Non-linear Approximated Value Adjustments for Derivatives Under Multiple Risk Factors

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Abstract. We develop a numerical method to approximate the adjusted value of a European contingent claim in a market model where the underlying's price is correlated with the stochastic default intensities of the two parties of the contract.

When the close-out value of the contract is chosen as a fraction of the adjusted value, the latter verifies a non linear, not explicitly solvable BSDE. In a Markovian setting, this adjusted value is a deterministic function of the state variable verifying a non-linear PDE.

We develop here a numerical method to approximate the PDE solution, as an alternative choice to the commonly used Monte Carlo simulations, which require large computational times, especially when the number of the state variables grows.

We construct the approximated solution by the simple method of lines and we show the method to be accurate and efficient in a simplified cases. We show numerical results in the case of both constant intensities and the situation where only one is diffusive.

Keywords: Value adjustments · Backward stochastic differential equation · Nonlinear valuation · Credit risk · Numerical method

1 Introduction

Before the financial crisis of 2007−08, the value of a financial contract, such as an option, was determined by taking the conditional expectation of the discounted expected price, under a risk-neutral measure, without taking into account any credit risk (*default-free*). After the crisis, the need to introduce the effect of potential parties' defaults in the contract price became of great importance, and a theory of *Value Adjustments* was developed.

The first to be introduced was the CVA (*Credit Valuation Adjustments*), a positive quantity to be subtracted from the default-free price accounting for the possible losses due to the counterparty's default.

Over the years, the role of CVA increased considerably and became crucial when trading derivatives in the OTC markets, stimulating a lot of research in the field: for some examples, see [8,10,14].

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When evaluating products with cashflows in both directions (such as forwards or swaps), in order to take into account default of either part, the need to consider further adjustments arose, such as DVA (*Debt Valuation Adjustments*), FVA (*Funding Valuation Adjustments*), LVA (*Liquidity Valuation Adjustments*), were introduced and we refer the reader to the papers by Brigo et al. [5–8] for a detailed discussion.

The adjusted value of a financial contract under a risk-neutral measure turn on the so-called "close-out value", which is a portion of a contractually agreed price to be paid as partial compensation when the default of one of the parties occurs. This adjusted value can be represented as the solution of a BSDE (backward stochastic differential equation).

If the close-out value is as a portion of the price of the defaultable contract itself, this BSDE equation becomes non-linear, hence not explicitly solvable.

The principal choice to a numerical approximation of the solution of this BSDE is Monte Carlo simulation techniques but usually have very long computational times. So this, we are looking for alternative methods.

Under the Markovian assumption of all market processes, it is possible, thanks to Feynman-Kac's formula, to express the derivative's value as a deterministic function of the state variables that satisfies a PDE with the final condition given by the product's payoff [9]. We can use an alternative numerical approach to solve the problem by discretizing this PDE.

More in detail, we consider the case of a European call option subject to liquidity, funding, and collateralization risks. We assume both parties to be defaultable with stochastic intensities chosen as CIR processes in order to ensure their positivity, being certainly a desirable feature.

Among computational papers about this topic, we also quote the recent [3,4]. Both follow the works by Piterbarg [19] and Burgard and Kiaer [9] that obtain PDE formulation by hedging arguments on suitable portfolios and the application of the Ito lemma for jump-diffusion processes. They apply finite elements methods and contraction methods to approximate the PDE solutions, but they are able to treat only deterministic intensities or, in the linear case, a single Gaussian intensity.

We used the simple method of lines to approximate the PDE with a system of ODEs, which can be solved by one-step methods, such as the Euler one. We implemented the method by Matlab software and we used Monte Carlo simulations as benchmark. In the more complex case, the presence of a 3-dimensional vector of processes makes the Monte Carlo simulations computationally prohibitive, while the method of the lines still provides a numerical solution in highly competitive times. The numerical results were extremely satisfactory as they matched the Monte Carlo method in accuracy with remarkably shorter computational times.

The paper is structured as follows. In the first section, we briefly describe the modeling of the problem, and we introduce the BSDE characterizing the adjusted contract's value. Next, we consider the specific case of a European call, and we obtain the associated non-linear PDE. The third section is dedicated to the method of the lines and its implementation in a simplified case.

2 Evaluation of European Claims Under the Intensity Approach

We consider a finite time interval $[0, T]$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ that represent the market generated by the processes S_t, r_t and other possible stochastic factors, where r_t is the risk-free interest rate process, determining the money market account and S_t representing the asset price (underlying). We assume to be in absence of arbitrage with $\mathbb{P}$ representing a risk-neutral measure selected by some criterion.

