A Quantum Annealing Solution to the Job Shop Scheduling Problem

Riad Aggoune[1(✉)] and Samuel Deleplanque[2]

[1] ITIS Department, Luxembourg Institute of Science and Technology, 5 Av. des Hauts-Fourneaux, L-4362 Esch-sur-Alzette, Luxembourg
riad.aggoune@list.lu

[2] CNRS, Centrale Lille, JUNIA, Univ. Lille, Univ. Valenciennes, UMR 8520 IEMN, 41 boulevard Vauban, 59046 Lille Cedex, France
samuel.deleplanque@junia.com

Abstract. We consider in this paper the job shop scheduling problem. We first present a mathematical formulation of the problem, namely a disjunctive model. Then, we show how the model can be reformulated as a Quadratic Unconstrained Binary Optimization (QUBO) problem, that can be solved by an analog quantum computer.

Keywords: quantum computing · optimisation · QUBO · job shop scheduling

1 Introduction

Quantum optimization which is the use of quantum computers and algorithms for solving complex optimization problems, is one of the topics in quantum computing with the highest potential. Two types of approaches are generally used to solve combinatorial optimization problems with quantum computers: exact methods such as the Grover's search algorithm [8], and meta-heuristics such as the Quantum Annealing [9] and the Quantum Approximate Optimization Algorithm (QAOA) [6]. Exact methods and variational methods like QAOA make use of universal gate-based quantum computers, such as the ones developed by the IBM company. Quantum annealing is designed for analog quantum computers developed by D-Wave. Solving a combinatorial optimization problem with heuristics generally requires a transformation of the problem to a format suitable for the quantum computer. While some existing frameworks[1] can cope with Constrained Quadratic Models (CQM), Quadratic Unconstrained Binary Optimization (QUBO) is the best option to map an optimization problem to a quantum computer or simulator. We consider in this paper the job shop scheduling problem. We first review in the following Sect. 2 the quantum-based methods recently proposed in the literature. Then in Sect. 3 we propose a QUBO formulation of the scheduling problem. Finally, in Sect. 4 numerical results obtained with D-Wave quantum annealing machines and conclusions are presented.

[1] https://cloud.dwavesys.com/leap.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
O. Gervasi et al. (Eds.): ICCSA 2023 Workshops, LNCS 14104, pp. 421–428, 2023.
https://doi.org/10.1007/978-3-031-37105-9_28

2 Problem Definition

The job shop scheduling problem can be stated as follows: A set of n jobs $J = \{J_1, J_2, \ldots, J_n\}$ has to be processed on a set of m machines $M = \{M_1, \ldots, M_m\}$. Each job J_i consists of a linear sequence of n_i operations $(O_{i1}, O_{i2}, \ldots, O_{in_i})$. Each machine can process only one operation at a time and each operation O_{ij} with a processing time of p_{ij} time units needs only one machine. Each job visits the machines according to its own predefined routing. This problem generalizes the flow shop scheduling problem, in which all the jobs are processed following the same routing $(M_1, M_2, \ldots, M_m)$. The objective is to determine the starting date of each operation O_{ij} so that the makespan noted C_{max} is minimized. The problem is NP-hard for $n > 2$ and $m > 2$ [7].

The traditional solution approaches to solve job shop scheduling problems include heuristics and meta-heuristics as well as exact methods, such as branch-and-bound and constraint programming [3].

The linear disjunctive model [11] for the job shop scheduling problem can be expressed as follows. The starting times are represented by the integer variable vector, denoted by x. We use z to denote the binary variable vector, which satisfies the following conditions:

$$z_{ijk} = \begin{cases} 1 \text{ if the job } j \text{ precedes job } k \text{ on machine } i, \\ 0 \text{ otherwise.} \end{cases}$$

We note by $(\sigma_1^j, \ldots, \sigma_h^j, \ldots, \sigma_m^j)$ the processing order of job j through the machines. The minimization of the objective function (1) forces all the jobs to be finished as soon as possible.

