu}u(\eta)]'-1\right|<1, \quad \eta \in \Sigma$$

which concludes the desired assertion.

Theorem 2. *Let* $\kappa \geq 0$ *and*

$$\mathrm{Re}\left(1+\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}\right)>-\kappa, \quad \eta \in \Sigma. \tag{17}$$

(i) If $k \geq 2\kappa+3$, *then* $\mathrm{Re}\,([\mathfrak{T}_k^{\mu,\nu}u(\eta)]')>0$, $\eta \in \Sigma$.
(ii) If $k \geq 2\kappa+2$, *and* $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]/\eta \neq 0$, *then*

$$\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]} \prec \frac{1}{(1-\eta)}.$$

Proof. (i) Let

$$[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'=\frac{1}{(1-w(\eta))^{\frac{2(\kappa+1)}{k-1}}}, \quad k>1 \tag{18}$$

where $0<2(1+\kappa)/(k-1)\leq 1$. As in the above result, $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' \neq 0$ for all $\eta \in \Sigma$, which yields $w(\eta)=b_{k-1}\eta^{k-1}+\cdots$ is analytic in Σ with $w(0)=0$. Additionally, by (18), a logarithmic differentiation leads to

$$\left(1+\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}\right)=1+\frac{2(\kappa+1)(\eta w'(\eta))}{(k-1)(1-w(\eta))}. \tag{19}$$

Assume that $\eta_0 \in \Sigma$ is such that $\max_{|\eta|\leq|\eta_0|}|w(\eta)|=|w(\eta_0)|=1$. Then according to Lemma 1, we have $\eta_0 w'(\eta_0)=qw(\eta_0)$, $(q \geq k-1; w(\eta_0)=e^{i\theta}; \theta \in \mathbb{R})$. Thus,

$$\begin{aligned}\mathrm{Re}\left(1+\frac{\eta_0[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]'}\right)&=1+\frac{2(\kappa+1)}{k-1}\,\mathrm{Re}\left(\frac{qe^{i\theta}}{1-e^{i\theta}}\right)\\&=1-\frac{q(\kappa+1)}{k-1}\leq-\kappa,\end{aligned}$$

which contradicts (17). That is $|w(\eta)|<1, \eta \in \Sigma$. Also, (18) implies that

$$|\arg([\mathfrak{T}_k^{\mu,\nu}u(\eta)]')|=\frac{2(\kappa+1)}{k-1}|\arg((1-w(\eta))|<\frac{2(\kappa+1)}{k-1}\frac{\pi}{2}\leq\frac{\pi}{2},$$

where $k \geq 2\kappa+3$. Hence, $\mathrm{Re}\,([\mathfrak{T}_k^{\mu,\nu}u(\eta)]')>0$, $\eta \in \Sigma$.

(ii) Formulate a function w by

$$\frac{\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{[\mathfrak{T}_k^{\mu,\nu}u(\eta)]} = \frac{1}{1-w(\eta)}. \tag{20}$$

Then $w(\eta) = c_{k-1}\eta^{k-1} + ...$ is analytic in Σ with $\max_{|\eta|\leq|\eta_0|}|w(\eta)| = |w(\eta_0)| = 1$. In view of Lemma 1,

$$\eta_0 w'(\eta_0) = qw(\eta_0); \quad (q \geq k-1; w(\eta_0) = e^{i\theta}; \theta \in \mathbb{R}).$$

Thus, (20) gives

$$\begin{aligned}\mathrm{Re}\left(1 + \frac{\eta_0[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]''}{[\mathfrak{T}_k^{\mu,\nu}u(\eta_0)]'}\right) &= \mathrm{Re}\left(\frac{\eta_0 w'(\eta_0)}{1-w(\eta_0)} + \frac{1}{1-w(\eta_0)}\right)\\ &= \mathrm{Re}\left(\frac{qe^{i\theta}+1}{1-e^{i\theta}}\right) = \frac{1-q}{2} \leq -\kappa,\end{aligned}$$

where $q-1 \geq (k-1)-1 = k-2 \geq (2\kappa+2)-2 = 2\kappa$. Which contradicts (17). Hence, $|w(\eta)| < 1, \eta \in \Sigma$. This completes the proof.

Next, we proceed to show that the $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'$ is dominated by the JSF. Let $M_{\wp_m}$ is the set of all holomorphic functions of the form $\wp(\eta) = 1 + \wp_m\eta^m + \wp_{m+1}\eta^{m+1}, ..., \quad \eta \in \Sigma$.