We consider exchanging some European claim between two financial parties (1 = Counterparty, 2 = Investor) with maturity T and payoff $\Phi(S_T)$, where Φ is a function as regular as needed.

We take the perspective of the investor with the objective to compute the contract value, that we denote by $\bar{V}_t$, taking into account all the cashflows (see [7] for more details).

Both parties might default, and we denote by τ_1, τ_2 the random variables representing their default times.

In general, those r.v.'s are not necessarily stopping times with respect to the filtration $\mathcal{F}_t$, generated by the market observables. To price the defaultable contract, we first need to extend $\mathcal{F}_t$, to $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^1 \vee \mathcal{H}_t^2$, where $\mathcal{H}_t^i = \sigma(1_{\{\tau_i \le s\}}, s \le t)$, $i = 1, 2$, which is the smallest filtration making the random variables τ_i stopping times and then project the results back on $\mathcal{F}_t$. To do so, we assume that

- every $\mathcal{F}_t$-martingale remains a $\mathcal{G}_t$-martingale.;
- $\tau_i = \inf\{t \ge 0 | \int_0^t \lambda_u^i du > \xi_i\}$, where ξ_i are two independent standard exponential random variables, and λ^i for $i = 1, 2$ are two $\mathcal{F}_t$-predictable positive processes (see [7,17]);
- In this setting, $\lambda := \lambda^1 + \lambda^2$, is the default intensity of $\tau = min(\tau_1, \tau_2)$, and we set $\bar{\tau} = min(\tau, T)$.

Following Brigo et al. [7], we can write the $\mathcal{G}_t$-adapted value process of the defaultable derivative, $\bar{V}_t$, as the sum of the discounted default-free price and the adjustments due to default, funding, and collateralization risks which gets characterized as the solution of the following BSDE

$$\begin{aligned}\bar{V}_t = {} & \mathbb{E}_t^{\mathcal{G}}\left[1_{\{\tau>T\}} e^{-\int_t^T r_s ds}\Phi(S_T) + \int_t^{\bar{\tau}} e^{-\int_t^u r_s ds}\pi_u du\right] - \mathbb{E}_t^{\mathcal{G}}\left[\int_t^{\bar{\tau}} e^{-\int_t^u r_s ds}(c_u - r_u)C_u du\right] \\ & - \mathbb{E}_t^{\mathcal{G}}\left[\int_t^{\bar{\tau}} e^{-\int_t^u r_s ds}(\bar{f}_u - r_u)(\bar{V}_u - C_u)du\right] - \mathbb{E}_t^{\mathcal{G}}\left[\int_t^{\bar{\tau}} e^{-\int_t^u r_s ds}(r_u - \bar{h}_u)\bar{H}_u du\right] \\ & + \mathbb{E}_t^{\mathcal{G}}\left[e^{-\int_t^T r_s ds} 1_{\{t\le\tau\le T\}}\left(\epsilon_\tau - 1_{\{\tau_1<\tau_2\}} LGD_1(\epsilon_\tau - C_\tau)^+ + 1_{\{\tau_2<\tau_1\}} LGD_2(\epsilon_\tau - C_\tau)^-\right)\right].\end{aligned} \tag{1}$$

Some terms in the above are predefined by the contract's agreements, others depend on the price evolution. We summarize their meaning and measurability properties in Table 1.

Table 1. Summary of cashflows and their measurability properties

Symbol	Role	Assumption
$\Phi()$	Payoff at maturity	Lipschitz function of S_T
π	Contract dividends	$\mathcal{F}$-predictable
C	Collateral process	$\mathcal{F}$-predictable
$\bar{H}$	Hedging process	$\mathcal{G}$-predictable
ϵ	Close-out value	$\mathcal{F}$-predictable
c	Collateral rate	$\mathcal{F}$-predictable
$\bar{f}$	Funding rate	$\mathcal{G}$-predictable
$\bar{h}$	Hedging rate	$\mathcal{G}$-predictable
LGD_i, $i = 1, 2$	Loss Given Default	Constant

The close-out value ϵ_u is usually taken as portion of the default-free price or as the price of the defaultable claim: the first choice gives a solvable linear BSDE, while the second ($\epsilon_u = \bar{V}_u$) determines a non-linear BSDE, not explicitly solvable.

It is well-known that the default times are not market observables, so we have to translate in terms of $\mathcal{F}$ any risk-neutral evaluation, which would take place in the $\mathcal{G}$. That means that we must project on to the market filtration $\mathcal{F}_t$ the theoretical price represented by Eq. (1).