$$\sum_{j \in J} x_{\sigma_m^j j} \tag{1}$$

The makespan C_{max} can be minimized by adding the following constraints:

$$C_{max} \geq \sum_{j \in J} x_{\sigma_m^j j} + p_{\sigma_m^j j} \tag{2}$$

Constraints (3) forbid consecutive operations of one job to start before the previous one is finished.

$$x_{\sigma_h^j j} \geq x_{\sigma_{h-1}^j j} + p_{\sigma_{h-1}^j j} \qquad \forall j \in J, h = 2..m \tag{3}$$

Constraints (4) and (5) forbid to have more than one operation at a time on a given machine.

$$x_{ij} \geq x_{ik} + p_{ik} - M z_{ijk} \qquad \forall j, k \in J, j < k, i \in M \tag{4}$$

$$x_{ik} \geq x_{ij} + p_{ij} - M(1 - z_{ijk}) \qquad \forall j, k \in J, j < k, i \in M \tag{5}$$

3 Related Works

The scientific literature on quantum solutions for hard combinatorial optimization problems is becoming significant and diversified. The studies of shop scheduling problems are quite recent and scarce. They can be classified according to the types of quantum computers and algorithms used to solve the problems: analog, universal computers, and simulators. In general, the solution approaches consist in first mapping the decision variables of the considered problems to the qubits of the quantum computer. Then, quantum algorithms are applied to make the qubits value evolve until solutions are found. Solving optimization problems with quantum computers is therefore strongly limited by the number of qubits available, among other hardware constraints.

The first paper in the domain is from Venturelli *et al.* [15]. The authors proposed a QUBO formulation and a quantum annealing solution for the job shop scheduling problem with the makespan objective. The method was implemented on a D-Wave quantum annealer, with 509 working qubits. The authors also introduced window shaving and immediate selections techniques to reduce the number of considered variables. Their model has then been re-used in several studies, as listed below.

In [10] the authors have developed a hybrid quantum annealing heuristic to solve a particular instance of the job shop scheduling problem on the D-Wave 2000Q quantum annealing system that consists of 2041 qubits and a maximum of 6 connections between qubits. The proposed approach includes variable pruning techniques and a processing window heuristic. In [4], job shop instances with unitary operations have been tested on the D-Wave Advantage machine, built upon 5640 qubits and 15 possible connections between qubits. Extensive experiments with the reverse annealing procedure and comparisons with simulated annealing are also described. In [1], a QUBO formulation is proposed for the minimisation of the total completion time in a job shop. The model is solved using the D-Wave hybrid solver and Advantage quantum annealing computer.

A generalization of the job shop scheduling problem with pools of parallel machines available for processing operations was considered in [5]. The authors proposed a QUBO derived from the one of [15] and an iterative procedure to solve relatively large size instances on a specialized hardware[2]. Using the QUBO formulations proposed in [5], the authors in [14], tackle the flexible job shop scheduling problem with the D-Wave solvers. Another QUBO formulation is proposed in [12] for assigning dispatching rules to the machines and scheduling the operations in a flexible job shop system. The problem is solved using the leap hybrid solver. In [13], the authors propose a QUBO formulation for the job shop scheduling with worker assignment considerations. Possible ways to approximate the makespan are discussed and instances solved with the Fujitsu

[2] https://www.fujitsu.com/global/services/business-services/digital-annealer.

Digital Annealer are described. In the same environment, the authors in [16] efficiently solve large instances of the job shop scheduling problem with a hybrid approach that combines constraints programming and QUBO models for one-machine problems.

Finally, the only study involving gate-based computers is due to [2]. The authors have proposed four variational quantum heuristics for solving a job shop scheduling problem with early and late delivery as well as production costs, adapted from a steel manufacturing process. They have compared the performance of the heuristics on two-machine flow shop instances using IBM quantum processors with 5 to 23 qubits.