Theorem 3. *Consider the following assumptions:*

(i) $-1 \leq \flat < \aleph \leq 1$, $-1 \leq \jmath < \imath \leq 1$, *and* $\Bbbk\jmath \leq 0$.
(ii) $0 < \wp_1' = 2\wp_1/(\aleph - \flat) \leq 2$.
(iii)

$$\begin{aligned}&\Big((\imath-\jmath)(1-\flat)^2 + 16\Bbbk\jmath(\aleph-\flat)\Big)^2 + (\imath-\jmath)^2(1+\flat)^4(2-\wp_1')^2\\ &\leq 16\Bbbk^2(\aleph-\flat)^2 + 2(\imath-\jmath)(1+\flat)^2\big((\imath-\jmath)(1-\flat)^2 - \Bbbk\jmath(\aleph-\flat)\big)4(2-\wp_1').\end{aligned} \tag{21}$$

If

$$1 + \Bbbk\eta[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'' \prec \frac{1+\imath\eta}{1+\jmath\eta}, \tag{22}$$

then

$$[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' \in \mathbb{J}[\aleph, \flat].$$

Proof. Define $\eth : \Sigma \to \mathbb{C}$ by

$$\eth(\eta) = \frac{(\aleph-1) + (1-\flat)[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}{(\aleph+1) - (1+\flat)[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'}, \tag{23}$$

where $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' \in \mathcal{H}_{\wp_1}$. Clearly, $\eth$ is analytic on Σ and

$$\eth(\eta) = 1 + 2\wp_1'\eta^m + \eth_{m+1}\eta^{n+1} + \cdots \in \mathcal{H}_{\wp_1'},$$

whenever $0 < \wp_1' = 2\wp_1/(\aleph - \flat) \leq 2$. By (22) and (23), we have

$$\Re\left\{\frac{(\imath-\jmath)\big((1+\flat)\eth(\eta)+(1-\flat)\big)^2+2\Bbbk(1-\jmath)(\aleph-\flat)\eta\eth'(\eta)}{(\imath-\jmath)\big((1+\flat)\eth(\eta)+(1-\flat)\big)^2-2\Bbbk(1+\jmath)(\aleph-\flat)\eta\eth'(\eta)}\right\}>0.$$

Define $\top : \mathbb{C}^2 \to \mathbb{C}$ by

$$\top(r,s)=\frac{(\imath-\jmath)\big((1+\flat)r+(1-\flat)\big)^2+2\Bbbk(1-\jmath)(\aleph-\flat)s}{(\imath-\jmath)\big((1+\flat)r+(1-\flat)\big)^2-2\alpha(1+\jmath)(\aleph-\flat)s}.$$

Then $\top$ is continuous of r and s on $\triangle =: \mathbb{C}^2 - \{(r,s) : M(r,s) = 0\}$, where

$$M(r,s)=(\imath-\jmath)\big((1+\flat)r+(1-\flat)\big)^2-2\alpha(1+\jmath)(\aleph-\flat)s.$$

Note that $(1,0) \in \triangle$ and $\Re(\top(1,0)) > 0$. Moreover, for all $(ia,b) \in \triangle$,

$$\Re\{\top(ia,b)\}=\Re\left\{\frac{(\imath-\jmath)(1-\flat)^2+2(1-\flat^2)(\imath-\jmath)i\rho-(\imath-\jmath)(1+\flat)^2a^2+2\Bbbk(1-\jmath)(\Bbbk-\flat)b}{(\imath-\jmath)(1-\flat)^2+2(1-\flat^2)(\imath-\jmath)ia-(\imath-\jmath)(1+\flat)^2a^2-2\Bbbk(1+\jmath)(\aleph-\flat)b}\right\}.$$

Denoting by

$$x=(\imath-\jmath)(1-\flat)^2,\quad y=2\alpha(1-\jmath)(\aleph-\flat),$$
$$z=-(\imath-\jmath)(1+\flat)^2,\quad w=2(\imath-\jmath)(1-\flat^2),\quad v=-2\Bbbk(1+\jmath)(\aleph-\flat).$$

