# Computational Model for Fluid and Elastic Solid Interaction Based on Symmetric Hyperbolic Thermodynamically Compatible Systems Theory

Evgeniy Romenski[1(✉)] and Galina Reshetova[2]

[1] Sobolev Institute of Mathematics SB RAS, Novosibirsk 630090, Russia
evrom@math.nsc.ru
[2] Institute of Computational Mathematics and Mathematical Geophysics SB RAS, Novosibirsk 630090, Russia
kgv@nmsf.sscc.ru

**Abstract.** A computational model of interaction of a compressible fluid and deformable elastic solid is presented. The model is derived from the general solid-fluid two-phase mixture model and its derivation is based on the Symmetric Hyperbolic Thermodynamically Compatible (SHTC) systems theory. The governing equations form a symmetric hyperbolic system of partial differential equations of the first order, the solutions of which satisfy the thermodynamic law of conservation of energy. These properties allow the direct application of advanced high accuracy computational methods to solve model equations and ensure the reliability of numerically obtained solutions. Some preliminary results of numerical simulation are presented, showing the applicability of the model for studying complex problems of the solid-fluid interaction.

**Keywords:** Fluid solid interaction · Symmetric Hyperbolic Thermodynamically Compatible system · Runge-Kutta WENO method · Diffuse interface

## 1 Introduction

The solid-fluid interaction is a complex multiphysics problem of interest for various industrial applications, see, for example, [1]. One of the main challenges in modeling this type of problems is the numerical treatment of the interface between a fluid and a deformable moving solid. The difficulty here consists in

The development of the model is supported by the state contract of the Sobolev Institute of Mathematics (project no. FWNF-2022-0008), implementation of numerical methods and simulations are supported by the Russian Science Foundation (project no. 19-77-20004).

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022
O. Gervasi et al. (Eds.): ICCSA 2022, LNCS 13376, pp. 303–313, 2022.
https://doi.org/10.1007/978-3-031-10450-3_26

the formulation of a moving curvilinear interface boundary condition and its numerical implementation. In recent decades, a diffuse interface approach has emerged that has an advantage of being able to use a rectangular mesh in the 2D case and a parallelepiped mesh in the 3D case. An application of various high-order numerical methods combined with a diffuse interface approach to multi-material problems can be found in a recently published review paper [2]. The main idea of this approach is to formulate a general multiphase model, rather than to combine separate models for each material through the boundary interaction. In this case, the interface is taken into account as a jump in the volume fraction of the phases, and the interaction of the phases is automatically described by the conservation laws of the general multiphase model. Such an approach is highly convenient for numerical implementation and high-performance numerical simulations.

The objective of the present paper is to develop a computational model for fluid interaction with a deformable solid combining a high accuracy Runge-Kutta WENO numerical method and diffuse interface approach. The mathematical model is based on a general theory of Symmetric Hyperbolic Thermodynamically Compatible (SHTC) equations [3–6], which has been successfully applied to modeling processes in complex media and development of the unified model of continuum [7–9]. We consider a compressible solid-fluid mixture theory [10] formulated within the class of SHTC equations. In this formulation, fluid and solid constituents are characterized by their own set of state variables, material constants, and a thermodynamic equation of state. The closure relations of the model depend on the volume and mass fraction of the phases according to some mixture rules. Thus, for the case of a vanishing phase volume fraction, one can obtain the governing equations for a pure fluid or pure elastic solid phases. This property is highly attractive, because it allows one to simulate in domains of a complex structure by the single PDE system with the use of diffuse interfaces. This approach has already been used in [11] to simulate the propagation of small amplitude waves, and in this paper we apply it to nonlinear processes.

The rest of the paper is organized as follows. Section 2 formulates the governing equations of the model and discusses their properties. Section 3 briefly describes high accuracy Runge-Kutta WENO numerical method, and Sect. 4 presents some preliminary results of solving a numerical test problem.

