

Remarks on the Zeros of Quadratic Coquaternionic Polynomials

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Abstract. In this paper we focus on the study of monic quadratic polynomials whose coefficients are coquaternions and present several new results concerning the number and nature of its zeros. Examples specially constructed to illustrate the diversity of cases that can occur are also presented.

Keywords: Coquaternions · Polynomial computation · Zeros

1 Introduction

Coquaternions, also known in the literature as split quaternions, were introduced in 1849 by the English mathematician Sir James Cockle [4]. This four dimensional hypercomplex real algebra generalizes complex numbers and recent years have witnessed an increased interest among mathematicians and physicists on applications of these numbers [1–3,11–18,20].

One can find also in recent literature results concerning polynomials whose coefficients are coquaternions [7,8,19]. In particular, in [8], the structure of the zero-set of such polynomials was described, revealing the existence of non-isolated zeros of two different natures, namely the so-called hyperboloidal and linear zeros.

These results motivated the authors of this paper to start studying the dynamics of the quadratic map in the coquaternionic setting [5,6,10] leading to preliminary results which can be considered as even richer than the ones obtained in the well known complex case. A complete understanding of the dynamics requires a deep knowledge of the quadratic equation: number of zeros; coexistence of zeros of different nature, etc. For this reason we study, in this paper, the zeros of the equation $x^2 + bx + c$, where b and c are coquaternions.

The paper is organized as follows: Sects. 2 and 3 contain some background on the algebra of coquaternions and the basic results concerning the ring of polynomials with coquaternionic coefficients. Section 3 describes the zero-set of monic quadratic polynomials by adapting the results of [8, Theorem 3.14] to the

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present context. Some aspects concerning the existence/coexistence of hyperboloidal zeros and linear zeros are also clarified. The last section details all the possible situations that may occur concerning the number and nature of the zeros together with several examples clearly illustrating each case.

2 Basic Results and Notations

Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$. The algebra of real coquaternions, which we denote by $\mathbb{H}_{\text{coq}}$, is generated by the product given according to the following rules

$$\mathbf{i}^2 = -1,\ \mathbf{j}^2 = \mathbf{k}^2 = 1,\ \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

In analogy with the complex and quaternionic case, the *conjugate* $\overline{q}$ of a coquaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined as $\overline{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$; the number q_0 is called the *real part* of q and denoted by $\operatorname{Re} q$ and the *vector part* of q, denoted by $\underline{q}$, is given by $\underline{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. We identify the set of elements in $\mathbb{H}_{\text{coq}}$ whose vector part is zero with the set $\mathbb{R}$ of real numbers. The quantities $\operatorname{tr} q = q - \underline{q} = 2\operatorname{Re} q$ and $\det q = q\,\overline{q} = q_0^2 + q_1^2 - q_2^2 - q_3^2$ are called the *trace and* determinant of q, respectively. Finally, it can be shown that a coquaternion q is invertible or nonsingular if and only if $\det q \neq 0$. In that case, we have, $q^{-1} = \frac{\overline{q}}{\det q}$.

The notion of quasi-similarity for two coquaternions will play an important role in this paper. We say that two coquaternions p and q are *quasi-similiar* if $\operatorname{Re} p = \operatorname{Re} q$ and $\det \underline{p} = \det \underline{q}$. Naturally, quasi-similarity is an equivalence relation in $\mathbb{H}_{\text{coq}}$ and the corresponding equivalence class of q is called the *quasi-similarity class* of q and denoted by $[\![q]\!]$. We can identify

$$[\![q]\!] = \left\{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_0 = q_0 \text{ and } x_1^2 - x_2^2 - x_3^2 = q_1^2 - q_2^2 - q_3^2\right\}$$

with a hyperboloid in the hyperplane $x_0 = q_0$, which will be a hyperboloid of two sheets if $\det \underline{q} > 0$, a hyperboloid of one sheet if $\det \underline{q} < 0$ and a cone, if $\det \underline{q} = 0$.