4 QUBO Formulation

The boolean variable x_{ij}^t takes the value 1 if the operation i of the job j starts in period t, with $i = 1..n_i$, $j = 1..n$, $t = 1..T$, and takes the value 0 otherwise. We note $M_{ij}, i = 1..n_i, j = 1..n$, the required machine for the operation i of the job j. The minimization of the Objective function (6) forces the last operations of all jobs to start globally as soon as possible.

$$\sum_j \sum_t t.x_{n_i j}^t \tag{6}$$

We force each operation to start exactly once with the following set of constraints (7) to be relaxed in the Objective function.

$$(\sum_t x_{ij}^t - 1)^2, \qquad i = 1..n_i, j = 1..n \tag{7}$$

Constraints (8) forbid to have more than one operation at a time on a given machine.

$$\begin{gathered} x_{ij}^t x_{i'j'}^{t'} = 0, \\ \forall (i,j,t) \cup (i',j',t') : i,i' = 1..n_i, j,j' = 1..n, (i,j) \neq (i',j'), \\ M_{ij} = M_{i'j'}, (t,t') \in T^2, 0 \leq t' - t < p_{ij} \end{gathered} \tag{8}$$

Constraints (9) forbid consecutive operations to start before the previous one is finished.

$$x_{ij}^t x_{i+1j}^{t'} = 0, \quad i = 1..(n_i - 1), j = 1..n, (t,t') \in T^2, t + p_{ij} > t' \tag{9}$$

The Quadratic Unconstrained Binary Optimisation (QUBO) model is given with the Objective function (10) to minimize with 4 multipliers, λ_1, λ_2, λ_3 and λ_4, balancing the relaxation of the 4 sets of constraints.

$$\begin{aligned} &\sum_j \sum_t t.x^t_{n_i j} \\ +\lambda_1 &\sum_j \sum_i (\sum_t x^t_{ij} - 1)^2 \\ +\lambda_2 &\sum_{(i,j,t)\cup(i',j',t')\in T1} x^t_{ij} x^{t'}_{i'j'} \\ +\lambda_3 &\sum_{(i,j,t,t')\in T2} x^t_{ij} x^{t'}_{i+1j} \end{aligned} \tag{10}$$

with:

$$T1 = (i,j,t) \cup (i',j',t') : i,i' = 1..n_i, j,j' = 1..n, (i,j) \neq (i',j'),$$
$$M_{ij} = M_{i'j'}, (t,t') \in T^2, 0 \leq t' - t < p_{ij}$$
$$T2 = (i,j,t,t') : i = 1..(n_i - 1), j = 1..n, (t,t') \in T^2, t + p_{ij} > t'$$

5 Computational Experiments

As an early result, we present in Fig. 1 the solution we obtained with the D-Wave Hybrid Solution on an instance from the literature [4]. In this instance, 4 jobs with 4 operations have to be scheduled on 4 different machines. The next step will be the direct implementation of the QUBOs for the job shop instances and their solution with the full quantum machines.

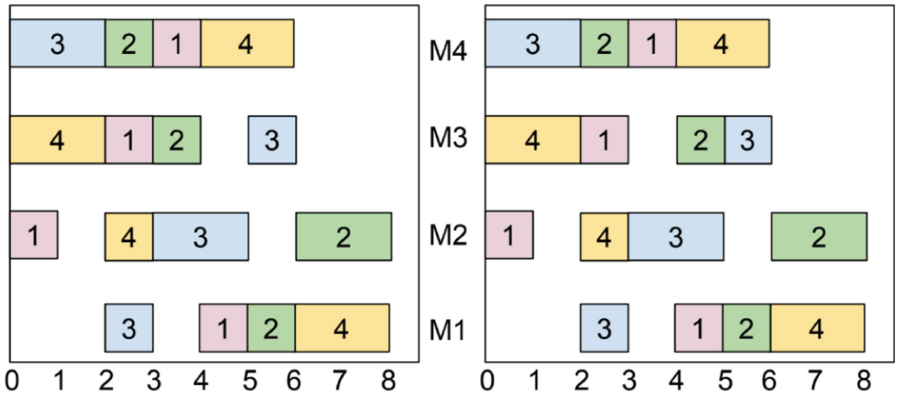


Fig. 1. Representation of the optimal solution from the literature (left, [4]) and the solution obtained with the D-Wave Hybrid Solution (right). The instance has 4 machines (M1, M2, M3, M4), 4 jobs (each represented by a different color), and 4 operations per job.

