

Solving 3SAT and MIS Problems with Analog Quantum Machines

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Abstract. This work considers the use of analog quantum machines to solve the boolean satisfiability problem 3SAT by taking Quadratic Unconstrained Binary Optimization models (QUBO) as input. With the aim of using real quantum computers instead of emulators to solve instances of the problem, we choose the D-Wave quantum machines, which have a static topology and limited connectivity. Therefore, the choice of the problem formulation must take these important constraints into account. For this reason, we propose to solve 3SAT instances through polynomial-time reduction to the Maximum Independent Set problem. This is because the resulting graph is less dense and requires lower connectivity than the one that would be produced by directly modeling 3SAT into a QUBO.

Keywords: Quantum Computing · Quantum Annealing · 3SAT · Maximum Independent Set · Combinatorial optimization

1 Introduction

Analog quantum machines are currently the most advanced quantum computers for solving small to medium-sized instances of combinatorial optimization problems. Quantum universal gate-based computers, such a s the IBM numerical machines, are based on NISQ technology (Noisy Intermediate-Scale Quantum) and have an error rate too high to solve anything other than very small instances. Additionally, the number of quantum bits (qubits) in available machines is insufficient in 2023.

There are several types of analog quantum computers, including Pasqal and D-Wave machines. The former is expected to release in 2023 a real (i.e., not an emulator) computer based on Rydberg atoms, but today only the latter offers real quantum machines with a large number of qubits. Both consider transverse-field Ising models, and users can employ Quadratic Unconstrained Binary Optimization models (QUBO) since they are isomorphic to Ising models. The main differences between the two types of analog quantum computers are as follows: while a Pasqal machine can dynamically create a qubits network according to the QUBO, D-Wave machines have a static topology that must be taken into

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account since the qubits graphs can have a connectivity and topology that do not necessarily correspond to the QUBO graph. From its input (QUBO), the D-Wave machine automatically transforms the model to map it into the qubits graph. This process, called the "embedding process," is another optimization problem and can be very time-consuming. As a result, the number of qubits is larger than the number of vertices from the QUBO, and an embedding solution might not even be found.

In this work, we consider several static topologies from the three latest D-Wave machines: Chimera, Pegasus, and Zephyr, for solving the 3SAT problem. This problem, which we will define in this paper, could directly be modeled by a QUBO by relaxing the clause satisfaction constraints in the objective function. However, it is costly in terms of the number of expressions in the objective function, and this implies a difficult embedding problem. The main contribution of this work is to experiment with the polynomial-time reduction from the 3SAT problem to the Maximum Independent Set problem (MIS) for solving 3SAT instances with a quantum computer. The QUBO of the MIS is relatively simple and seems to be more adaptable to the topologies of the D-Wave machines. In these topologies, the connectivity (i.e., the vertex degrees) is limited.

Even if such a transformation implies that the 3SAT instance is no more difficult than the MIS instance we obtain, the density of the graph might be more convenient to be mapped into the qubits graph compared to the one we would obtain directly from a 3SAT QUBO.

The remainder of this paper is organized as follows. Section 2 introduces the machines used in this work. The 3SAT problem and the MIS optimization problem are described in Sect. 3. The method to transform a 3SAT instance into a MIS instance and the process for obtaining a solution to the 3SAT from a solution to the MIS are presented in Sect. 4. In the final section, Sect. 5, we report on quantum computational experiments.

2 Quantum Annealing and D-Wave Machines

In this work, we focus on the D-Wave quantum machines. These quantum computers are available through the cloud and have up to 5,000 qubits. Although they are not programmable like universal gate-based machines (e.g., IBM quantum machines), their technology directly optimizes Quadratic Unconstrained Binary Optimization models (QUBO). In short, they attempt to reach the ground state of an Ising spin glass system configured in a way that corresponds (indirectly) to the search for the minimum value of a QUBO (Ising models and QUBO are isomorphic; you can refer to [4] and [10] for more information about modeling general optimization problems with QUBO and Ising models, respectively).

The resolution process is adiabatic[1] and is based on quantum annealing [7], which is theoretically proven to be more efficient than the simulated annealing

[1] See [1] for more information about adiabatic systems.

meta-heuristic [9] where quantum fluctuations replace temperature changes. The machine intrinsically executes quantum annealing through its hardware. Like universal gate-based quantum machines, quantum annealing machines execute the process several times (the "shots" are here called anneals). The number of anneals is part of the input and is equal to the number of solutions we obtain in the output. An anneal time can also be given as input, and since the machine can be used for approximately one second in total, the number of anneals and the annealing time must respect this bound (some non-linear supplementary times also contribute to the total).