Given a quasi-similarity class $[\![q]\!] = [\![q_0 + \underline{q}]\!]$, its *characteristic polynomial* is the polynomial $\Psi_{[\![q]\!]}$ defined as

$$\Psi_{[\![q]\!]}(x) = x^2 - 2q_0\,x + \det q. \tag{1}$$

Since $\Psi_{[\![q]\!]}$ is a second degree monic polynomial with real coefficients, it can be written as

$$\Psi_{[\![q]\!]} = \begin{cases} (x-\alpha)(x-\bar{\alpha}),\ \alpha \in \mathbb{C}\setminus\mathbb{R}, & \text{if } \det \underline{q} > 0 \\ (x-r_1)(x-r_2),\ r_1, r_2 \in \mathbb{R}, & \text{if } \det \underline{q} \leq 0 \end{cases} \tag{2}$$

Reciprocally, if Q is a second degree monic polynomial with real coefficients, it is easy to see that it is the characteristic polynomial of a uniquely defined quasi-similarity class, namely

$$Q = \begin{cases} \Psi_{[\![\alpha]\!]}, & \text{if the roots of } Q \text{ are } \alpha, \bar{\alpha} \in \mathbb{C}\setminus\mathbb{R} \\ \Psi_{[\![\frac{r_1+r_2}{2} + \frac{r_1-r_2}{2}\mathbf{j}]\!]}, & \text{if the roots of } Q \text{ are } r_1, r_2 \in \mathbb{R} \end{cases} \tag{3}$$

3 Polynomials in $\mathbb{H}_{\mathbf{coq}}[x]$

We denote by $\mathbb{H}_{\text{coq}}[x]$ the set of polynomials P whose coefficients are only located on the left of the variable, i.e. P is of the form

$$P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,\ c_i \in \mathbb{H}_{\text{coq}}. \tag{4}$$

The addition and multiplication of such polynomials are defined as in the commutative case, where the variable is assumed to commute with the coefficients. This is a ring, referred to as the ring of (left) *one-sided* polynomials in $\mathbb{H}_{\text{coq}}$.

Given a polynomial P of the form (4), its *conjugate polynomial* is the polynomial defined by $\overline{P}(x) = \overline{c}_n x^n + \overline{c}_{n-1}x^{n-1} + \cdots + \overline{c}_1 x + \overline{c}_0$, and its *companion polynomial* is the polynomial (with real coefficients) given by

$$\mathcal{C}_P(x) = P(x)\overline{P}(x).$$

If $P(z) = 0$, we say that z is a *zero* (or a root) of the polynomial P and we denote by $Z(P)$ the *zero-set* of P.

In a previous work [8] we gave a complete characterization of the zero-set $Z(P)$ of a given coquaternionic polynomial P, distinguishing between three types of zeros z:

1. *isolated zero*, if $[\![z]\!]$ contains no other zeros of P;
2. *hyperboloidal zero*, if $[\![z]\!] \subseteq Z(P)$;
3. *linear zero*, if z is neither an isolated zero nor a hyperboloidal zero of P.

We also proved that the zeros of any nth degree polynomial belong to, at most, $n(2n-1)$ similarity classes. In what follows, we will treat all the zeros belonging to the same quasi-similarity class as forming a single zero, i.e., we will refer to a whole hyperboloid of zeros or a line of zeros simply as a hyperboloidal zero or a linear zero, respectively. We end this section by presenting two important results in the context of the present work. Full details and proofs can be found in [8].

Theorem 1. *If $z \in \mathbb{H}_{\text{coq}}$ is a zero of a polynomial P of the form (4), then $\Psi_{[\![z]\!]}$ is a divisor of $\mathcal{C}_P$.*

Denote by $\alpha_1, \overline{\alpha}_1, \ldots, \alpha_m, \overline{\alpha}_m \in \mathbb{C}\setminus\mathbb{R}$ and $r_1, r_2, \ldots, r_\ell \in \mathbb{R}$, where $\ell = 2(n-m)$, $(0 \leq m \leq n)$, the $2n$ roots of $\mathcal{C}_P$. It is easy to see (cf. (3)) that the characteristic polynomials which divide $\mathcal{C}_P$ are the ones associated with the classes:

$$[\![\alpha_k]\!];\ k = 1, \ldots, m, \tag{5}$$

$$[\![r_{ij}]\!];\ i = 1, \ldots, \ell - 1,\ j = i+1, \ldots, \ell, \tag{6}$$

with

$$\mathsf{r}_{ij} = \frac{r_i + r_j}{2} + \frac{r_i - r_j}{2}\mathsf{j}. \tag{7}$$

This leads to the next result concerning the zero-set of P.

Theorem 2. *Let P be a polynomial of the form (4). Then:*

$$Z(P) \subseteq \bigcup_k [\![\alpha_k]\!] \bigcup_{i,j} [\![r_{ij}]\!],$$

where $[\![\alpha_k]\!]$ and $[\![r_{ij}]\!]$ are the quasi-similarity classes defined by (5)–(7).

Last result underpins the use of the designation *admissible classes* (with respect to the zeros) of the polynomial P, for classes given by (5)–(7).