It should be noted that the solution obtained with our model and the hybrid solver is not an active schedule. Indeed, Job J_2 on machine M_3 could be started earlier.

Five instances with four machines and four operations per job had been solved with T taking values in $\{19, 13, 11, 9, 8, 7, 6, 5\}$. In Table 1, we report the objective function of the best solution obtained with one call of the D-Wave Hybrid Solution, where *n.s.e.* and *s.n.f.* mean *no solution exists* and *solution not found*, respectively. The chart in Fig. 2 shows the evolution of the results according to the number of periods.

Table 1. Results on 5 instances (4 jobs, 4 machines and 4 operations per job) solved by the D-Wave Hybrid Solution.

	T = 19	T = 13	T = 11	T = 9	T = 8	T = 7	T = 6	T = 5
inst1	41	30	25	26	25	24	21	*n.s.e.*
inst2	38	29	26	26	25	21	22	*s.n.f.*
inst3	46	29	27	24	20	20	18	18
inst4	55	38	38	31	29	28	*s.n.f.*	*n.s.e.*
inst5	42	28	25	21	21	20	19	18

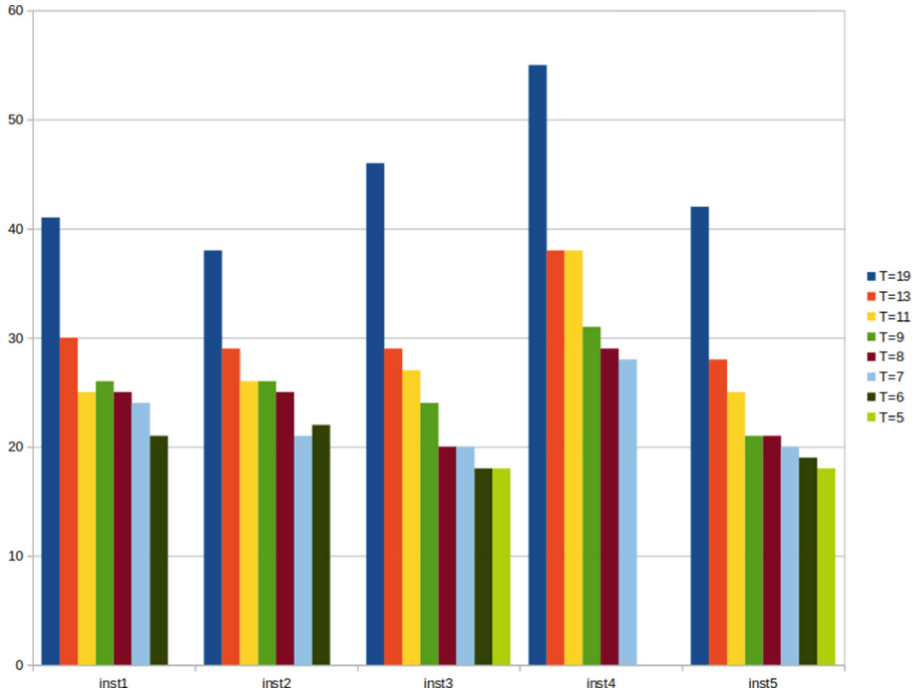


Fig. 2. Objective function evolution according to the number of periods T for 5 instances.