## 2 Model Formulation and Governing Equations

The mathematical model presented in the paper for simulations the interaction of an elastic solid and a compressible fluid can be formulated using a solid-fluid two-phase two-pressure two-velocity model from [10]. The derivation is based on the one velocity and one pressure approximation of the above model and a special choice of the volume fraction balance equation. Denote the fluid and solid phase parameters by indexes 1 and 2, respectively. Assume that the solid-fluid mixture is characterised by the volume fractions of the fluid $\alpha_1$ and the solid $\alpha_2 = 1 - \alpha_1$. The other state variables characterizing the medium are: $v^i$ -

velocity of the medium (common for both phases), $\rho_1, \rho_2$ - phase mass densities of the fluid and solid, respectively, $A_{ij}(i,j=1,2,3)$ - the distortion field for the entire mixture.

The resulting system of equations consists of the total momentum conservation law, phase densities conservation laws for the fluid and solid phases and evolution equation for the distortion, which can be obtained from the PDE system presented in [10] assuming that the phase velocities of the mixture are equal to $v^i$ and the phase pressure is equal to the common value $p_1(\rho_1) = p_2(\rho_2) = p$. We should also add an additional balance equation for the volume fraction, which in our case differs from that presented in [10]. The equation for the volume fraction in physically consistent two-phase models contains a transport term and a source term which is responsible for the phase pressures relaxation. In our case, we neglect the mentioned source term and consider only a term describing transport with the velocity of the medium. This means that the transition zone between the phases is not correctly simulated from the point of view of physics, and the volume fraction simply plays the role of a color function. Nevertheless, such an approach for modeling a two-phase flow with interfaces was proposed in [12] and was successfully applied to diffuse interphase simulations in many multimaterial problems.

Thus, the complete system of the governing PDEs reads as

$$\begin{aligned}
&\frac{\partial \rho v^i}{\partial t} + \frac{\partial (\rho v^i v^k + p\delta_{ik} - \sigma_{ik})}{\partial x_k} = 0,\\
&\frac{\partial \alpha_1 \rho_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 v^k}{\partial x_k} = 0,\\
&\frac{\partial \alpha_2 \rho_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 v^k}{\partial x_k} = 0,\\
&\frac{\partial A_{ik}}{\partial t} + \frac{\partial A_{im} v^m}{\partial x_k} + v^j \left( \frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k} \right) = 0,\\
&\frac{\partial \rho \alpha_1}{\partial t} + \frac{\partial \rho \alpha_1 v^k}{\partial x_k} = 0.
\end{aligned} \tag{1}$$

Here $\rho = \alpha_1\rho_1 + \alpha_2\rho_2$ is the mixture total density, $\rho_1, \rho_2$ are the phase densities, $p = p_1(\rho_1) = p_2(\rho_2)$ is the pressure of the mixture, coinciding with the pressures of both phases, $A_{ik}$ is the distortion, $\sigma_{ik}(A_{mn}, \alpha_1)$ is the shear stress, whose definition is given below.

Note that the energy conservation law holds for system (1)

$$\frac{\partial \rho (E + v^i v^i/2)}{\partial t} + \frac{\partial (\rho v^k (E + v^i v^i/2) + v^k p - v^i \sigma_{ik})}{\partial x_k} = 0, \tag{2}$$

that means that the model is thermodynamically compatible. Here $E(\alpha_1, \rho_1, \rho_2, A_{ij})$ is the specific internal energy of the mixture.

System (1) has an additional important property. An additional stationary conservation law (constraint) holds

$$\frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k} = 0 \tag{3}$$

for all the time moments if it holds in the initial data at $t = 0$.

We emphasize that system (1) can be used to simulate a "pure" compressible fluid flow if $\alpha_2 = \epsilon, \alpha_1 = 1 - \epsilon$, where $\epsilon$ is a very small number. And vice versa, it can be used to simulate the finite deformations of a"pure" elastic medium if $\alpha_1 = \epsilon, \alpha_2 = 1 - \epsilon$, with a very small $\epsilon$. This can be done by a special choice of closing relations and material constants. Note that it is possible to take into account the non-elastic deformation of a solid medium by introducing a source term into the right-hand side of the equation for distortion $A_{ik}$, as is described, for example, in [7].