By the key lemma and its extensions (see [1,2,5,7,17]), if $\bar{\mathcal{V}}_t$ is a $\mathcal{G}$-adapted process then there exists an $\mathcal{F}_t$-adapted process $\mathcal{V}_t$ such that $1_{\{t<\tau\}}\mathcal{V}_t = 1_{\{t<\tau\}}\bar{\mathcal{V}}_t$. We may therefore project Eq. (1) on $\mathcal{F}_t$, obtaining the following $\mathcal{F}$-BSDE:

$$V_t = \mathbb{E}_t^{\mathcal{F}}\Bigg[e^{-\int_t^T (r_s+\lambda_s)ds}\Phi(S_T) + \int_t^T e^{-\int_t^u (r_s+\lambda_s)ds}\Big(\pi_u - (c_u - r_u)C_u - (f_u - r_u)(V_u - C_u) - (r_u - h_u)H_u + V_u\lambda_u - LGD_1\lambda_u^1(V_u - C_u)^+ + LGD_2\lambda_u^2(V_u - C_u)^-\Big)du\Bigg], \tag{2}$$

where V_t, f_u, h_u and H_u are $\mathcal{F}$-adapted processes such that $1_{\{t<\tau\}}V_t = 1_{\{t<\tau\}}\bar{V}_t$, $1_{\{u<\tau\}}f_u = 1_{\{u<\tau\}}\bar{f}_u$, $1_{\{u<\tau\}}h = 1_{\{u<\tau\}}\bar{h}_u$, $1_{\{u<\tau\}}H_u = 1_{\{u<\tau\}}\bar{H}_u$.

If $\mathcal{F}_t$ is generated by a (possibly multidimensional) Brownian motion driving the market assets prices, by the martingale representation theorem, taking for granted the necessary integrability conditions, we can rewrite Eq. (2) as

$$V_t = \Phi(S_T) + \int_t^T \Big(\pi_u + (f_u - c_u)C_u - f_uV_u - (r_u - h_u)H_u - LGD_1\lambda_u^1(V_u - C_u)^+ + LGD_2\lambda_u^2(V_u - C_u)^-\Big)du - \int_t^T Z_u dW_u + \mathcal{M}_t \tag{3}$$

with W_t is a (vector) Brownian motion, Z_t, possibly square integrable, an $\mathcal{F}$-adapted (vector) process, and $\mathcal{M}_t$ is a martingale orthogonal to $\int_t^T Z_u \cdot dW_u$, possibly depending on further stochastic factors.

Missing a closed form solution for Eq. (3), one may try to construct an appropriate approximation procedure. In the literature, the most widespread method is Monte Carlo simulations (possibly coupled with deep learning-techniques as in [13]), which have very long computational times. It is then worth looking for alternative, less costly methods. This is possible when the underlying processes are diffusions, generating a Markovian vector.

3 Markovian BSDE and PDE for V_t

Under appropriate conditions, Eq. (3) has a unique adapted solution (V, Z) (see [7,18]) and in a Markovian setting, this is a deterministic function of the state variables. That is the case when $(S, \lambda^1, \lambda^2)$ are diffusion processes.

To analyze this case, we simplify the equation by assuming that

- the claim pays no dividends, hence $\pi = 0$;
- the rates r, f, c, h are deterministic bounded functions of time;
- the collateral process is a fraction of the process V_u, namely $C_u = \alpha_u V_u$, where $0 \leq \alpha_u \leq 1$ is a function of time;
- the process $H_t = H(t, S_t, V_t, Z_t)$, where $H(u, x, v, z)$ is a deterministic, Lipschitz-continuous function in v, z, uniformly in u. Besides $H(u, x, 0, 0)$ is continuous in x. This means that we can included H_t in the dynamics, if we assume a δ-hedging of the product (see [1,2,7,8]);

Using the flow notation, we assume that the market model is given by S satisfying:

$$S_s^{t,x} = x + \int_t^s r_u S_u^{t,x} du + \int_t^s \sigma S_u^{t,x} dW_u, \quad \sigma > 0, \quad t \leq u \leq T,$$

with default intensities following a CIR dynamics

$$\lambda_s^{i,t,w} = w + \int_t^s \gamma_i(\psi_i - \lambda_u^{i,t,w}) du + \int_t^s \eta_i \sqrt{\lambda_u^{i,t,w}} dB_u^i, \quad i = 1, 2$$

with

$$W_t = \rho_1 B_t^1 + \rho_2 B_t^2 + \sqrt{1 - \rho_1^2 - \rho_2^2} B_t^3, \qquad \rho_1^2 + \rho_2^2 \leq 1, \quad -1 \leq \rho_i \leq 1$$

where (B_t^1, B_t^2, B_t^3) is 3-dimensional standard Brownian motion, and $\gamma_i, \psi_i, \eta_i \geq 0, \quad i = 1, 2$, that verify the *Feller condition*, $2\gamma_i\psi_i \geq \eta_i^2$, to ensure the processes' positivity.

