Construction of Torus of Revolution Generated from Epicycloids

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Abstract. Among the geometric bodies of revolution, the torus of revolution stands out, which can be generated from a circumference, lemniscate curves and the figure-of-eight curve. Given the classical definition used in Mathematics, interest arises in finding other curves that generate the torus of revolution when rotating around an axis. In this article, carrying out the respective analyzes and the necessary programming using the Mathematica software, allowed us to carry out the necessary calculations and geometric visualizations of the mathematical object: This is how a torus of revolution was built from epicycloid curves in their parametric form. The study was extended by determining curves that were on the torus generated by epicycloid curves, which when properly projected to planes, curves that present beautiful symmetries were obtained. When the points of these curves are taken correctly, special irregular polygons are obtained.

With the obtaining of these results, a satisfactory answer to the research question was obtained, as well as a way of defining it. In addition, it has shown us a wide path of research on the different curves that a torus of revolution can generate.

Keywords: Torus · revolution · epicycloides

1 Introduction

The word torus comes from the latin word torus which means curved elevation, it has a specific geometric shape. Topologically, it is a closed orientable surface of genus 1. Within the tori of revolution we already have three types of basic curves that generate tori of revolution, the first one based on a circumference that is well known, you can torus of revolution with Bernoulli lemniscates [1] and then it is possible to define spatial curves with curvatures and non-zero torsion that generate tori of revolution that in some cases coincide with the tori generated by circumferences. And thirdly, it is possible to generate torus of revolution by the curve of eight, in this case the equations are simpler than the lemniscatic torus [2]. Given what has been described, the research question

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arises: can toruses of revolution be generated from epicycloidal curves? Through an exhaustive analysis and the use of Mathematica software [3–5] as a graphic visualization tool, we were able to verify that new torus of revolution can be generated. The torus of revolution have multiple applications in different fields from making geometric designs of polygons, polyhedrons [6,7] and even antennas [8], particle accelerators with their quasi-toroidal shape, in optics with toric surfaces and in the world of the video games.

2 The Epicycloid

The present research work is based on the definition of an epicycloid. For this we will see the formal definition of the epitrochoid [9], taking into account Fig. 1.

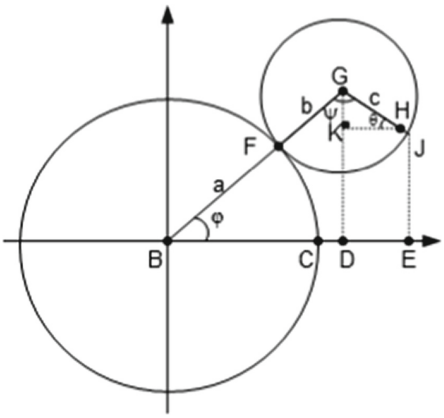


Fig. 1. Circle of radius b moving through another circle of radius a to generate an epitrochoid

They obtained the equation of the ep itrochoid.

$$epitro(t) = \left((a+b)cos(\varphi) - c\ \ cos\left(\tfrac{a+b}{b}\right)\varphi, (a+b)sin(\varphi) - c\ \ sin\left(\tfrac{a+b}{b}\right)\varphi\right)$$

If in the previous equation we make b=c we will have the Epicycloid equation.

$$epici(t) = \left((a+b)cos(\varphi) - b\ \ cos\left(\tfrac{a+b}{b}\right)\varphi, (a+b)sin(\varphi) - b\ \ sin\left(\tfrac{a+b}{b}\right)\varphi\right)$$

For research purposes we are interested in the case when $b > c$.
Let's look at a specific example. Previously we defined with Mathematica.

$$\mathbf{Epitro}[\mathbf{t}_,\{a_,b_,c_\}] := \left\{(a+b)\mathbf{Cos}[t] - c\ \ Cos\left(\frac{(a+b)}{b}t\right),(a+b)\mathbf{Sin}[t] - c\ \ \mathbf{Sin}\left(\frac{(a+b)}{b}t\right)\right\}$$

$$\mathbf{Epitro}[\ \mathbf{t},\{4,4,1.5\}]$$
$$\{10\ \ Cos[t] - Cos[5t], 10\ \ Sin[t] - Sin[5t]\}$$
$$\mathbf{Epitro}[\mathbf{t},\{8,1,2\}]$$
$$\{9\ \ Cos[t] - 2\ \ Cos[9t], 9\ \ Sin[t] - 2\ \ Sin[9t]\}$$

$$\mathbf{ParametricPlot}[\mathbf{Epitro}[\mathbf{t},\{4,4,1.5\}],\{\mathbf{t},0,2\pi\},\mathbf{PlotStyle} \rightarrow \mathbf{Hue}[\mathbf{Random}[]]]$$
$$\mathbf{ParametricPlot}[\mathbf{Epitro}[\mathbf{t},\{8,1,2\}],\{t,0,2\pi\},\mathbf{PlotStyle} \rightarrow \mathbf{Hue}[\mathbf{Random}[]]]$$