System (1) belongs to the class of the SHTC systems, i.e. it is thermodynamically compatible and can be converted to a symmetric hyperbolic form. These properties ensure the reliability of the numerically obtained solutions.

Closing relations for system (1) are the dependence of phase pressures on the density of the corresponding phases and the dependence of shear stress on distortion. These relations can be obtained from the internal energy thermodynamic potential $E$, which provides the closure of the general two-phase fluid-solid mixture model (see the details in [10]).

In this paper, we use the stiffened gas equation of state for each phase that means that the energy for each phase takes the following form

$$e(\rho) = \frac{C^2}{\gamma(\gamma-1)}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)} + \frac{\rho_0 C^2 - \gamma p_0}{\gamma\rho}, \tag{4}$$

where $C$ is the bulk sound velocity, $\gamma$ is the adiabatic constant, $\rho_0$ is the reference density and $p_0$ is the common reference pressure for two phases. Based on the thermodynamic definition of the pressure $p = \rho^2 \partial e/\partial\rho$, the pressure for each phase is computed as

$$p_i(\rho_i) = \frac{\rho_{i0} C_i^2}{\gamma_i}\left(\left(\frac{\rho_i}{\rho_{i0}}\right)^{\gamma_i} - 1\right) + p_0, \quad i = 1, 2, \tag{5}$$

where material constants for each phase are different. The shear stress is defined as (see [10])

$$\sigma_{ik}(A_{mn}) = -\frac{\rho c_2 C_s^2}{2}\left(g_{i\alpha}g_{\alpha k} - \frac{1}{3}g_{\alpha\beta}g_{\beta\alpha}\delta_{ik}\right), \tag{6}$$

where $g_{ij} = G_{ij}(detG)^{-1/3}$ is the normalized Finger tensor $G_{ij} = A_{\alpha i}A_{\alpha j}$, $C_s^2$ is the squared shear velocity of sound in the solid, $c_2 = \alpha_2\rho_2/\rho$ is the mass fraction of the solid phase. It can be seen from (6) that in the pure fluid ($c_2 = 0$) the shear stress vanishes, whereas in the pure solid ($c_2 = 1$) it corresponds to the Hooke law for the elastic medium.

## 3 Numerical Method

The governing equations of the model formulated in the previous section form a first order hyperbolic system, and therefore the modern high accuracy methods developed for solving such equations can be applied to numerical simulations. We consider two-dimensional processes and reduced two-dimensional equations in the coordinates $x_1 = x, x_2 = y$ in the presence of possible source terms (for example, gravity) can be written down as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} + B_1(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} + B_2(\mathbf{U})\frac{\partial \mathbf{U}}{\partial y} = \mathbf{S}(\mathbf{U}). \tag{7}$$

Here $\mathbf{U}$ is the vector of conservative variables

$$\mathbf{U} = (\rho v_1, \rho v_2, \alpha_1\rho_1, \alpha_2\rho_2, A_{11}, A_{21}, A_{12}, A_{22}, \alpha_1)^T. \tag{8}$$

Note that in the two-dimensional case $A_{13} = A_{31} = 0, A_{33} = 1$.

The non-conservative terms in (7) with the matrices $B_1(\mathbf{U}), B_2(\mathbf{U})$ correspond to the *curlA* term in the fourth equation of system (1). In this paper, we assume that the numerical approximation gives us the value of *curlA* close to zero due to compatibility condition (3) and therefore we neglect the non-conservative terms in (7). Of course, this assumption is made only to test the performance of the model and should be discarded in the future when developing more accurate methods.