The qubits network is static: the topology cannot be dynamically adapted to the QUBO (or Ising model) given as input. The connectivity of each topology, which can be viewed as the degree of the qubits graph, is crucial. For instance, if a QUBO formulation considers a complete graph, such a specific graph is not directly available in the topology. In such a case, the QUBO must be transformed into another problem, but this time able to be mapped into the machine. This embedding process is automatically done by the machine. The optimization is then performed on a larger graph. The population of qubits used, which is larger than the number of vertices related to the QUBO, sometimes fails to correspond exactly to the initial problem, especially for some qubit pairs called logic qubits that fail to take the same value. This problem is called *Chain Breaks*. The three representations in Fig. 1 show the three latest topologies of D-Wave machines, while Table 1 provides important information such as the number of qubits and the related connectivity.

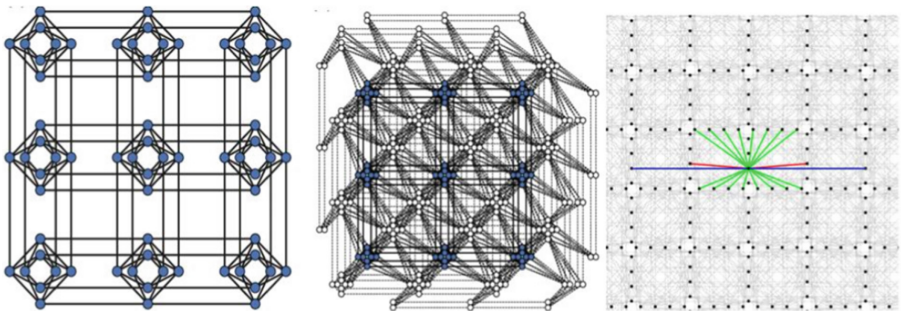


Fig. 1. From left to right: Chimera, Pegasus and Zephir D-Wave machine topologies. More information is given in Table 1.

The embedding process involves solving another optimization problem and can be very time-consuming, to the point of not being able to provide a mapping. To anticipate this significant issue, the user's machine can take it into account while formulating the QUBO. This is an important aspect of this work: we could create a QUBO to solve the 3SAT problem, but its direct formulation as a QUBO tends to have difficulties being embedded in the machine due to the topology of the qubits graph. Taking this into account, we use the polynomial-time reduction

Table 1. D-Wave quantum computing machines

Topology	Chimera	Pegasus	Zephyr
Name Machine	DW_2000Q	Advantage_System	Advantage2_prototype
Machine Version	6	6.1	1.1
Number of working Qubits	2041	5616	563
Connectivity	6	15	20
Annealing Time Range	[1,2000]	[0.5,2000]	[1,2000]

to the MIS to obtain a QUBO more suitable for the topology and, in turn, have a less time-consuming embedding process.

The QUBO model $f^{QUBO}(x)$, which we can provide as input to the machine, is given by the expression (1) with x as a binary vector and Q as the cost matrix.

$$f^{QUBO}(x) = x^T Q x = \sum_i Q_{i,i} x_i + \sum_{i<j} Q_{i,j} x_i x_j. \tag{1}$$

3 A Satisfaction Problem: 3SAT, and a Combinatorial Optimization Problem: MIS

The 3SAT is a Boolean satisfiability problem. The goal is to determine if there exists a solution satisfying a conjunction of clauses, where each clause is a disjunction of 3 literals (i.e., variables or negations of variables). The 3SAT problem is NP-Hard (the proof is given by [8]; you can also refer to [2] for a general survey on satisfiability problems). We denote c_i as a clause with $i = 1, \ldots, |\mathcal{C}|$ from the set of clauses $\mathcal{C}$, and $\mathcal{V}$ as the set of variables such that we denote each variable as $v_j, j = 1, \ldots, |\mathcal{V}|$.