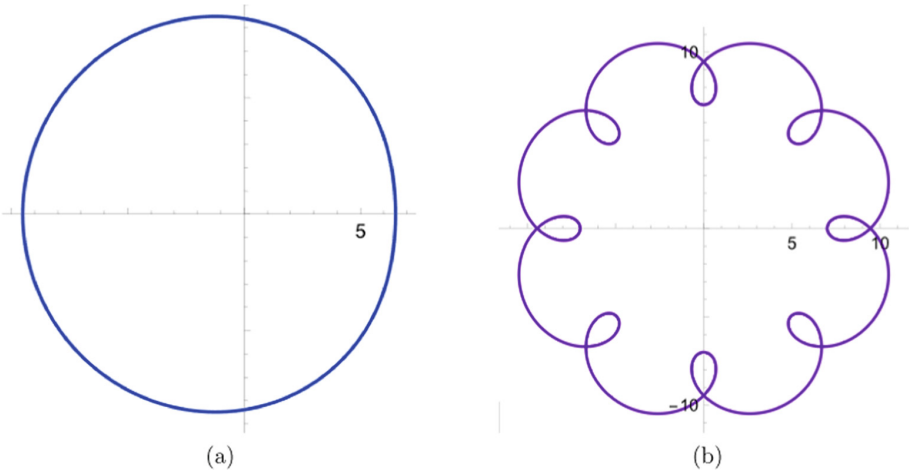


Fig. 2. (a) Epitro[t,4,4,1.5] and (b) Epitro[t,8,1,2]

3 Generation of the Torus of Revolution from an Epicycloid

If we have an epicycloid given in parametric form (Figs. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22):

$$\mathbf{Epici3D}(u_,(a_,b_),B) = (B,0,0) + \left((a+b)Cos[u] - b\ \ Cos\left[\frac{(a+b)}{b}u\right],0,(a+b)Sin[u] - b\ \ Sin\left[\frac{(a+b)}{b}u\right]\right)$$

Which obviously rests in the plane x = B, if it rotates around the "Z" axis then we would have the following equation:

$$\mathbf{ToroEpici}(u,v,a) = \begin{pmatrix} \mathbf{Cos}(v) & \mathbf{Sin}(v) & 0 \\ -\mathbf{Sin}(v) & \mathbf{Cos}(v) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B+(a+b)\mathbf{Cos}[u]-b & \mathbf{Cos}\left[\frac{a+b}{b}u\right] \\ 0 \\ (a+b)\mathbf{Sin}[u]-b & \mathbf{Sin}\left[\frac{a+b}{b}u\right] \end{pmatrix}$$

Performing the matrix calculations is as follows:

$$\begin{aligned} &\mathbf{ToroEpici}(u,v,a) = \left(\left(B+(a+b)\mathbf{Cos}[u]-b \ \ \mathbf{Cos}\left[\frac{(a+b)}{b}u\right]\right)\mathbf{Cos}[v],\right. \\ &\left.\left(B+(a+b)\mathbf{Cos}[u]-b \ \ \mathbf{Cos}\left[\frac{(a+b)}{b}u\right]\right)\mathbf{Sin}[v], (a+b)\mathbf{Sin}[u]-b \ \ \mathbf{Sin}\left[\frac{(a+b)}{b}u\right]\right) \end{aligned} \tag{1}$$

The domain of said parameterization is defined by:
$D: -\frac{\pi}{2} < u < \frac{\pi}{2} \ \ y \ \ 0 < v < 2\pi$
We graph with Mathematica.
Now, we define the **ToroEpici** with the Mathematica software taking into account the definition of rotation and Eq. (1)

$$\mathbf{Rotz}[\theta_] := \{\{\mathbf{Cos}[\theta], -\mathbf{Sin}[\theta], \theta\}, \{\mathbf{Sin}[\theta], \mathbf{Cos}[\theta], \theta\}, \{0,0,1\}\}$$
$$\mathbf{ToroEpici}[\{u_, v_\}, \{a_, b_\}, B] := \mathbf{Rotz}[v] \cdot \mathbf{Epici3D}[u, \{a,b\}, B]$$

Then explicitly with Mathematica we get it:

$$\begin{aligned} &TorEpici[\{u,v\},\{a,b\},B] \\ &\left\{\left(B+(a+b)\mathbf{Cos}[u]-b \ \ \left[\frac{(a+b)}{b}u\right]\right)\mathbf{Cos}[v],\right. \\ &\left(B+(a+b)\mathbf{Cos}[u]-b \ \ \mathbf{Cos}\left[\frac{(a+b)}{b}u\right]\right)\mathbf{Sin}[v], \\ &\left.(a+b)\mathbf{Sin}[u]-b \ \ \mathbf{Sin}\left[\frac{(a+b)}{b}u\right]\right\} \end{aligned} \tag{2}$$

3.1 Examples of Epicycloidal Torus

We will show three concrete examples:
Example 01
If from Eq. 2 we give the values $a = 1$, b= $\frac{1}{4}$ and B = 2 we obtain with the Mathematica software.

$$TorEpici[\{u, v\}, \{1, \frac{1}{4}\}, 2]$$
$$\left\{\left(2 + \frac{5\mathbf{cos}[u]}{4} - \frac{1}{4}\mathbf{Cos}[5u]\right)\mathbf{Cos}[v],\right.$$
$$\left.\left(2 + \frac{5\mathbf{cos}[u]}{4} - \frac{1}{4}\mathbf{Cos}[5u]\right)\mathbf{Sin}[v], \frac{5\mathbf{sin}[u]}{4} - \frac{1}{4}\mathbf{Sin}[5u]\right\}$$

By graphing this surface we have a geometric object that looks like a yo-yo.

$$\mathbf{g0} = \mathbf{ParametricPlot3D}[\mathbf{TorEpici}[\{u, v\}, \{1, 1/4\}, 2], \{u, 0, 2\pi\},$$
$$\{v, 0, 2\pi\}, \mathbf{PlotStyle} \to \mathbf{Opacity}[0.6], \mathbf{Mesh} \to \mathbf{False}, \mathbf{PlotPoints} \to 75]$$