4 The Zero Structure of Quadratic Polynomials in $\mathbb{H}_{\mathbf{coq}}[x]$

In what follows we consider the quadratic equation

$$x^2 + bx + c = 0, \ b, c \in \mathbb{H}_{\text{coq}}. \tag{8}$$

It is already known that equations of this form can have at most 6 zeros; what remains to be seen is how many zeros of each type can they have. For our purpose here we rewrite Theorem 3.14 of [8] for a quadratic polynomial in the following form:

Theorem 3. *Consider a quadratic polynomial P of the form (8). For each admissible class $[\![q]\!] = [\![q_0 + \underline{q}]\!]$ of P, denote by $A + Bx$ the polynomial*

$$A + Bx = P(x) - \Psi_{[\![q]\!]},$$

with $A, B \in \mathbb{H}_{\text{coq}}$. The nature of the zero of P in $[\![q]\!]$ (if any) is the following:

1. *If $\det B \neq 0$, then $[\![q]\!]$ contains only the **isolated zero** $-B^{-1}A$;*
2. *If $A = B = 0$, then $[\![q]\!]$ is a **hyperboloidal zero** of P;*
3. *If $B \neq 0, \det B = 0$, $\det \underline{q} \leq 0$ and $A = -\left(q_0 \pm \sqrt{-\det \underline{q}}\right)B$, then $[\![q]\!]$ contains a **linear zero** of P;*
4. *If $B \neq 0$, $\det B = 0$ and $A = -B\gamma$, for some $\gamma \in \mathbb{C} \setminus \mathbb{R}$, then the class $[\![q]\!]$ contains only one **isolated zero** of P;*
5. *If none of the above conditions holds, then there are **no zeros** of P in $[\![q]\!]$.*

We point out that the result in [8] is more complete since it also provides the expression for the zeros. Since we are interested only in the number of zeros of each type, we chose to present a simplified version of the result.

The following result clarify some aspects concerning the zeros of quadratic polynomials in $\mathbb{H}_{\text{coq}}[x]$.

Theorem 4. *Consider a quadratic polynomial P of the form (8).*

1. *If $[\![z]\!]$ is a hyperboloidal zero of P, then $\mathcal{C}_P$ has two double roots or one real root with multiplicity 4, if $\det \underline{z} \neq 0$ or $\det \underline{z} = 0$, respectively.*

2. *If* $[\![z]\!]$ *contains a linear zero, then the roots of* $\mathcal{C}_P$ *are all reals and at least one of them is also a root of* P.
3. P *cannot have both linear and hyperboloidal zeros.*

Proof. It is clear from Theorem 3 that $[\![z]\!]$ is a hyperboloidal zero of a quadratic polynomial P if and only if $\Psi_{[\![z]\!]} = P$. Taking into account relation (3), we conclude that

$$[\![z]\!] = [\![\alpha]\!],\ \alpha \in \mathbb{C} \setminus \mathbb{R} \text{ or } [\![z]\!] = [\![\tfrac{r_1+r_2}{2} + \tfrac{r_1-r_2}{2}\mathbf{j}]\!],\ r_1, r_2 \in \mathbb{R},$$

i.e. P is of one of the forms

$$P(x) = (x-\alpha)(x-\bar{\alpha}) \text{ or } P(x) = (x-r_1)(x-r_2). \tag{9}$$

This implies that

$$\mathcal{C}_P(x) = (x-\alpha)^2(x-\bar{\alpha})^2 \text{ or } \mathcal{C}_P(x) = (x-r_1)^2(x-r_2)^2.$$

Observing that $\frac{r_1+r_2}{2} + \frac{r_1-r_2}{2}\mathbf{j}$ has singular vector part if and only if $r_1 = r_2$, the result 1. follows.

From Theorem 3 (3.) it also follows that a class of the type (5) will never contains a linear zero, since $\det \underline{\alpha} > 0$, for all $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

Consider now a class $[\![\frac{r_i+r_j}{2} + \frac{r_i-r_j}{2}\mathbf{j}]\!]$ of the form (6)–(7). If such class contains a linear zero, then by Theorem 3 (3.), P can be written as

$$P(x) = (x-r_i)(x-r_j) + Bx + A,$$

with $B \neq 0$, $\det B = 0$ and $A = -\left(\frac{r_i+r_j}{2} \pm \frac{r_i-r_j}{2}\right)B$, i.e. P has one of the forms

$$P(x) = (x-r_i)(x-r_j) + B(x-r_i) \text{ or } P(x) = (x-r_i)(x-r_j) + B(x-r_j), \tag{10}$$

for some singular coquaternion $B \neq 0$. Therefore either r_i or r_j is a zero of P. This proves part 2.