Let's take an example where the 3SAT problem formulated in expression (2) has at least two solutions: ($v_1 = 1$, $v_2 = 1$, $v_3 = 1$) and ($v_1 = 0$, $v_2 = 0$, $v_3 = 0$).

$$(v_1 \vee v_2 \vee \neg v_3) \wedge (\neg v_1 \vee v_2 \vee v_3) \wedge (v_1 \vee \neg v_2 \vee v_3). \tag{2}$$

Not all variables must be part of each clause, but this simple example allows us to introduce the one-in-three 3SAT, where a solution satisfies all the clauses and each clause must be satisfied by exactly one variable. The simple example in expression (2) does not have such a solution.

The MIS problem is a combinatorial optimization problem considering a simple undirected graph $\mathcal{G}$ with a set of vertices $\mathcal{X}$ and a set of edges $\mathcal{E}$. In this problem, we search for the largest subset of vertices $\mathcal{S} \subseteq \mathcal{X}$, also called the maximum independent set, in such a way that no two vertices in $\mathcal{S}$ can be adjacent (i.e., no vertex of $\mathcal{S}$ can be directly connected by an edge of $\mathcal{E}$ to another vertex of the same set). The MIS problem is NP-Hard ([3]; please refer to [5] for an interesting review of this problem).

To model the MIS through a linear program, we denote x as a decision variable vector, where each element $x_k, k \in \mathcal{X}$, takes a $\{0; 1\}$ value such that:

$$x_k = \begin{cases} 1 \text{ if the vertex } k \text{ is in the independent set, i.e., } k \in \mathcal{S}, \\ 0 \text{ otherwise.} \end{cases} \tag{3}$$

Maximizing the cardinality of $\mathcal{S}$ means maximizing the objective function $f_{MIS}(x)$ defined by expression:

$$f_{MIS}(x) = \sum_{k \in \mathcal{X}} x_k. \tag{4}$$

The set of constraints, which, for each edge (k, l) in $\mathcal{E}$, forbids that both vertex k and vertex l are in $\mathcal{S}$, is given by the inequalities:

$$x_k + x_l \leq 1 \qquad \forall (k, l) \in \mathcal{E}. \tag{5}$$

In the scope of using a quantum machine that takes a QUBO as input, we reformulate the set of constraints (5) with the quadratic constraints:

$$x_k x_l = 0 \qquad \forall (k, l) \in \mathcal{E}. \tag{6}$$

We denote λ as the multiplier of the constraints (6) relaxed in the objective function of the QUBO. Considering a correct λ value, searching for a Maximum Independent Set in the undirected graph $\mathcal{G}$ can be done by minimizing the QUBO function $f_{MIS}^{QUBO}(x)$:

$$f_{MIS}^{QUBO}(x) = -\sum_{k \in \mathcal{X}} x_k + \lambda \sum_{(k,l) \in \mathcal{E}} x_k x_l. \tag{7}$$

The minimization of the first expression $-\sum_{k \in \mathcal{X}} x_k$ of $f_{MIS}^{QUBO}(x)$ tends to select the largest number of vertices in the set $\mathcal{S}$ since this is the opposite of the objective function (4). The minimization of the second expression $\lambda \sum_{(k,l) \in \mathcal{E}} x_k x_l$ corresponds to the relaxation of the quadratic constraints (6) weighted by the multiplier λ. We can easily see that, for each constraint not satisfied, which means for two adjacent vertices in $\mathcal{S}$, the objective function will have a penalty of λ.

4 Solving a 3SAT Instance on a Quantum Computer Using Polynomial-Time Reduction to the Maximum Independent Set Problem

Since the QUBO related to a 3SAT instance is difficult to embed into the qubits graph due to its topology, we searched for different methods to solve the boolean satisfiability problem, especially models with a less dense graph. The polynomial-time reduction from SAT to MIS, since $3SAT \leq_p MIS$, allows us to solve 3SAT by resolving MIS.

From an instance of the 3SAT problem, where each binary variable is denoted as $v_j, j = 1..|\mathcal{V}|$, and its negation as $\neg v_j$, we denote a binary variable $y_{i,j}$ with $i = 1..|\mathcal{C}|, j = 1..|\mathcal{V}|$, independently of the negation if applicable. It takes a value from $\{0, 1\}$ such that:

$$y_{i,j} = \begin{cases} 1 \text{ if the vertex } (i,j) \text{ is in the independent set } \mathcal{S}, \\ 0 \text{ otherwise.} \end{cases} \tag{8}$$

Each variable $y_{i,j}$ is represented by a vertex in the set $\mathcal{X}$ of a graph $\mathcal{G}$. Each edge e in the set $\mathcal{E}$ exists for one of the following two reasons:

- the two connected variables belong to the same clause $i, i = 1..|\mathcal{C}|$,
- the two connected variables correspond to the same original variable $j, j = 1..|\mathcal{V}|$, and one is the negation of the other.