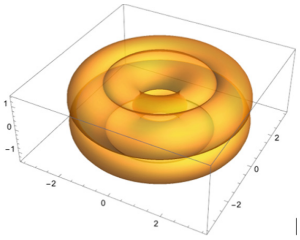
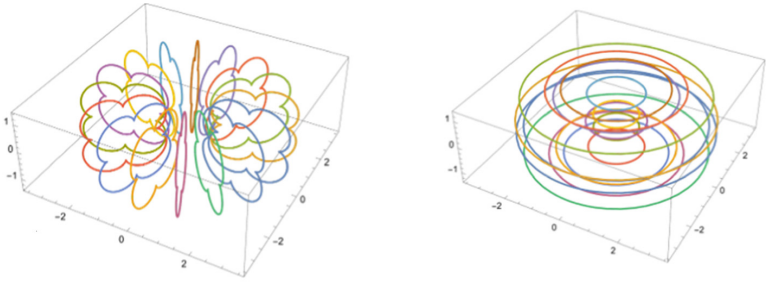


Fig. 3. Yo-yo surface.

We have the graphs of the u-parametric curves and the v-parametric curves.



(a) (b)

Fig. 4. (a) The parametric u-curves on the yoyo surface. (b) The parametric v-curves on the yoyo surface.

Example 02
If from Eq. 2 we give the values $a = 1$, $b = \frac{1}{6}$ and $B = 2$ we obtain with the Mathematica software.

$$\mathbf{TorEpici}[\{u, v\}, \{1, \frac{1}{6}\}, 2]$$

$$\left\{\left(2 + \frac{7\mathbf{Cos}[u]}{6} - \frac{1}{6}\mathbf{Cos}[7u]\right)\mathbf{Cos}[v],\right.$$
$$\left.\left(2 + \frac{7\mathbf{Cos}[u]}{6} - \frac{1}{6}\mathbf{Cos}[7u]\right)\mathbf{Sin}[v], \frac{7\mathbf{Sin}[u]}{6} - \frac{1}{6}\mathbf{Sin}[7u]\right\}$$

$$\mathbf{g1} = \mathbf{ParametricPlot3D}[\{u, v\}, \{1, \frac{1}{6}\}, 2], \{u, 0, 2\pi\}, \{v, 0, 2\pi\},$$
$$\mathbf{PlotStyle} \rightarrow \mathbf{Opacity}[0, 6], \mathbf{Mesh} \rightarrow \mathbf{False}, \mathbf{PlotPoints} \rightarrow 75]$$

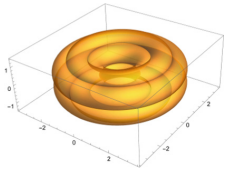
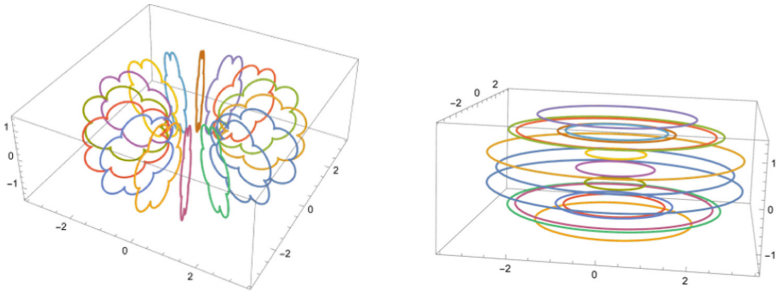


Fig. 5. *ToroEpici*$[\{u, v\}, \{1, \frac{1}{6}\}, 2]$.

We have the graphs of the u-parametric curves and the v-parametric curves.



(a) (b)

Fig. 6. (a) The parametric u curves of *ToroEpici*$[\{u, v\}, \{1, \frac{1}{6}\}, 2]$ and (b) The parametric v curves of *ToroEpici*$[\{u, v\}, \{1, \frac{1}{6}\}, 2]$

Example 03
If from Eq. 2 we give the values $a = 1$, $b = \frac{1}{16}$ and $B = 2$ we obtain with the Mathematica software

$$\mathbf{TorEpici}[\{u, v\}, \{1, \frac{1}{16}\}, 2]$$
$$\left\{\left(2 + \frac{17\mathbf{Cos}[u]}{16} - \frac{1}{16}\mathbf{Cos}[17u]\right)\mathbf{Cos}[v],\right.$$
$$\left.\left(2 + \frac{17\mathbf{Cos}[u]}{16} - \frac{1}{16}\mathbf{Cos}[17u]\right)\mathbf{Sin}[v], \frac{17\mathbf{Sin}[u]}{16} - \frac{1}{16}\mathbf{Sin}[17u]\right\}$$
$$\mathbf{g2} = \mathbf{ParametricPlot3D}[\{u, v\}, \{1, \frac{1}{16}\}, 2], \{u, 0, 2\pi\}, \{v, 0, 2\pi\},$$
$$\mathbf{PlotStyle} \rightarrow \mathbf{Opacity}[0, 6], \mathbf{Mesh} \rightarrow \mathbf{False}, \mathbf{PlotPoints} \rightarrow 75]$$

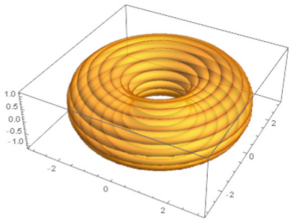
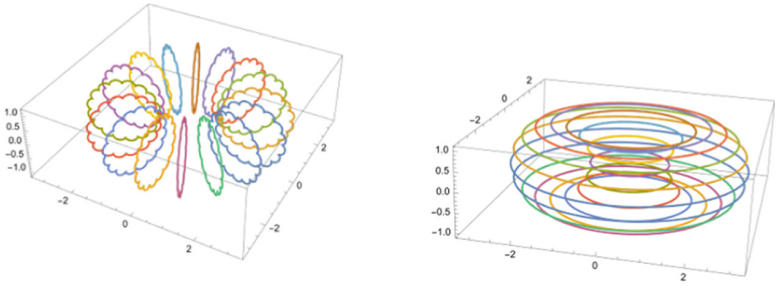


Fig. 7. *ToroEpici*$[\{u, v\}, \{1, \frac{1}{16}\}, 2]$.