We also introduce the vector of primitive variables

$$\mathbf{V} = (v_1, v_2, \rho_1, \rho_2, A_{11}, A_{21}, A_{12}, A_{22}, \alpha_1)^T. \tag{9}$$

The fluxes $\mathbf{F}(\mathbf{U})$ and $\mathbf{G}(\mathbf{U})$ in the $x$ and $y$ directions are computed as

$$\mathbf{F}(\mathbf{U}) = (\rho v_1^2 - \Sigma_{11}, \rho v_1 v_2 - \Sigma_{21}, \alpha_1\rho_1 v_1, \alpha_2\rho_2 v_1, A_{1j}v_j, A_{2j}v_j, 0, 0, \alpha_1 v_1)^T, \tag{10}$$

$$\mathbf{G}(\mathbf{U}) = (\rho v_1 v_2 - \Sigma_{12}, \rho v_2^2 - \Sigma_{22}, \alpha_1\rho_1 v_2, \alpha_2\rho_2 v_2, 0, 0, A_{1j}v_j, A_{2j}v_j, \alpha_1 v_2)^T, \tag{11}$$

where $\Sigma_{ij} = \sigma_{ij} - p\delta_{ij}, i, j = 1, 2$.

Finally, if the gravity is taken into account, the vector of source terms is computed as

$$\mathbf{S}(\mathbf{U}) = (0, -\rho g, 0, 0, 0, 0, 0, 0, 0)^T, \tag{12}$$

where $g = 9.8\,\mathrm{m/s}^2$ is the acceleration of gravity.

To solve system (7), we apply the standard Runge-Kutta WENO method [14], which is successfully used in modeling various problems of fluid dynamics and solid mechanics. The chosen order of accuracy of the method is the fifth order in space and time. Since we apply the diffuse interface method, the spatial approximation uses a square grid, which is highly convenient for the numerical implementation.

The numerical experiments described below were performed in the square domain $(x, y) \in [0, L] \times [0, L]$ with the cells $I_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$, where $x_{i\pm1/2} = x_i \pm h/2$, $y_{j\pm1/2} = y_j \pm h/2$, $h = L/N$, $N$ is the number of intervals along the axes $X$ and $Y$. The spatial finite difference approximation of system (7) for the mesh cell $I_{ij}$ reads as

$$\frac{d\mathbf{U}_{ij}}{dt} + \frac{\mathbf{F}_{i+1/2,j} - \mathbf{F}_{i-1/2,j}}{h} + \frac{\mathbf{G}_{i,j+1/2} - \mathbf{G}_{i,j-1/2}}{h} = \mathbf{S}_{ij}, \tag{13}$$

where $\mathbf{U}_{ij}$ is the value of the vector of conservative variables corresponding to the mesh cell $I_{ij}$, the fluxes $\mathbf{F}_{i\pm1/2,j}$, $\mathbf{G}_{i,j\pm1/2}$ are computed at the cell boundaries, and $\mathbf{S}_{ij}$ is the value of the source terms in this cell.

Thus, if the values of the fluxes at the cell boundaries are known, we obtain an ordinary differential equation for $\mathbf{U}_{ij}$ which can be written in the following form:

$$\frac{d\mathbf{U}_{ij}}{dt} = \mathcal{L}(\mathbf{U}), \tag{14}$$

where $\mathcal{L}(\mathbf{U}) = -(\mathbf{F}_{i+1/2,j} - \mathbf{F}_{i-1/2,j})/h - (\mathbf{G}_{i,j+1/2} - \mathbf{G}_{i,j-1/2})/h + \mathbf{S}_{ij}$. This means that if the cell boundary fluxes are known, we have the ordinary differential equation for $\mathbf{U}_{ij}$ which can be numerically solved by the high order Runge-Kutta method. In the present paper, we use the SSPRK (Strong Stability Preserving Runge-Kutta) method [15], whose formulae for ODE system (14) at the time moment $t_n$ read as

$$\begin{aligned}
&\mathbf{U}^{(0)} = \mathbf{U}^n_{ij},\\
&\mathbf{U}^{(k)} = \mathbf{U}^{(k-1)} + \frac{\Delta t}{2}\mathcal{L}(\mathbf{U}^{k-1}), \quad k = 1, ..., K-1\\
&\mathbf{U}^{(K)} = \sum_{m=0}^{K-2} \alpha_{K,m}\mathbf{U}^{(m)} + \alpha_{K,K-1}\left(\mathbf{U}^{(K-1)} + \frac{\Delta t}{2}\mathcal{L}(\mathbf{U}^{K-1})\right),\\
&\mathbf{U}^{(n+1)} = \mathbf{U}^{(K)},
\end{aligned} \tag{15}$$