Even though the total number of variables is always $3 * |\mathcal{C}|$, it is easy to see that the density of the resulting graph is related to the redundancy of each original 3SAT variable and its negation across different clauses. In short, instances with fewer variable occurrences than the graph's connectivity may have a better chance of fitting into the topology of the qubits graph.

The small 3SAT example defined by the expression (2) is transformed into a graph of the MIS problem in Fig. 2. It involves literals that are colored blue or orange, representing variables and their negations, respectively. The binary variables $y_{i,j}$ are associated with clause i and 3SAT variable j. The first clause C_1 is represented by the first K_3 complete subgraph. Each 3-vertex clique enforces a maximum of one y_{ij} to be equal to 1. However, it is notable that a solution to the MIS problem does not necessarily correspond to a one-in-three 3SAT instance since the negation of 3SAT variables is not considered in the MIS variables. Negations are taken into account through edges connecting different clauses. For instance, if a variable v_{SAT} appears in the first clause C_1, while its negation $\neg v_{SAT}$ appears in the second clause C_2, then there is an edge $(y_{C_1,v_{SAT}}, y_{C_2,\neg v_{SAT}})$ in $\mathcal{E}$.

If the resolution of the MIS provides an optimal solution, a solution to the 3SAT problem exists if the cardinality of set $\mathcal{S}$ is equal to the number of clauses. Otherwise, the 3SAT instance cannot be satisfied. In fact, if the MIS resolution does not have exactly one $y_{i,j}$ equal to 1 for each 3-vertex clique i, not all clauses will be satisfied. Assuming such an optimal solution exists, we can deduce the value of the 3SAT variables $v_j, j = 1..|\mathcal{V}|$, from each $y_{i,j} = 1$ as follows:

$$v_j = \begin{cases} 0 \text{ if the variable } v_j \text{ appears as a negation in the clause } i, \\ 1 \text{ otherwise.} \end{cases} \tag{9}$$

Continuing with the example given in expression (2) and the graph obtained in Fig. 2, we can deduce the values of the original variables as follows:

$$\begin{cases} x_1 = y_{1,1} = 1 \\ x_2 = y_{3,2} = 1 \\ x_3 = \neg y_{2,3} = 0. \end{cases} \tag{10}$$

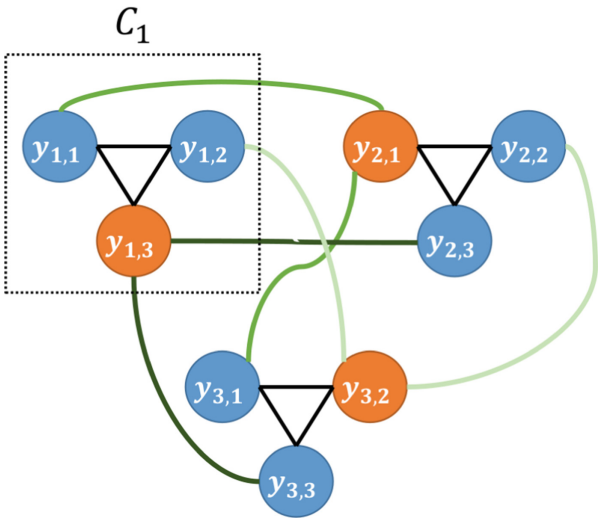


Fig. 2. The MIS graph representation obtained from the boolean satisfiability problem given by the expression (2).

Note that the solution may not be a solution of the one-in-three 3SAT problem, even if the solution of the MIS, with exactly one vertex in $\mathcal{S}$ per 3-vertex clique from the clauses, suggests that exactly one literal will satisfy each clause. For example, all literals satisfy the first clause of the expression (2).

Figure 3 reports the steps of the resolution process from a 3SAT instance to the solution. The main steps of the resolution scheme are as follows. User actions are colored in blue, while machine actions are in green. The 3SAT instance is first reduced to a MIS instance. The latter is formulated through a Quadratic Unconstrained Binary Optimization model (QUBO), which is then given as input to the quantum machine. The machine transforms the QUBO to embed it into the topology of the qubits network. If such an embedding is found, the problem is solved multiple times (i.e., several anneals) by the quantum annealer, which then reconstitutes the MIS solutions. The user can then take all the optimal solutions found. If the number of vertices in the independent set is equal to the number of clauses, all of these solutions, once transformed, satisfy the 3SAT instance.