We have the graphs of the u-parametric curves and the v-parametric curves.



(a) (b)

Fig. 8. (a) The parametric u curves of *ToroEpici*$[\{u, v\}, \{1, \frac{1}{16}\}, 2]$ and (b) The parametric v curves of *ToroEpici*$[\{u, v\}, \{1, \frac{1}{16}\}, 2]$

4 Parametric Curves in Epicycloidal Torus

Taking Eq. 2 into account, an infinite number of curves are defined in the epicyclic torus in the following way, we assume that we have a parametric function $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ and we substitute in Eq. 2 $u = \alpha_1(t)$ y $v = \alpha_2(t)$, with the Mathematica software it is defined in the following way.

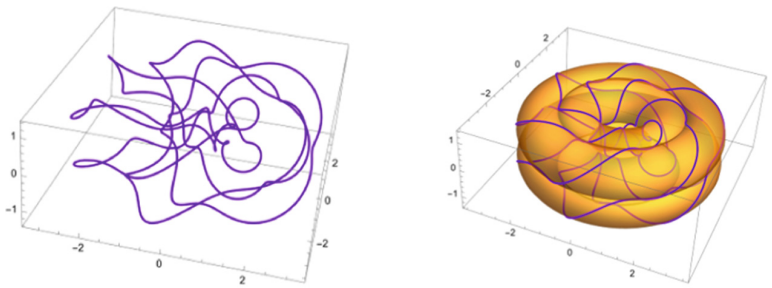
$$\begin{aligned} &\mathbf{Curva1}[\mathbf{t_}, \{\mathbf{a_}, \mathbf{b_}, \mathbf{B_}\}, \alpha : \{_, _\}] := \\ &\mathbf{TorEpici}[\{\mathbf{u}, \mathbf{v}\}, \{\ \mathbf{a}, \mathbf{b}\}, \mathbf{B}\}]/.\{\mathbf{u} \to \alpha\mathbf{[[1]]}, \mathbf{v} \to \alpha\mathbf{[[2]]}\} \end{aligned} \tag{3}$$

4.1 The Curve $Curve1[t, \{1, 1/4, 2\}, \{6t, 5t\}]$

If from Eq. 3 we give the values $/a = 1$, $b = 1/4$ *and* $B = 2, \alpha_1(t) = 6t$, $\alpha_2(t) = 5t$, we obtain with the Mathematica software.

$$\begin{aligned} &\mathbf{Curva1}[\mathbf{t}, \{\mathbf{1}, \mathbf{1/4}, \mathbf{2}, \{\mathbf{6t}, \mathbf{5t}\}] \\ &\left\{Cos[5t]\left(2 + \frac{5}{4}Cos[6t] - \frac{1}{4}Cos[30t]\right), \right. \\ &\left.\left(2 + \frac{5}{4}Cos[6t] - \frac{1}{4}Cos[30t]\right) Sin[5t], \frac{5}{4}Sin[6t] - \frac{1}{4}Sin[30t]\right\} \end{aligned}$$

The graph is obtained with The graph of the curve alone is shown and the other the curve resting on the epicyclic torus.



(a) (b)

Fig. 9. (a) Curve1[t,1,1/4,2,6t,5t] (b) Curva1 resting on the epicilloidal torus that contains it.

We project this curve onto the z=0 plane and call it $\beta_1(t)$.

$$\begin{aligned} &\beta 1[t_] := \mathbf{Delete}[\mathbf{Curva1}[t, \{\mathbf{1}, \mathbf{1/4}, \mathbf{2}\}, \mathbf{6t}, \mathbf{5t}], 3] \\ &\beta 1[t] \\ &\left\{Cos[5t]\left(2 + \frac{5}{4}Cos[6t] - \frac{1}{4}Cos[30t]\right), \left(2 + \frac{5}{4}Cos[6t] - \frac{1}{4}Cos[30t]\right) Sin[5t]\right\} \end{aligned}$$

The graph is obtained by

$$pc\alpha 1 = ParametricPlot[\beta 1[t], \{t, 0, 2\pi\}, PlotStyle \to Hue[0.75], Axes \to False]$$

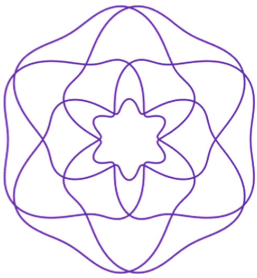


Fig. 10. $Curva1 = \beta_1(t)$

The curve is symmetric about the ordinate axes and the origin.

Some Polygons on Curva1[t]. In this curve are immersed equilateral triangles, regular hexagons, regular dodecahedrons, dodecaangulos etc.