We take the fifth order SSPRK method with $K = 6$ and coefficients $\alpha_{K,0} = 1/9$, $\alpha_{K,1} = 2/5$, $\alpha_{K,2} = 0$, $\alpha_{K,3} = 4/9$, $\alpha_{K,4} = 0$, $\alpha_{K,5} = 2/45$.

The time step $\Delta t$ must be chosen in accordance with the Courant-Friedrichs-Lewy stability condition $\Delta t = C_{CFL} \times \min(h/V_{ij})$ with $0 < C_{CFL} < 1$, where $V_{ij}$ is the velocity of fastest wave in the mesh cell $I_{ij}$.

For the evaluation of fluxes on the cell boundaries we use the fifth order WENO reconstruction for primitive variables in conjunction with the Lax-Friedrichs method [13]. We do not give here the formulas for the WENO reconstruction, which can be found in [14]. This gives us the values of all variables on the left and on the right of the cell boundary for each spatial direction. After the reconstruction, we can compute the fluxes as follows. If the reconstructed values of the variables $U_L$ to the left and $U_R$ to the right of the boundary located at

the point $x_{i+1/2,j}$ between cells in the $x$-direction are known, the flux on this boundary is computed as

$$\mathbf{F}_{i+1/2,j} = \frac{1}{2}(\mathbf{F}(\mathbf{U}_L) + \mathbf{F}(\mathbf{U}_R)) - \frac{1}{2}\frac{h}{\Delta t}(\mathbf{U}_R - \mathbf{U}_L). \tag{16}$$

Similarly, the flux in the $y$-direction for the intercell boundary located at $y_{i,j+1/2}$ is computed as

$$\mathbf{G}_{i,j+1/2} = \frac{1}{2}(\mathbf{G}(\mathbf{U}_L) + \mathbf{G}(\mathbf{U}_R)) - \frac{1}{2}\frac{h}{\Delta t}(\mathbf{U}_R - \mathbf{U}_L), \tag{17}$$

where $U_L, U_R$ are reconstructed values from the left and the right sides of the boundary.

Finally, it should be noted that the numerical approximation of the equation for the volume fraction should be done separately in order to avoid spurious oscillations caused by a jump in the volume fraction in the initial data. We take the following approximation:

$$\begin{aligned}
&\frac{d(\alpha_1)_{i,j}}{dt} = \\
&-\frac{1}{2h}\left((v_1)^n_{i,j}\left((\alpha_1)^L_{i+1/2,j} + (\alpha_1)^R_{i+1/2,j}\right) - \frac{h}{\Delta t}\left((\alpha_1)^R_{i+1/2,j} - (\alpha_1)^L_{i+1/2,j}\right)\right) \\
&+\frac{1}{2h}\left((v_1)^n_{i,j}\left((\alpha_1)^L_{i-1/2,j} + (\alpha_1)^R_{i-1/2,j}\right) - \frac{h}{\Delta t}\left((\alpha_1)^R_{i-1/2,j} - (\alpha_1)^L_{i-1/2,j}\right)\right) \\
&-\frac{1}{2h}\left((v_2)^n_{i,j}\left((\alpha_1)^L_{i,j+1/2} + (\alpha_1)^R_{i,j+1/2}\right) - \frac{h}{\Delta t}\left((\alpha_1)^R_{i,j+1/2} - (\alpha_1)^L_{i,j+1/2}\right)\right) \\
&+\frac{1}{2h}\left((v_2)^n_{i,j}\left((\alpha_1)^L_{i,j-1/2} + (\alpha_1)^R_{i,j-1/2}\right) - \frac{h}{\Delta t}\left((\alpha_1)^R_{i,j-1/2} - (\alpha_1)^L_{i,j-1/2}\right)\right)
\end{aligned} \tag{18}$$

This approximation can be derived from fulfilling the requirement of maintaining the homogeneity of the phase density fields in a medium with a discontinuity only in the volume concentration (see, for example [16]). When deriving, specific formulas (16), (17) for the Lax-Friedrichs flux must be used.