5 Quantum Computational Experiments

Preliminary results have been obtained on small instances by testing the Chimera and Pegasus topologies. Figure 4 highlights the importance of topology, even if the available connectivity is higher than that required by the graph represented by the QUBO. For instance, the 4 variables and 6 clauses 3SAT instance gives a small graph with 18 variables. We can see from Fig. 4 that the degree of the

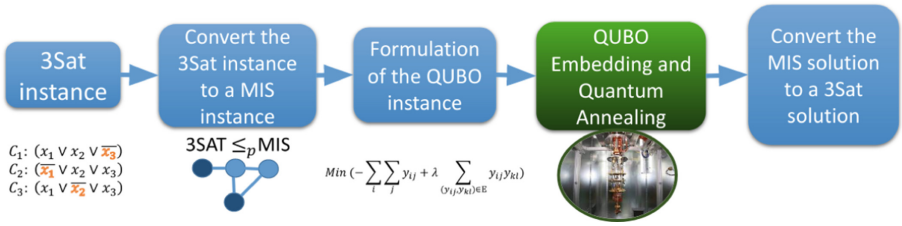


Fig. 3. Main steps of the resolution scheme.

vertices (see A and C) does not exceed 6, which is expected since each vertex is adjacent to the two other vertices in its 3-vertex clique and can also be adjacent to its negation through the 5 other clauses (since all variables have here more than one negation, the degree cannot be 7). Table 1 shows that Chimera has a connectivity of 6 (which seems sufficient for our instance), while Pegasus has a connectivity of 15. The embedding process in the 2000Q machine uses significantly more qubits (61) with its Chimera topology compared to Pegasus of the Advantage machine. After the embedding process, 1000 anneals were executed on the machines, and both gave all the optimal solutions of the MIS and, consequently, all the solutions of the 3SAT instance. The Zephyr topology was also tested with similar success, although not all data about the embedding process were available.

Focusing on the Pegasus topology, we conducted experiments on a challenging instance of the 3SAT problem with 11 clauses and 4 variables. The $3SAT \leq_p MIS$ reduction gives a graph with 33 variables. The graph cannot be considered a sparse graph, but rather a dense graph, since the number of edges is high: all edges for the 11 3-vertex cliques and other edges connecting the same variables and their negations between the different 3-vertex cliques. We used several anneal times balanced by the number of anneals to study the best configuration for such a specific graph. The results are reported in Table 2, which contains the *Chain Breaks Rate* and the number of optimal solutions obtained through the anneals. These results had been obtained on the quantum machine Advantage 6.1 (Pegasus topology).

Table 2. Results on a 4 variables and 11 Clauses 3SAT instance according to the *Annealing Time* et the number of Anneals.

Annealing Time (µs)	|Anneals|	Chain Breaks Rate	|Optimal Solutions|
0.5	7000	High	92
1	7000	Average	6
10	7000	Average	2
100	2500	Average	3
1000	500	Average	1
2000	250	High	0