Dodecaangle

This polyhedron is generated from the points:

$$P_1 = \left(-sin\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}cos\left[\frac{\pi}{8}\right]+\frac{1}{4}sin\left[\frac{\pi}{8}\right]\right), -cos\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}cos\left[\frac{\pi}{8}\right]+\frac{1}{4}sin\left[\frac{\pi}{8}\right]\right)\right)$$

$$P_2 = \left(sin\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}cos\left[\frac{\pi}{8}\right]+\frac{1}{4}sin\left[\frac{\pi}{8}\right]\right), -cos\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}cos\left[\frac{\pi}{8}\right]+\frac{1}{4}sin\left[\frac{\pi}{8}\right]\right)\right)$$

These points are generated with Mathematica as follows:

$$pol0 = \beta 1/@\{11\pi/16, 5\pi/16\}$$

$$\left\{\left\{-Sin\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}Cos\left[\frac{\pi}{8}\right]+\frac{1}{4}Sin\left[\frac{\pi}{8}\right]\right), -Cos\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}Cos\left[\frac{\pi}{8}\right]+\frac{1}{4}Sin\left[\frac{\pi}{8}\right]\right)\right\},\right.$$
$$\left.\left\{Sin\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}Cos\left[\frac{\pi}{8}\right]+\frac{1}{4}Sin\left[\frac{\pi}{8}\right]\right), -Cos\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}Cos\left[\frac{\pi}{8}\right]+\frac{1}{4}Sin\left[\frac{\pi}{8}\right]\right)\right\}\right\}$$

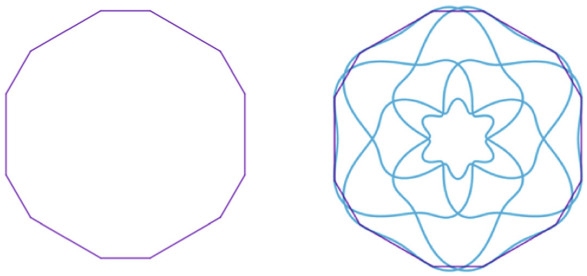
The distance of these points from the origin is

$$Norm/@polo0//Simplify$$

To generate the other points:

$$\mathbf{rpol0} = \mathbf{Partition}[\mathbf{Flatten}[\mathbf{Table}[\mathbf{Table}[\mathbf{RotationMatrix}[\theta]\cdot pol0[[i]], \{i, 2\}]$$
$$\{\theta, 0, 2\pi - \frac{\pi}{3}, \frac{\pi}{3}\}]], 2];$$

The length of the sides in alternating form is
$\{1.26829, 208969, 1.26829, 208969, 1.26829, 208969, 1.26829, 208969,$
$1.26829, 208969, 1.26829, 208969, \}$
The graph of the first polygon with the curve $\beta_1[t]$ is shown below.



(a) (b)

Fig. 11. (a) Dodecangle (b) Dodecangle on the curve $\beta_1(t)$

24 sided polygon

This polyhedron is generated from the points:

$$P_1 = \left(-sin\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}cos\left[\frac{\pi}{8}\right]+\frac{1}{4}sin\left[\frac{\pi}{8}\right]\right), -cos\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}cos\left[\frac{\pi}{8}\right]+\frac{1}{4}sin\left[\frac{\pi}{8}\right]\right)\right)$$

$$P_2 = \left(sin\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}cos\left[\frac{\pi}{8}\right]+\frac{1}{4}sin\left[\frac{\pi}{8}\right]\right), -cos\left[\frac{\pi}{16}\right]\left(2+\frac{5}{4}cos\left[\frac{\pi}{8}\right]+\frac{1}{4}sin\left[\frac{\pi}{8}\right]\right)\right)$$

These points are generated with Mathematica as follows

$$\textbf{pol1} = \beta 1/@\{7\pi/24, 17\pi/24\}$$

$$\left\{\left\{-\left(\left(2+\frac{3}{2\sqrt{2}}\right)Sin\left[\frac{\pi}{24}\right]\right), -\left(\left(2+\frac{3}{2\sqrt{2}}\right)Cos\left[\frac{\pi}{24}\right]\right)\right\},\right.$$
$$\left.\left\{\left(2+\frac{3}{2\sqrt{2}}\right)Sin\left[\frac{\pi}{24}\right], -\left(\left(2+\frac{3}{2\sqrt{2}}\right)Cos\left[\frac{\pi}{24}\right]\right)\right\}\right\}$$

The distance of these two points from the origin is:

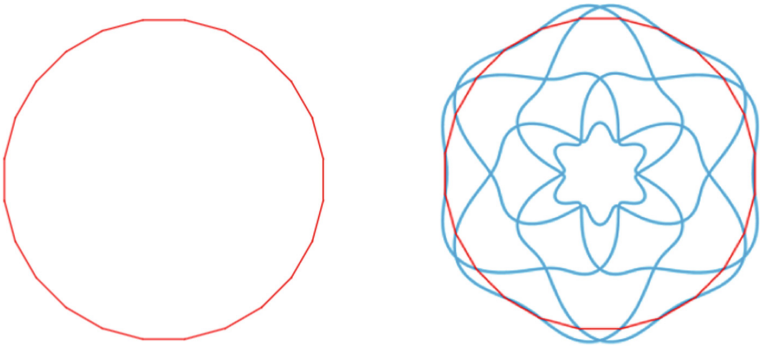
$$\textbf{Norm/@pol1//Simplify}$$

$$\left\{\frac{1}{2}\sqrt{\frac{41}{2}+12\sqrt{2}}, \frac{1}{2}\sqrt{\frac{41}{2}+12\sqrt{2}}\}\right\}$$

We generate the other points as follows

$$\textbf{rpol1} = \textbf{Partition}[\textbf{Flatten}[\textbf{Table}[\textbf{Table}[\textbf{RotationMatrix}[\theta] \cdot \textbf{pol1}[[i]], \{i, 2\}\ \{\theta, 0, 2\pi - \pi/6, \pi/6\}]], 2];$$

The length of the sides is 0.798993.
The graph of the second polyhedron with the curve is shown below.