6 Conclusion

In this paper, we first reviewed the recent solutions proposed to solve the job shop scheduling problem on quantum computers, with a focus on QUBO formulations that help integrating constraints that are relevant in practice. We have also proposed a QUBO model that has been solved using D-Wave quantum annealing machines. While the existing hardware are still limited in their ability to handle the number of variables generated, it remains important to progress on the modeling of practical problems. Indeed, the quantum solutions are not yet competitive as compared to classical solutions approaches, but this might change when the number of qubits available is increased. Future works include investigating mechanisms for keeping the number of variables low while integrating efficiently the constraints.

References

1. Aggoune, R., Deleplanque, S.: Solving the job shop scheduling problem: QUBO model and quantum annealing. In: 21st EU/ME meeting - Emerging Optimization Methods: From Metaheuristics to Quantum Approaches, April 2023, Troyes (2023). ⟨hal-04037312⟩
2. Amaro, D., Rosenkranz, M., Fitzpatrick, N., Hirano, K., Fiorentini, M.: A case study of variational quantum algorithms for a job shop scheduling problem. EPJ Quantum Technol. **9**, 100–114 (2022)
3. Da Col, G., Teppan, E.C.: Industrial-size job shop scheduling with constraint programming. Oper. Res. Perspect. **9**, 100249 (2022)
4. Carugno, C., Ferrari Dacrema, M., Cremonesi, P.: Evaluating the job shop scheduling problem on a D-wave quantum annealer. Sci. Rep. **12**, 6539 (2022). https://doi.org/10.1038/s41598-022-10169-0
5. Denkena, B., Schinkel, F., Pirnay, J., Wilmsmeier, S.: Quantum algorithms for process parallel flexible job shop scheduling. CIRP J. Manuf. Sci. Technol. **33**, 12142 (2020)
6. Farhi, E., Goldstone, J., Gutmann, S.: A quantum approximate optimization algorithm (2014). https://doi.org/10.48550/arxiv.1411.4028
7. Garey, Michael R. and Johnson, David S. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman and Co., USA, 1979
8. Grover, L.K.: A fast quantum mechanical algorithm for database search. In: Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing, pp. 212–219 (1996)
9. Tadashi, K., Nishimori, H.: Quantum annealing in the transverse ising model. Phys. Rev. E **58**(5), 5355 (1998)
10. Kurowski, K., Węglarz, J., Subocz, M., Różycki, R., Waligóra, G.: Hybrid quantum annealing heuristic method for solving job shop scheduling problem. In: Krzhizhanovskaya, V.V., et al. (eds.) ICCS 2020. LNCS, vol. 12142, pp. 502–515. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-50433-5_39
11. Manne, A.S.: On the job-shop scheduling problem. Oper. Res. **8**(2), 223 (1960). https://doi.org/10.1287/opre.8.2.219

12. Rao, P.U., Sodhi, B.: Scheduling with multiple dispatch rules: a quantum computing approach. In: Groen, D., de Mulatier, C., Paszynski, M., Krzhizhanovskaya, V.V., Dongarra, J.J., Sloot, P.M.A. (eds.) ICCS 2022. LNCS, vol. 13353, pp. 233–246. Springer, Cham (2022). https://doi.org/10.1007/978-3-031-08760-8_20
13. Shimada, D., Shibuya, T., Shibasaki, T.: A decomposition method for makespan minimization in job-shop scheduling problem using ising machine. In: 2021 IEEE 8th International Conference on Industrial Engineering and Applications, PP. 307–314 (2021)
14. Schworm, P., Wu, X., Glatt, M., Aurich, J.C.: Solving flexible job shop scheduling problems in manufacturing with Quantum Annealing. Prod. Eng. Res. Dev. **17**, 105–115 (2023). https://doi.org/10.1007/s11740-022-01145-8
15. Venturelli, D., Marchand, D.J., Rojo, G.: Quantum annealing implementation of job-shop scheduling. arXiv preprint:1506.08479, 2015
16. Zhang, J., Lo Bianco, G., Beck, J.C.: Solving job-shop scheduling problems with QUBO-based specialized hardware. In: Proceedings of the International Conference on Automated Planning and Scheduling, vol. 32, no. 1, pp. 404–412 (2022)