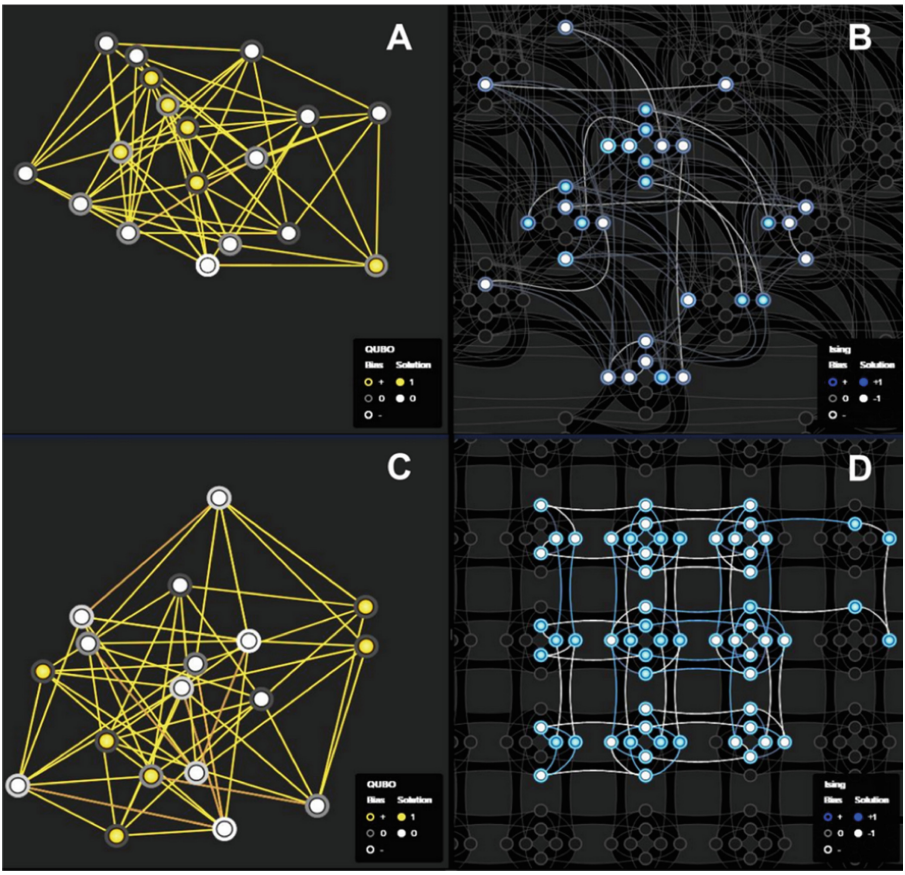


Fig. 4. Results obtained on two different quantum machines: Advantage and D-Wave 2000Q. Graph representations (A) and (C) show the QUBO graphs of the same instance for the two experiments, while qubit graphs (B) and (D) show the result of the embedding process from the QUBO according to the topology of the Advantage machine (Pegasus) and the 2000Q machine (Chimera), respectively. The 3SAT instance has 4 variables and 6 clauses. After transforming the instance to a MIS instance, the problem has 18 variables in the QUBO (3 variables and 6 clauses). We can see the importance of topology: the more recent machine (Advantage) requires 31 qubits (B), while the previous generation (2000Q) needs 61 (D).

Analyzing the results in Table 2, we observed some surprising outcomes. For an annealing time of $\{0.5, 1, 10\}$, the machine allows the execution of 7000 anneals for these three cases, and the number of optimal solutions obtained is higher with a shorter annealing time (0.5μs) compared to the longest (10μs) of the three. Since an adiabatic process is related to a system remaining in its state (here: the ground state) by giving slow enough perturbations, we would expect that a longer annealing time would help to stay in the ground state (i.e., give the

optimal solution after measurement). However, for these experiments, the *Chain Breaks Rate* reported in Table 2 shows that the machine had more difficulties keeping qubits coupled with the shortest and longest annealing time.

Finally, we conducted experiments to solve small 3SAT literature instances. We tested the 20 variables and 90 clauses of [6]. Although the set of 270 vertices of the MIS graph we obtained from these instances seems small compared to the 5616 qubits available in the Advantage machine, no embedding was found by the machine for any of these instances.

6 Conclusion

In this work, the 3SAT problem is tackled using an analog quantum computer from D-Wave, employing a polynomial-time reduction to the MIS. The aim of this resolution scheme is to generate a QUBO that is easier to map to the qubit graph. This mapping, known as "embedding" is time-consuming due to the static topologies of the qubit graph. When a 3SAT problem instance is reduced to an MIS instance, the embedding seems to have fewer difficulties finding a QUBO mapping, as the degree of the related MIS graph is smaller compared to a direct transformation from a 3SAT instance to a QUBO. The vertex degree is directly linked to the occurrences of the related variables and their negations. Experiments suggest that using such polynomial-time reductions, typically employed in complexity theory, increases the likelihood of solving a 3SAT instance. However, the static topology of D-Wave machines presents challenges not only in terms of degree but also in terms of the topology itself (e.g., a chain of vertices does not necessarily have a corresponding chain in the qubits graph).

Two projects seem to emerge for future work. First, the resolution of combinatorial optimization problems for which a polynomial-time reduction to the MIS exists should be studied, as this problem yields a QUBO that quantum machines can handle relatively easily (e.g., clique, coloring, and cover problems).

Second, other analog computers already exist, and more will be released in the coming years. For example, Pasqal machines also take a QUBO as input but do not have a static topology. It could be interesting to generate benchmarks on both machines based on the resolution scheme presented in this paper.

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