## 4 Numerical Simulations

The numerical approach is tested on a two-dimensional model of a "pure" elastic ball falling in a "pure" compressible fluid under the gravity action. For the simulation, we use system (1) and fifth order in space and time Runge-Kutta WENO numerical techniques discussed in the previous sections. Note that to include the gravity force in the model, it is necessary to put the source term $-\rho g$ into the equation for $v_2$, where $g$ is the gravitational acceleration. In the

simulations, we took an artificial acceleration value $g = 900.8 \times 10^3$ to speed up the falling process.

To describe this test model in terms of the solid-fluid mixture, we put a small value of fluid with the volume fraction $\alpha_1 = \epsilon = 10^{-6}$ ($\alpha_2 = 1 - \epsilon$) in the solid phase and vice versa put the same value of the solid phase $\alpha_2 = \epsilon = 10^{-6}$ ($\alpha_1 = 1 - \epsilon$) in the fluid phase. The material parameters of the fluid phase (Fluid) and the solid phase (Solid) are given in Table 1. The adiabatic constant in formula (5) is equal to $\gamma = 2$.

The computational domain is 4-m square with uniform $400 \times 400$ mesh, and the initial ball with radius 0.3 m located at $x = 2$ m and $y = 0.7$ m. The simulation is performed with a Courant number of 0.8.

Figure 1 shows the results of numerical simulations at the initial time (the upper snapshot) and after each 1000 time steps. Figure represents the distribution of the fluid volume fraction $\alpha_1$. We can observe the numerical diffusion of the sharp interface between the ball and the fluid, but the diffusion is not crucial and the overall shape is preserved (Fig. 2). It is possible to reduce this numerical diffusion by employing the Tangent of Hyperbola INterface Capturing (THINC) method [17,18] which we plan to incorporate in our numerical techniques.

**Table 1.** Physical parameters used for simulation.

| State | Property | Parameters | Value | Unit |
|---|---|---|---|---|
| Fluid (water): | Fluid density | $\rho_f = \rho_{10}$ | 1040 | kg/m$^3$ |
| | Sound velocity | $c_f$ | 1500 | m/s |
| | Bulk modulus | $K_f = K_1 = \rho_f c_f^2$ | 2.34 | GPa |
| Solid (granite): | Solid density | $\rho_s = \rho_{20}$ | 2500 | kg/m$^3$ |
| | P-wave velocity | $v_p$ | 6155 | m/s |
| | Bulk velocity | $c_s$ | 4332 | m/s |
| | Shear velocity | $c_{sh}$ | 3787 | m/s |
| | Bulk modulus | $K_s = K_2 = \rho_s c_s^2$ | 46.91 | GPa |
| | Shear modulus | $\mu_s = \mu_2 = \rho_s c_{sh}^2$ | 35.85 | GPa |