Finally, assume that P has one hyperboloidal zero $[\![z_1]\!]$ and one linear zero in $[\![z_2]\!]$. The use of 1. and 2. allows to conclude that $[\![z_1]\!] = [\![\frac{r_1+r_2}{2} + \frac{r_1-r_2}{2}\mathbf{j}]\!]$ and, from (9),

$$\mathcal{C}_P(x) = (x-r_1)^2(x-r_2)^2,$$

with $r_1 \neq r_2$ (otherwise P would have just one zero). The admissible classes that can contain linear zeros are $[\![r_1]\!]$ and $[\![r_2]\!]$. Assume first that $[\![z_2]\!] = [\![r_1]\!]$. The use of Theorem 3 gives

$$P(x) = (x-r_1)(x-r_2) = \Psi_{[\![r_1]\!]}(x) + Bx + A = (x-r_1)^2 + Bx + A,$$

which implies that $B = r_1 - r_2$. Since $r_1 \neq r_2$, it follows that the necessary condition $\det B = 0$ for the existence of linear zeros does not hold. The same reasoning leads to the conclusion that $[\![r_2]\!]$ is not also a linear zero. □

5 The Number and Nature of the Zeros

We now discuss in detail all the cases that may occur concerning the zeros of the companion polynomial of the quadratic polynomial P – a 4th degree polynomial with real coefficients – having in mind the correspondence between these zeros and the admissible classes of P, which we know already are at most 6 (cf. Sect. 3).

To illustrate each case, we construct polynomials, by using a set of Mathematica functions – `CoqPolynomial` – specially design to deal with coquaternionic polynomials [9].

Case 1: $\mathcal{C}_P(x) = (x-\alpha)(x-\bar{\alpha})(x-\beta)(x-\bar{\beta}),\ \alpha,\beta \in \mathbb{C}\setminus\mathbb{R}$;

In this case, the number of admissible classes is 1 ($[\![\alpha]\!]$, if $\alpha=\beta$) or 2 ($[\![\alpha]\!]$ and $[\![\beta]\!]$, if $\alpha\neq\beta$). From Theorem 4, it follows that there are no linear zeros, but P can have: one hyperboloidal zero $[\![\alpha]\!]$, if $\alpha=\beta$; or at most two isolated zeros.

Example 1. Consider the polynomials

$$P_1(x) = x^2 - 2x + 4 \qquad P_2(x) = x^2 - 3\mathbf{i}x - 3 + \mathbf{i}$$
$$P_3(x) = x^2 + (2-4\mathbf{i})x - 3 - 4\mathbf{i} \qquad P_4(x) = x^2 + 1 - \mathbf{i} - \mathbf{j}.$$

The details about the zero-sets of each polynomial and its companion polynomial are as follows:

1. $Z(\mathcal{C}_{P_1})$: $1 \pm \sqrt{3}\mathbf{i}$, $1 \pm \sqrt{3}\mathbf{i}$
   $Z(P_1)$: **one** hyperboloidal zero $[\![1+\sqrt{3}\mathbf{i}]\!]$
2. $Z(\mathcal{C}_{P_2})$: $-1 \pm 2\mathbf{i}$, $1 \pm \mathbf{i}$
   $Z(P_2)$: **two** isolated zeros $-1+2\mathbf{i}$ and $1+\mathbf{i}$
3. $Z(\mathcal{C}_{P_3})$: $-1 \pm 2\mathbf{i}$, $-1 \pm 2\mathbf{i}$
   $Z(P_3)$: **one** isolated zero $-1+2\mathbf{i}$
4. $Z(\mathcal{C}_{P_4})$: $\pm\mathbf{i}$, $\pm\mathbf{i}$
   $Z(P_4)$: **no** zeros

Case 2: $\mathcal{C}_P(x) = (x-\alpha)(x-\bar{\alpha})(x-r_1)(x-r_2),\ \alpha \in \mathbb{C}\setminus\mathbb{R}, r_1, r_2 \in \mathbb{R}$

In this situation it follows at once from Theorem 4 that there are only (if any) isolated roots. Since there are two admissible classes $[\![\alpha]\!]$ and $[\![\frac{r_1+r_2}{2} + \frac{r_1-r_2}{2}\mathbf{j}]\!]$, the polynomial P has at most 2 isolated zeros.