(a) (b)

Fig. 12. (a) Dodecangle (b) Dodecangle on the curve $Curva1 = \beta_1(t)$

Obviously this 24-sided polygon has dodecagons, regular hexagons, and equilateral triangles. ***12-sided polygon.*** We generate this polygon with the following pair of points FALTA

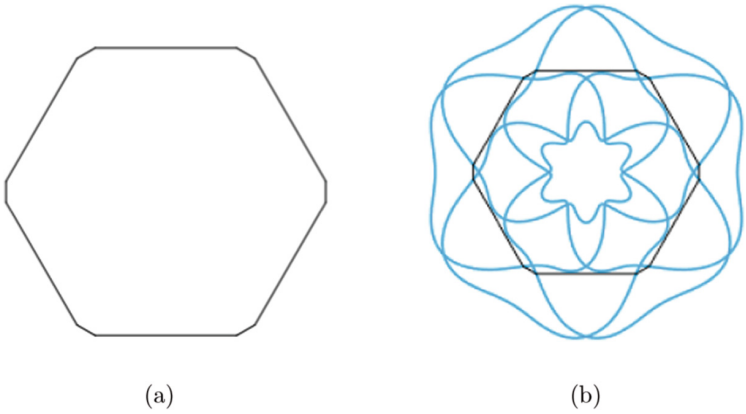
pol2=β1/@{17 π/16,13 π/48 }//Simplify

$$\left\{\left\{\frac{1}{4}\left(-8+Cos\left[\frac{\pi}{8}\right]-5Sin\left[\frac{\pi}{8}\right]\right)Sin\left[\frac{3\pi}{16}\right],\frac{1}{4}Cos\left[\frac{3\pi}{16}\right]\left(-8+Cos\left[\frac{\pi}{8}\right]-5sin\left[\frac{\pi}{8}\right]\right)\right\}\right.$$
$$\left.\left\{\frac{1}{4}\left(-8+Cos\left[\frac{\pi}{8}\right]-5Sin\left[\frac{\pi}{8}\right]\right)Sin\left[\frac{7\pi}{48}\right],\frac{1}{4}Cos\left[\frac{7\pi}{48}\right]\left(-8+Cos\left[\frac{\pi}{8}\right]-5sin\left[\frac{\pi}{8}\right]\right)\right\}\right\}$$

We generate the other points as follows:

$$rpol2 = Partition[flatten[Table[Table[RotationMatriz[\theta] \cdot pol2[[i]], \{i, 2\}]\ \{\theta, 0, 2\pi - \pi/3, \pi/3\}]], 2];$$

The length of the sides in alternating form is 0.293972 and 1.98799
The graph of the third polyhedron with the curve is shown below.



(a) (b)

Fig. 13. (a) Dodecangle (b) Dodecangle on the curve Curve $\beta_1(t)$

4.2 The Curve $Curva1[t, 1, 1/6, 2, 6t, 5t]$

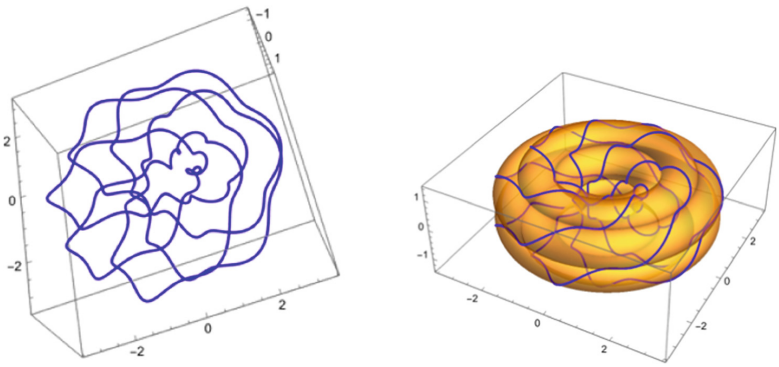
We define the curve as follows

$$\textbf{Curva1}[t, \{1, 1/6, 2\}, \{6t, 5t\}]//\textbf{Simplify}$$

$$\left\{-\frac{1}{6}Cos[5t](-12 - 7Cos[6t] + Cos[42t]),\right.$$

$$\left.-\frac{1}{6}(-12 - 7Cos[6t] + Cos[42t])Sin[5t], \frac{1}{6}(7Sin[6t] - Sin[42t])\right\}$$

The graph of the curve and the torus that contains it is shown in the following figure.



(a) (b)

Fig. 14. (a) $Curve1[t, 1, 1/6, 2, 6t, 5t]$ (b) Curve1 resting on the epicyclic torus that contains it

We project it to the plane z=0 with.

$$\beta 2[t_] := \mathbf{Delete}[\mathbf{Curva1}[\mathbf{t}, \{\mathbf{1}, \mathbf{1/6}, \mathbf{2}\}, \{\mathbf{6t}, \mathbf{5t}\}], 3]$$

$$\left\{Cos[5t]\left(2+\frac{7}{6}Cos[6t]-\frac{1}{6}Cos[42t]\right), \left(2+\frac{7}{6}Cos[6t]-\frac{1}{6}Cos[42t]\right)Sin[5t]\right\}$$

And we get its graph.

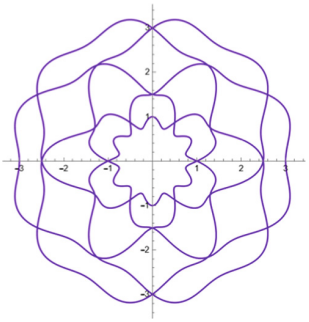


Fig. 15. Curve $\beta_2(t)$

The graph is obviously symmetric to the ordered axes and the origin.