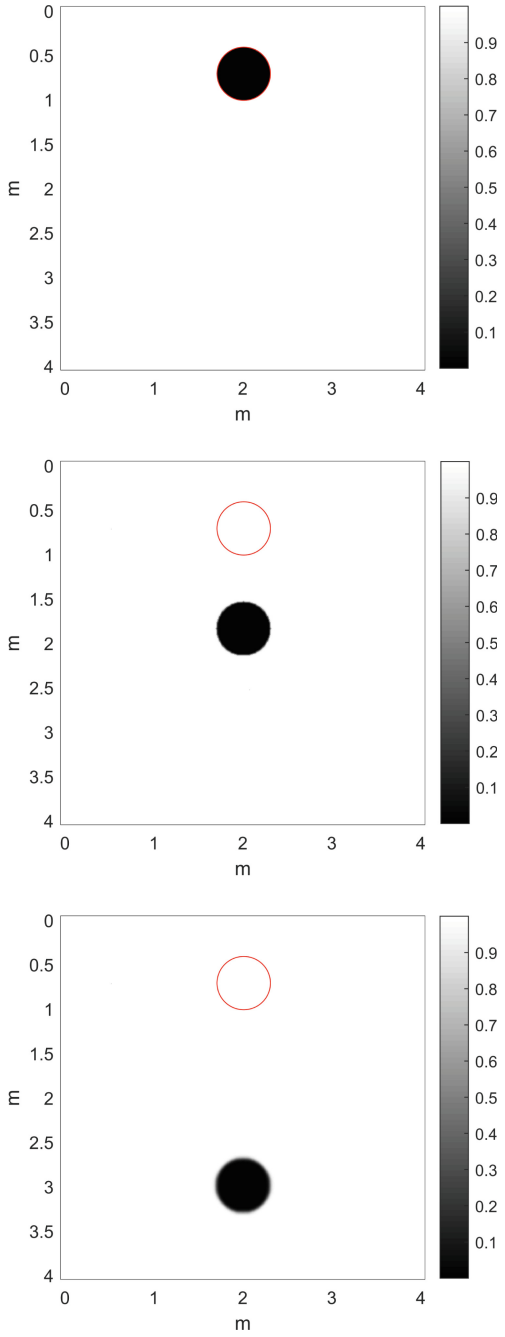



**Fig. 1.** The snapshots of the fluid volume fraction $\alpha_1$ at the initial time (upper figure), after 1000 time steps (middle figure) and after 2000 time steps (lower figure).

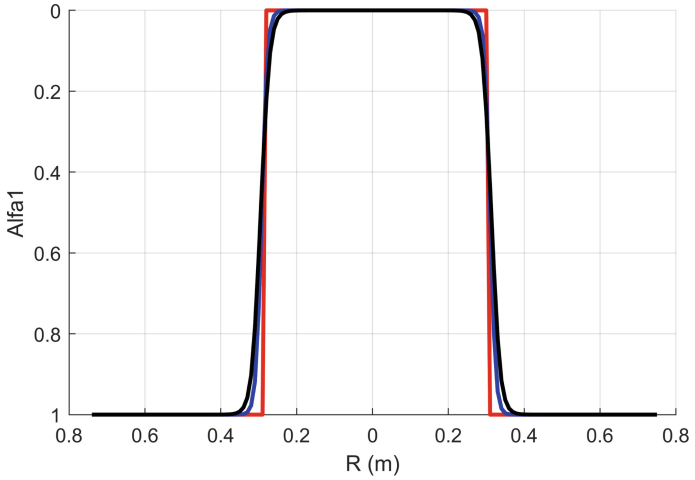



**Fig. 2.** Comparison of diffusion of the fluid volume fraction $\alpha_1$ near the sharp interface between the ball of radius R and the fluid. Red - at the initial time, Blue - after 1000 time steps and Black - after 2000 time steps. (Color figure online)

## 5 Conclusions and Prospects

We have presented a new computational model applicable for the simulation of the interaction of a compressible fluid and an elastic deformable solid. Its derivation is based on the theory of SHTC systems. The governing equations of the model are well-posed (they can be transformed to a symmetric hyperbolic system, and existence and uniqueness of the solution is provided at least locally in time) and thermodynamically correct (the first and second laws of thermodynamics are satisfied). The interfaces between the fluid and the solid are represented by a jump in the phase volume fraction in the single system of PDEs, which allows one to use a diffuse interface model and rectangular (2D) or parallelepiped (3D) meshes. A numerical example is presented that illustrates the applicability of the model for simulation the problem of the solid-fluid interaction. Further research will involve the development of more accurate methods for the diffuse interface, such as THINK [17,18], as well as the application of the method to solve various problems of the interaction of a fluid and a solid.