Example 2. The following polynomials illustrate Case 2.

$$P_1(x) = x^2 + (1+\mathbf{i}+\mathbf{k})x + 1 + \mathbf{i} - \mathbf{j} - \mathbf{k}$$
$$P_2(x) = x^2 + (1+\mathbf{i}+\mathbf{k})x + 1 - \mathbf{i} - \mathbf{j} - \mathbf{k}$$
$$P_3(x) = x^2 + (1+\mathbf{i}+\mathbf{j}+\mathbf{k})x + 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$$

1. $Z(\mathcal{C}_{P_1})$: $\pm\sqrt{3}\mathbf{i}$, 0, -2
   $Z(P_1)$: **two** isolated zeros $-4\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and $-1+\mathbf{i}+\mathbf{j}-\mathbf{k}$
2. $Z(\mathcal{C}_{P_2})$: $-1 \pm \sqrt{7}\mathbf{i}$, 0, -1
   $Z(P_2)$: **one** isolated zero $\frac{1}{2}(-1+3\mathbf{i}+3\mathbf{j}+\mathbf{k})$

3. $Z(cal\, C_{P_3})$: $-1 \pm 2\mathbf{i}$, 0, 0
   $Z(P_3)$: **no** zeros

Case 3: $\mathcal{C}_P(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4),\ r_1, r_2, r_3, r_4 \in \mathbb{R}$

In this case, we have to distinguish between 5 different cases.

Case 3.1: $r_1 = r_2 = r_3 = r_4$

In this case there is just one admissible class $[\![r_1]\!]$, which means that there is at most one zero of P. In case of existence, such zero can be of any of the three different natures, as it is shown in Example 3 (cf. [8, Example 4.5]).

Example 3. The companion polynomial of the following quadratic polynomials

$$\begin{aligned} P_1(x) &= x^2 - 2x + 1 \\ P_2(x) &= x^2 - (2 + \mathbf{i} + \mathbf{j})x + 1 + \mathbf{i} + \mathbf{j} \\ P_3(x) &= x^2 - (2 + 6\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})x + 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \\ P_4(x) &= x^2 - 2x + 1 + 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}. \end{aligned}$$

has 1 as a zero of multiplicity 4. However the situation concerning the zeros of the correspondent polynomials is quite different.

1. $Z(P_1)$: **one** hyperboloidal zero $[\![1]\!]$
2. $Z(P_2)$: **one** linear zero $\{1 + \lambda(\mathbf{i} + \mathbf{j}) : \lambda \in \mathbb{R}\}$
3. $Z(P_3)$: **one** isolated zero $\frac{1}{2}(1 + 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$
4. $Z(P_4)$: **no** zeros

Case 3.2: $r_1 = r_2 = r_3 \neq r_4$.

The admissible classes are $\mathcal{C}_1 = [\![r_1]\!]$ and $\mathcal{C}_2 = [\![\frac{r_1+r_4}{2} + \frac{r_1-r_4}{2}\mathbf{j}]\!]$. Since there are no hyperboloidal zeros, the possiblities for the zeros of P are: 2 linear zeros, 1 linear zero and/or 1 isolated zero or only isolated zeros (at most 2).

We are going to show that $\mathcal{C}_1$ contains a linear zero if and only if $\mathcal{C}_2$ contains a linear zero.

Assume first that $\mathcal{C}_1$ contains a linear zero. From (10) we know that r_1 itself is a zero of P. Following Theorem 3 for the class $\mathcal{C}_2$, we obtain

$$P(x) = (x - r_1)(x - r_4) + Bx + A$$

and since $P(r_1) = 0$, it follows that $A = -r_1 B$. Observe that for any $q = q_0 + \underline{q} \in \mathcal{C}_2$ we have that $q_0 = r_1$ and $\det \underline{q} = 0$. To conclude that $\mathcal{C}_2$ contains a linear zero it remains to prove that $\det \overline{B} = 0$. But, if B is nonsingular, then the first part of Teorem 3 gives that the only zero of P in $\mathcal{C}_2$ is r_1 which clearly is not possible ($r_1 \in \mathcal{C}_2 \Leftrightarrow r_1 = r_4$).