First Polygon in the curve $\beta_2[t]$
We generated this first polygon with:

$$\mathbf{Pol00} = \beta 2/@\{95\pi/64, 97\pi/64\}//Simplify$$

$$\left\{\left\{-\frac{2}{3}\left(3+2Cos\left[\frac{\pi}{8}\right]\right)Sin\left[\frac{5\pi}{64}\right], -\frac{2}{3}Cos\left[\frac{5\pi}{64}\right]\left(3+2Cos\left[\frac{\pi}{8}\right]\right)\right\},\right.$$

$$\left.\left\{\left(2+\frac{4}{3}Cos\left[\frac{\pi}{8}\right]\right)Sin\left[\frac{5\pi}{64}\right], -\frac{2}{3}Cos\left[\frac{5\pi}{64}\right]\left(3+2Cos\left[\frac{\pi}{8}\right]\right)\right\}\right\}$$

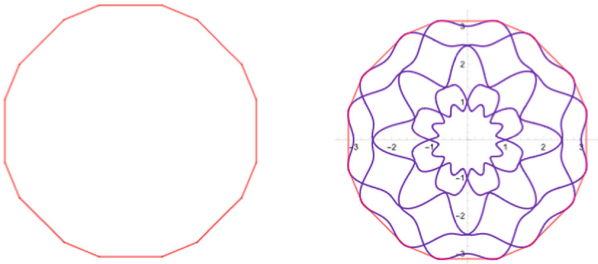
Then we generate 16 points in total

$$\mathbf{N}[\mathbf{rPol00} =$$
$$\mathbf{Partition}[\mathbf{Flatten}[\mathbf{Table}[\mathbf{Table}[\mathbf{RotationMatrix}[\theta]\cdot\mathbf{Pol00}[[i]], \{i, 2\}], \{\theta, 0, 2\pi-\pi/4, \pi/4\}]], 2], 3]$$

$$\{\{-0.785, -3.13\}, \{0.785, 3.13\}, \{1.66, -2.77\}, \{2.77, -1.66\}, \{3.13, -0.785\},$$
$$\{3.13, 0785\}, \{2.77, 1.66\}, \{1.66, 2.77\}, \{0.785, 3.13\}, \{-0.785, 3.13\},$$
$$\{-1.66, 2.77\}, \{-2.77, 1.66\}, \{-3.13, 0.785\}, \{-3.13, -0.785\}, \{-2.77, -1.66\}$$
$$\{-1.66, -2.77\}\}$$

The internal angles have a measure of $\theta_1 = 2.74889$ radians, and the alternating side lengths are 1.57055 and 0.948419.

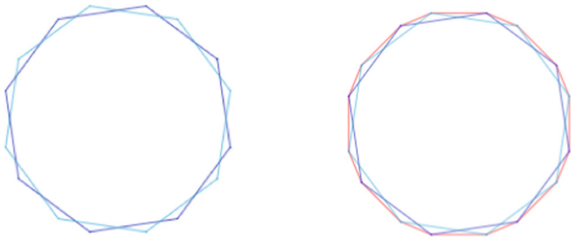
The graph of this polygon is:



(a) (b)

Fig. 16. (a) curve $\beta 2[t]$ (b) Curve $\beta 2$ and the first 16-sided polygon

The first polygon in the curve $\beta_2[t]$ has embedded two regular octagons whose angles in radians measure 2.35619 and the length of the side is 2.47354.
The two regular octagons are shown, and also the 16-sided polygon next to the two regular octagons.



(a) (b)

Fig. 17. (a) The octagons (b) Octagons and the 16-sided polygon

Second polygon on the $\beta 2[t]$ curve.
We generate this polynomial with the points

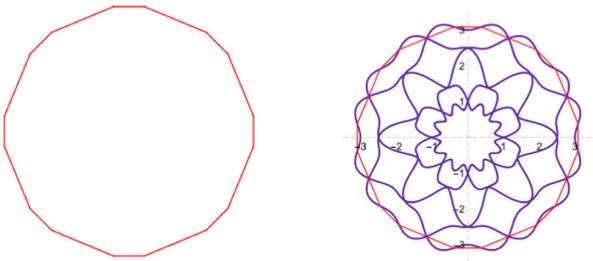
Pol2 β 2 /@ {191Pi/128,193Pi/128} // Simplify

$$\left\{\left\{-\frac{1}{6}Sin\left[\frac{5\pi}{128}\right]\left(12+7Cos\left[\frac{\pi}{16}\right]-Sin\left[\frac{\pi}{16}\right]\right),\right.\right.$$
$$\left.-\frac{1}{6}Cos\left[\frac{5\pi}{128}\right]\left(12+7Cos\left[\frac{\pi}{16}\right]-Sin\left[\frac{\pi}{16}\right]\right)\right\},$$
$$\left\{\frac{1}{6}Sin\left[\frac{5\pi}{128}\right]\left(12+7Cos\left[\frac{\pi}{16}\right]-Sin\left[\frac{\pi}{16}\right]\right),\right.$$
$$\left.\left.-\frac{1}{6}Cos\left[\frac{5\pi}{128}\right]\left(12+7Cos\left[\frac{\pi}{16}\right]-Sin\left[\frac{\pi}{16}\right]\right)\right\}\right\}$$

The rest of the points are generated with

N[**rPol2** =
Partition[**Flatten**[**Table**[**Table**[**RotationMatrix**[θ] · **Pol2**[[i]], $\{i, 2\}$] ,
$\{\theta, 0, 2\pi - \pi/4, \pi/4\}$]], 2], 2]

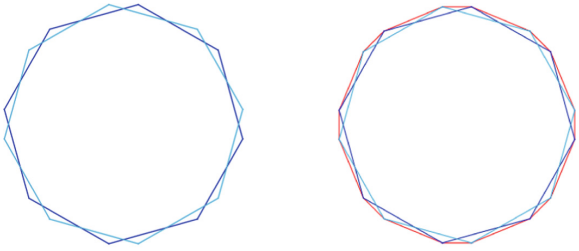
nternal angles have a measure of $\theta_1 = 2.74889$ radians. And the alternating side lengths are 0.7619 and 1.65988. Next, we show the graph.