## References

1. Peksen, M.: Multiphysics modelling of interactions in systems. In: Peksen, M. (ed.) Multiphysics Modelling, pp. 139–159. Academic Press, Cambridge (2018)
2. Maltsev, V., Skote, M., Tsoutsanis, P.: High-order methods for diffuse-interface models in compressible multi-medium flows: a review. Phys. Fluids **34**, 021301 (2022)
3. Godunov, S.K., Romenskii, E.I.: Elements of Continuum Mechanics and Conservation Laws. Kluwer Academic/Plenum Publishers, New York (2003)

4. Godunov, S.K., Mikhaĭlova, T.Y., Romenskii, E.I.: Systems of thermodynamically coordinated laws of conservation invariant under rotations. Siberian Math. J. **37**, 690–705 (1996)
5. Romenski, E.: Hyperbolic systems of thermodynamically compatible conservation laws in continuum mechanics. Math. Comput. Model. **28**, 115–130 (1998)
6. Peshkov, I., Pavelka, M., Romenski, E., Grmela, M.: Continuum mechanics and thermodynamics in the Hamilton and the Godunov-type formulations. Continuum Mech. Thermodyn. **30**(6), 1343–1378 (2018). https://doi.org/10.1007/s00161-018-0621-2
7. Dumbser, M., Peshkov, I., Romenski, E., Zanotti, O.: High order ADER schemes for a unified first order hyperbolic formulation of continuum mechanics: viscous heat-conducting fluids and elastic solids. J. Comput. Phys. **314**, 824–862 (2016)
8. Dumbser, M., Peshkov, I., Romenski, E., Zanotti, O.: High order ADER schemes for a unified first order hyperbolic formulation of Newtonian continuum mechanics coupled with electro-dynamics. J. Comput. Phys. **348**, 298–342 (2017)
9. Peshkov, I., Dumbser, M., Boscheri, W., Romenski, E., Chiocchetti, S., Ioriatti, M.: Simulation of non-Newtonian viscoplastic flows with a unified first order hyperbolic model and a structure-preserving semi-implicit scheme. Comput. Fluids **224**, 104963 (2021)
10. Romenski, E., Reshetova, G., Peshkov, I., Dumbser, M.: Modeling wavefields in saturated elastic porous media based on thermodynamically compatible system theory for two-phase solid-fluid mixtures. Comput. Fluids **206**, 104587 (2020)
11. Reshetova, G., Romenski, E.: Diffuse interface approach to modeling wavefields in a saturated porous medium. Appl. Math. Comput. **398**, 125978 (2021)
12. Allaire, G., Clerc, S., Kokh, S.: A five-equation model for the simulation of interfaces between compressible fluids. J. Comput. Phys. **181**, 577–616 (2002)
13. Toro, E.F.: Riemann Solvers and Numerical Methods for Fluid Dynamics. A Practical Introduction, 3rd edn. Springer, Heidelberg (2009). https://doi.org/10.1007/b79761
14. Jiang, G.-S., Shu, C.-W.: Efficient implementation of weighted ENO schemes. J. Comput. Phys. **126**, 202–228 (1996)
15. Gottlieb, S.: On high order strong stability preserving Runge-Kutta and multi step time discretizations. J. Sci. Comput. **25**, 105–128 (2005)
16. Saurel, R., Abgrall, R.: A simple method for compressible multifluid flows. SIAM J. Sci. Comput. **21**, 1115–1145 (1999)
17. Deng, X., Inaba, S., Xie, B., Shyue, K.-M., Xiao, F.: High fidelity discontinuity-resolving reconstruction for compressible multiphase flows with moving interfaces. J. Comput. Phys. **371**, 945–966 (2018)
18. Xiao, F., Honma, Y., Kono, T.: A simple algebraic interface capturing scheme using hyperbolic tangent function. Int. J. Numer. Meth. Fluids **48**(9), 1023–1040 (2005)