We prove now that the converse is also true. If $\mathcal{C}_2$ contains a linear zero, then either r_1 or r_4 is a zero of P (see Theorem 4). Clearly r_4 can not be a zero of P, because it is not (at least) a double zero of its companion polynomial. To classify the zero (if any) in the class $\mathcal{C}_1$, we write

$$P(x) = (x - r_1)^2 + Bx + A.$$

Since r_1 is a zero of P, then $A = -r_1 B$. We argue now that B is singular. If B is nonsingular, then the first part of Theorem 3 gives that the only zero of P in $\mathcal{C}_1$ is r_1. But

$$P(x) = (x - r_1)(x - r_1 + B)$$

which means that $r_1 - B$ is a zero of P, which in turn implies that $r_1 - B \in \mathcal{C}_1$ or $r_1 - B \in \mathcal{C}_2$. In the first case we get $\det B = 0$, while in the second one we obtain $\operatorname{Re} B = \frac{r_1 - r_4}{2}$ and $\det \underline{B} = \frac{(r_1 - r_4)^2}{4}$. This also leads to $\det B = 0$.

In conclusion, the possibilities for the zeros of P are: 2 linear zeros or only isolated zeros (at most 2), as is illustrated in Example 4.

Example 4. Consider the polynomials:

$$\begin{aligned} P_1(x) &= x^2 + (1 + 5\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})x - 2 + 10\mathbf{i} + 6\mathbf{j} + 10\mathbf{k} \\ P_2(x) &= x^2 + (1 + \mathbf{i} + 2\mathbf{k})x + 1 + \mathbf{j} + \mathbf{k} \\ P_3(x) &= x^2 + (-5 + \mathbf{i} + \mathbf{j} + \mathbf{k})x + 6 + \mathbf{i} - 3\mathbf{j} + \mathbf{k} \end{aligned}$$

The details about the zero-set of each polynomial and its companion polynomial are as follows:

1. $Z(\mathcal{C}_{P_1})$: -2, -2, -2, 4

   $Z(P_1)$: **two** linear zeros
   $\{-2 + \lambda(17\mathbf{i} + 15\mathbf{j} + 8\mathbf{k}) : \lambda \in \mathbb{R}\}$ and $\{1 - 3\mathbf{j} + \lambda(\mathbf{i} + \mathbf{k}) : \lambda \in \mathbb{R}\}$
2. $Z(\mathcal{C}_{P_2})$: -1, -1, -1, 1
   $Z(P_2)$: **two** isolated zeros $-1 - \mathbf{i} - \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
3. $Z(\mathcal{C}_{P_3})$: 1, 3, 3, 3
   $Z(P_3)$: **one** isolated zero $1 + 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

Case 3.3: $r_1 = r_2 \neq r_3 = r_4$

The admissible classes are $\mathcal{C}_1 = [\![r_1]\!]$, $\mathcal{C}_2 = [\![r_3]\!]$ and $\mathcal{C}_3 = [\![\frac{r_1 + r_3}{2} + \frac{r_1 - r_3}{2}\mathbf{j}]\!]$. It is clear that, in case of existence, the only hyperboloidal zero would be $\mathcal{C}_3$; in this case the polynomial is $P(x) = (x - r_1)(x - r_3)$ which means that there are also two isolated zeros: r_1 and r_3.

Concerning the linear zeros we prove now that neither $\mathcal{C}_1$ nor $\mathcal{C}_2$ can contain such kind of zeros. In fact, if $\mathcal{C}_1$ contains a linear zero, then

$$P(x) = (x - r_1)^2 + B(x - r_1) = (x - r_1)(x - r_1 + B),$$

for some singular B. Therefore

$$\mathcal{C}_P(x) = (x - r_1)^3(x - r_1 + 2\operatorname{Re} B),$$

and r_1 would be a zero of $\mathcal{C}_P$ with multiplicity 3. The same reasoning leads to the conclusion that $\mathcal{C}_2$ can not contain a linear zero. This means that linear zeros, if any, are in $\mathcal{C}_3$. If such kind of zeros exists, then

$$P(x) = (x - r_1)(x - r_3 + B) \quad \text{or} \quad P(x) = (x - r_3)(x - r_1 + B),$$

for some singular B, which also shows that there is one isolated zero in $\mathcal{C}_1$ and another one in $\mathcal{C}_2$. In fact, if P is of the first form, then it has the roots $r_1 \in \mathcal{C}_1$ and $r_3 - B$. Moreover

$$\mathcal{C}_P(x) = (x - r_1)^2(x - r_3)(x - r_3 + 2\mathrm{Re}\, B),$$

which leads to the conclusion that $\mathrm{Re}\, B = 0$ and the zero $r_3 - B \in \mathcal{C}_2$. If P is of the second form, we use similar arguments to prove that $r_1 - B \in \mathcal{C}_1$.

In conclusion, the possibilities in the case under consideration are: one hyperboloidal zero and 2 isolated zeros; one linear zero and 2 isolated zeros or only isolated zeros (at most 3).