Then $\Re\top(ia,b) = \Re\big\{(x+by+za^2+wia)/(x+vb+za^2+wia)\big\}$. For $\Re\top(ia,b) < 0$, we have to prove

$$x^2+x(y+v)b+yvb^2+\big(2xz+w^2+z(y+v)b\big)a^2+z^2a^4<0.$$

But $b \leq -1/2$, and $\Bbbk\jmath < 0$, then, we have

$$\begin{aligned}2xz+w^2+(y+v)zb &\geqslant 2xz+w^2-\frac{1}{2}(y+v)c\\ &=2(\imath-\jmath)(1+\flat)^2\left((\imath-\jmath)(1-\flat)^2-\Bbbk\jmath(\Bbbk-\flat)\right)>0. \quad (24)\end{aligned}$$

In addition, we have $a^2 \leq -\big(\frac{2(2+\wp_1')b}{(2+\wp_1')+(2-\wp_1')}+1\big)$, where $\wp_1' = 2\wp/(\Bbbk-\flat)$; thus if (24) holds, then we get

$$\begin{aligned}&x^2+x(y+v)b+yvb^2+\big(2xz+w^2+(y+v)zb\big)a^2+z^2a^4\\ &\leq\Big(yv-\frac{2(2+\wp_1')(y+v)z}{(2+\wp_1')+(2-\wp_1')}+\frac{4(2+\wp_1')^2z^2}{\big((2+\wp_1')+(2-\wp_1')\big)^2}\Big)b^2\\ &\quad+\Big((x-z)(y+v)+(2xz+w^2)\left(\frac{-2(2+\wp_1')}{(2+\wp_1')+(2-\wp_1')}\right)\\ &\quad+\frac{4(2+\wp_1')z^2}{(2+\wp_1')+(2-\wp_1')}\Big)b+\big(x^2-2xz-w^2+z^2\big). \quad (25)\end{aligned}$$

According to (25) when $\ell(b) =: rb^2 + sb + t$, where

$$r = yv - \frac{2(2+\wp_1')(y+v)z}{(2+\wp_1')+(2-\wp_1')} + \frac{4(2+\wp_1')^2 z^2}{\big((2+\wp_1')+(2-\wp_1')\big)^2},$$
$$s = (x-z)(y+v) + (2xz+w^2)\left(\frac{-2(2+\wp_1')}{(2+\wp_1')+(2-\wp_1')}\right) + \frac{4(2+\wp_1')z^2}{(2+\wp_1')+(2-\wp_1')},$$
$$t = x^2 + z(z-2x) - w^2.$$

In view of the relation

$$\ell(b) \leq \max_{b\leq -\frac{1}{2}} \ell(b) = \frac{4t-2s+r}{4}, \tag{26}$$

we have $\max_{\sigma\leq -\frac{1}{2}} \ell(b) \leq 0$ is equivalent to

$$\begin{aligned} 4t-2s+r = {} & \big(4x^2+yv-2x(y+v)\big)\big((2+\wp_1')+(2-\wp_1')\big)^2 \\ & -2\big(2w^2+4xz-(y+v)z\big)\big((2+\wp_1')+(2-\wp_1')\big)(2-\wp_1') \\ & +4z^2\big(2-\wp_1'\big)^2 \leq 0. \end{aligned}$$

Replacing x, y, z, w and v in above formula, we get

$$\begin{aligned} \ell(b) \leq {} & 4\Big(\big((\imath-\jmath)(1-\flat)^2+\alpha\jmath(\aleph-\flat)\big)^2 - 16\Bbbk^2(\aleph-\flat)^2\Big) + 4(\imath-\jmath)^2(1+\flat)^4(2-\wp_1') \\ & -8(\imath-\jmath)(1+\flat)^2\big((\imath-\jmath)(1-\flat)^2-\Bbbk\jmath(\aleph-\flat)\big)4(2-\wp_1') \leq 0. \end{aligned}$$

If inequality (21) occurs, then this yields that $\top(r,s)$ achieves the conditions of Lemma 2, which ends the proof.

5 Conclusion

In a complex domain, an extension of the Tremblay differential operator is derived using the $k-$symbol fractional calculus. This operator is used to look at the analytical solutions of fractional differential equation. Our strategy is founded on a number of concepts, including Jack Lemma, subordination, and superordination. Some of the geometric properties are investigated and explored for the normalized operator, such as univalency and convexity. The bounded turning property is also studied to provide an analytic solution of a class of fractional differential equations, using the expression $[\mathfrak{T}_k^{\mu,\nu}u(\eta)]'$. Our last result is about the upper bound of the expression in terms of the Janowski starlike functions

$$[\mathfrak{T}_k^{\mu,\nu}u(\eta)]' \prec \frac{1+\aleph\eta}{1+\flat\eta}, \quad \eta\in\Sigma.$$

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