(a) (b)

Fig. 18. (a) curve $\beta 2[t]$ and $\beta 2[t]$ and the second 16-sided polygon

The second polygon on the curve $\beta_2[t]$ has embedded two regular octagons whose angles in radians measure 2.35619 and the length of the side is 2.38162. The two regular octagons are shown, and also the second 16-sided polygon next to the two regular octagons.



(a) (b)

Fig. 19. (a) Octagons. (b) Octagons and the second 16-sided polygon.

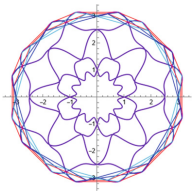


Fig. 20. The curve β_2 , the octagons and the two 16-sided polygons.

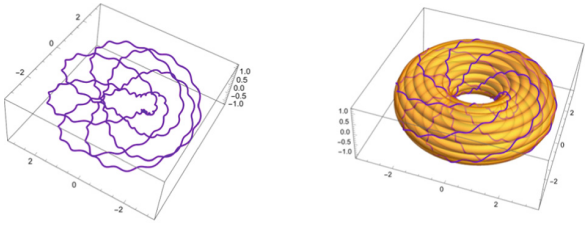
4.3 La Curva Curva1[t,1,1/16,2,6t,5t].

We take a third curve.

Curva1[t,{1,1/16,2},{6t,5t}]//Simplify

$$\left\{\frac{1}{16}Cos[5t]\,(32+17Cos[6t]-Cos[102t])\,,\right.$$
$$\left.\frac{1}{16}\,(32+17Cos[6t]-Cos[102t])\,Sin[5t],\frac{1}{16}\,(17Sin[6t]-Sin[102t])\right\}$$

The graph of the curve and the torus that contains it is shown in the following figure.



(a) (b)

Fig. 21. (a) $Curve1[t,\{1,1/16,2\},\{6t,5t\}]$ (b) Curve 1 alone resting on the epicilloidal torus that contains it.

We project it to the plane z=0 with.

$$\beta 3[t] := \mathbf{Delete}[\mathbf{Curva1}[t, \{1, 1/16, 2\}, \{6t, 5t\}], 3]$$

$$\beta 3[t]$$

$$\left\{Cos[5t]\left(2 + \frac{17}{16}Cos[6t] - \frac{1}{16}Cos[102t]\right), \left(2 + \frac{17}{16}Cos[6t] - \frac{1}{16}Cos[102t]\right)Sin[5t]\right\}$$

$$gbeta3 = ParametricPlot[\beta 3[t], \{t, 0, 2\pi\} PlotStyle \rightarrow Hue[0.75], PlotPints \rightarrow 75]$$

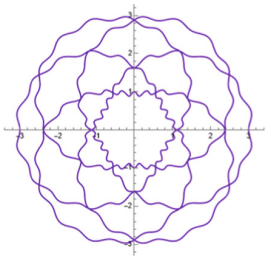


Fig. 22. The curve $\beta_2[t]$.

As can be seen, the graph is symmetric with respect to the axes and the origin.

5 Conclusions

According to the results found, it is concluded that a torus of revolution can be built in its generic form from the epicyclic curve:

$$\mathbf{ToroEpici}(u, v, a) = \left(\left(B + (a+b)\mathbf{Cos}[u] - b\ \ \mathbf{Cos}\left[\frac{(a+b)}{b}u\right]\right)\mathbf{Cos}[v],\right.$$

$$\left.\left(B + (a+b)\mathbf{Cos}[u] - b\ \ \mathbf{Cos}\left[\frac{(a+b)}{b}u\right]\right)\mathbf{Sin}[v], (a+b)\mathbf{Sin}[u] - b\ \ \mathbf{Sin}\left[\frac{(a+b)}{b}u\right]\right)$$

$D : -\frac{\pi}{2} < u < \frac{\pi}{2}\ \ y\ \ 0 < v < 2\pi$

Taking into account the equation: $\mathbf{Curva1}[\mathbf{t_}, \{\mathbf{a_}, \mathbf{b_}, \mathbf{B_}\}, \alpha : \{_, _\}] := \mathbf{TorEpici}[\{\mathbf{u}, \mathbf{v}\}, \{\ \mathbf{a}, \mathbf{b}\}, \mathbf{B}\}]/.\{\mathbf{u} \rightarrow \alpha\mathbf{[[1]]}, \mathbf{v} \rightarrow \alpha\mathbf{[[2]]}\}$

An infinite number of curves are defined in the epicyclic torus. Some existing spatial curves on the torus generated by epicycloidal curves were analyzed, taking into account the projections to the XY plane, observing the existing symmetries. Points were taken from the curves that would be the vertices of special polygons such as: 24-sided polygons, 16-sided polygons, 12-sided polygons, octagons.

It is recommended to deepen the study of the polygons that are generated with the projection of special curves that are on the torus generated by an epicyclic curve.

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