Example 5. To illustrate Case 3.3, we have chosen the following polynomials:

$$\begin{aligned}
P_1(x) &= x^2 + 4x + 1\\
P_2(x) &= x^2 + (3 + 5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})x + 2 + 10\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}\\
P_3(x) &= x^2 + (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})x - \tfrac{1}{6}(9 + 10\mathbf{i} + \mathbf{j} + 6\mathbf{k})\\
P_4(x) &= x^2 + (-1 + \tfrac{1}{2}\mathbf{i} + \mathbf{j} + \tfrac{1}{2}\mathbf{k})x + \tfrac{1}{2} + \mathbf{i} + \tfrac{1}{2}\mathbf{j} - \mathbf{k}\\
P_5(x) &= x^2 + (1 - 3\mathbf{i} - 3\mathbf{j} - \mathbf{k})x + \tfrac{1}{2}(1 - 3\mathbf{i} - 3\mathbf{j} - \mathbf{k})
\end{aligned}$$

1. $Z(\mathcal{C}_{P_1})$: $-2 \pm \sqrt{3}$, $-2 \pm \sqrt{3}$
   $Z(P_1)$: **one** hyperboloidal zero $[\![-2 + \sqrt{3}\mathbf{j}]\!]$
   two isolated zeros $-2 + \sqrt{3}$ and $-2 - \sqrt{3}$

2. $Z(\mathcal{C}_{P_2})$: -2, -2, -1, -1
   $Z(P_2)$: **one** linear zero $\{-\frac{3}{2} - \frac{2}{5}\mathbf{j} + \frac{3}{10}\mathbf{k} + \lambda(5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) : \lambda \in \mathbb{R}\}$
   two isolated zeros -2 and $-\frac{3}{2} + \frac{1}{2}\mathbf{j}$
3. $Z(\mathcal{C}_{P_3})$: -2, -2, 1, 1
   $Z(P_3)$: **three** isolated zeros $-2 - \frac{29}{24}\mathbf{i} - \frac{5}{6}\mathbf{j} - \frac{7}{8}\mathbf{k}$, $-\frac{1}{2} - \frac{4}{3}\mathbf{i} + \frac{1}{6}\mathbf{j} - 2\mathbf{k}$ and $1 + \frac{5}{12}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{1}{4}\mathbf{k}$
4. $Z(\mathcal{C}_{P_4})$: 0, 0, 1, 1
   $Z(P_4)$: **two** isolated zeros $\frac{1}{2} - \frac{3}{2}\mathbf{i} - \frac{3}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$ and $1 - \mathbf{i} - \mathbf{j}$
5. $Z(\mathcal{C}_{P_5})$: -1, -1, 0, 0
   $Z(P_5)$: **one** isolated zero $-\frac{1}{2} + \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$

Case 3.4: $r_1 = r_2 \neq r_3 \neq r_4$

The admissible classes are $\mathcal{C}_1 = [\![r_1]\!]$, $\mathcal{C}_2 = [\![\frac{r_1+r_3}{2} + \frac{r_1-r_3}{2}\mathbf{j}]\!]$, $\mathcal{C}_3 = [\![\frac{r_1+r_4}{2} + \frac{r_1-r_4}{2}\mathbf{j}]\!]$ and $\mathcal{C}_4 = [\![\frac{r_3+r_4}{2} + \frac{r_3-r_4}{2}\mathbf{j}]\!]$. It follows at once that there are no hyperboloidal zeros and arguments similar to those used in Case 3.3 lead to the conclusion that, in case of existence, the linear zeros are contained in $\mathcal{C}_2$ and $\mathcal{C}_3$ and in this case there are also two isolated zeros, one in each of classes $\mathcal{C}_1$ and $\mathcal{C}_4$.

Finally, the polynomial P can have only isolated zeros (up to 4). It is easy to see that when P has an isolated zero in the class $\mathcal{C}_2$, then there is also an isolated zero of P in $\mathcal{C}_3$, being the converse also true.

Example 6. To illustrate this situation, we use the polynomials:

$$P_1(x) = x^2 + (-\tfrac{7}{2} + \mathbf{i} + \tfrac{1}{2}\mathbf{j} + \mathbf{k})x + \tfrac{5}{2} - \mathbf{i} - \tfrac{1}{2}\mathbf{j} - \mathbf{k}$$
$$P_2(x) = x^2 + (-1 + \mathbf{j} + 2\mathbf{k})x - 2 + 6\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$
$$P_3(x) = x^2 + (1 - 2\mathbf{i} - \mathbf{j} + 2\mathbf{k})x - 2 - 2\mathbf{j}$$
$$P_4(x) = x^2 + 2x + 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

The corresponding details are as follows:

1. $Z(\mathcal{C}_{P_1})$: 1, 1, 2, 3
   $Z(P_1)$: **two** linear zeros

   $$\{\tfrac{3}{2} - \tfrac{3}{10}\mathbf{j} + \tfrac{2}{5}\mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})\} \text{ and } \{2 - \mathbf{k} + \lambda(\mathbf{i} + \mathbf{k})\}\ (\lambda \in \mathbb{R})$$

   two isolated zeros 1 and $\frac{5}{2} - \mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$
2. $Z(\mathcal{C}_{P_2})$: -2, 0, 0, 4

   $Z(P_2)$: **four** isolated zeros $-1 + 8\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$, $-5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$, $1 - 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ and $2 - \mathbf{i} - 2\mathbf{j} - \mathbf{k}$
3. $Z(\mathcal{C}_{P_3})$: -2, -2, 0, 2
   $Z(P_3)$: **three** isolated zeros $-2 + \mathbf{i} - \mathbf{k}$, $-1 - \mathbf{j}$ and $1 + \mathbf{j}$
4. $Z(\mathcal{C}_{P_4})$: -1, -1, $-1 \pm \sqrt{2}$
   $Z(P_4)$: **two** isolated zeros $-1 - \frac{1}{\sqrt{2}} + \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$ and $-1 + \frac{1}{\sqrt{2}} - \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$

Case 3.5: $r_1 < r_2 < r_3 < r_4$

In this last situation, there are 6 admissible classes and all the zeros are isolated.

Example 7. The zeros of the following polynomials are all isolated.

$$P_1(x) = x^2 + (-2 + \mathbf{i} - 2\mathbf{j} - 2\mathbf{k})x - 2 + 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$
$$P_2(x) = x^2 + (-\mathbf{i} - 3\mathbf{j} - \mathbf{k})x + 1 + \mathbf{i} - \mathbf{j} + \mathbf{k}$$
$$P_3(x) = x^2 + 3\mathbf{k}x + 1 + \mathbf{k}$$

1. $Z(\mathcal{C}_{P_1})$: -2, 0, 1, 5
   $Z(P_1)$: **six** isolated zeros $-1 + \mathbf{j}$, $\frac{1}{2}(-1 - \mathbf{i} + 3\mathbf{j} - \mathbf{k})$, $\frac{1}{2}(1 + 3\mathbf{i} - \mathbf{j} - 3\mathbf{k})$, $\frac{1}{6}(9 + 5\mathbf{i} + 21\mathbf{j} + 5\mathbf{k})$, $\frac{1}{2}(5 - 21\mathbf{i} - 5\mathbf{j} + 21\mathbf{k})$ and $\frac{1}{9}(27 - 29\mathbf{i} + 3\mathbf{j} + 34\mathbf{k})$
2. $Z(\mathcal{C}_{P_2})$: -2, -1, 0, 3
   $Z(P_2)$: **five** isolated zeros $-1 - \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\frac{1}{2}(-1 - \mathbf{i} + \mathbf{j} - \mathbf{k})$, $\frac{1}{2}(1 + \mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$, $\frac{1}{5}(5 + 2\mathbf{i} + 10\mathbf{j} + 2\mathbf{k})$ and $\frac{1}{2}(3 - 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$
3. $Z(\mathcal{C}_{P_3})$: -2, -1, 0, 3
   $Z(P_3)$: **four** isolated zeros $-1 - \mathbf{k}$, $1 - 2\mathbf{k}$, $\frac{1}{2}(-1 - \mathbf{k})$, $\frac{1}{2}(1 - 5\mathbf{k})$,

Table 1. Summary of the results

Case	Number of zeros (max)
1	1 HZ or 2 IZ
2	2 IZ
3.1	1 HZ or 1 LZ or 1 IZ
3.2	2 LZ or 2 IZ
3.3	1 HZ + 2 IZ or 1 LZ + 2 IZ or 3 IZ
3.4	2 LZ + 2 IZ or 4 IZ
3.5	6 IZ

6 Conclusions

Table 1 summarizes the main contributions of this paper concerning the maximum number of isolated (IZ), linear (LZ) and hyperboloidal zeros a quadratic coquaternionic polynomial may have. We believe that the results obtained are interesting from a theoretical point of view, but may also be useful for applications.

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