Here, we choose the two default intensities independent of each other to simplify calculations, but this assumption may be easily removed by adding a further parameter to the correlation structure.

As a shown in [1] assuming a δ-hedging of this product, an appropriate change of probability, we can rewrite in flow notation Eq. (3) as

$$\begin{aligned} dV_s^{t,x,y,z} = &- \Big[(1-\alpha_t)\Big[-f_t V_s^{t,x,y,z} - LGD_1 \lambda_s^{1,t,y} V_s^{t,x,y,z,+} + LGD_2 \lambda_s^{2,t,y} V_s^{t,x,y,z,-}\Big] \\ &- \alpha_t c_t V_s^{t,x,y,z} - (r_t - h_t)\frac{\partial V_s^{t,x,y,z}}{\partial S} S_s^{t,x}\Big] dt + Z_s^{t,x,y,z} dW_t + d\mathcal{M}_t \\ V_T^{t,x} = &\, \Phi(S_T^{t,x}). \end{aligned} \tag{4}$$

Since the triple $(S^{t,x}, \lambda^{1,t,y}, \lambda^{2,t,z})$ is Markovian, $V_s^{t,x,y,z}$ is a deterministic function of the state variables, $u(s, S_s^{t,x}, \lambda_s^{1,t,y}, \lambda_s^{2,t,z})$, with $u(t,x,y,z) \in C^{1,2}([0,T] \times \mathbb{R}^3_+)$.

By applying Ito's formula and comparing the two expressions, it can be shown that u verifies the non-linear PDE

$$\begin{cases} \mathcal{L}(u)(t,x,y,z) + (1-\alpha)\Big[-LGD_1 y u(t,x,y,z)^+ + LGD_2 z u(t,x,y,z)^-\Big] = 0 \\ u(T,x,y,z) = \Phi(x), \end{cases} \tag{5}$$

where

$$\begin{aligned} &\mathcal{L}(u)(t,x,y,z) = \partial_t u(t,x,y,z) + \gamma_1(\psi_1 - y)\partial_y u(t,x,y,z) + \gamma_2(\psi_2 - z)\partial_z u(t,x,y,z) \\ &+ hx\partial_s u(t,x,y,z) + \frac{1}{2}\eta_1^2 y \partial_y^2 u(t,x,y,z) + \frac{1}{2}\eta_2^2 z \partial_z^2 u(t,x,y,z) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 u(t,x,y,z) \\ &+ \frac{1}{2}\rho_1 \sigma x \eta_1 \sqrt{y}\partial_{xy} u(t,x,y,z) + \frac{1}{2}\rho_2 \sigma x \eta_2 \sqrt{z}\partial_{xz} u(t,x,y,z) - \alpha c u(t,x,y,z) \\ &- (1-\alpha) f u(t,x,y,z). \end{aligned}$$

When considering a triple of stochastic processes, Monte Carlo simulations to approximate Eq. (2) are bound to be extremely costly in terms of machine time, thus we suggest a discretization of (5) that seems to work efficiently in terms of computational times and accuracy.

4 The Method of Lines

The method of lines [11,12,15,16] is a numerical method to solve PDEs by approximating the spatial derivatives with finite differences, so generating a system of ODEs at each point of the discretization grid that can be solved by some method of time integration, such as the one-step Euler method.

The spatial domain $\mathbb{R}^3_+$ is unbounded, so we need to restrict it to an appropriate bounded rectangle $[a_x, b_x] \times [a_y, b_y] \times [a_z, b_z] \subset \mathbb{R}^3_+$. This truncation requires defining appropriate boundary conditions, which can be done exploiting the Black Scholes's pricing function $\phi(t,x;w,\sigma)$, with adjusted rates to include the default intensities

$$\begin{aligned} &u(t,a_x,y,z) = 0, && u(t,b_x,y,z) = \phi(t,b_x;r+\lambda,\sigma), \\ &u(t,x,a_y,z) = \phi(t,x;r+\lambda,\sigma), && u(t,x,b_y,z) = \phi(t,x;r+\lambda,\sigma), \\ &u(t,x,y,a_z) = \phi(t,x;r+\lambda,\sigma), && u(t,x,y,b_z) = \phi(t,x;r+\lambda,\sigma), \end{aligned} \tag{6}$$

We equispaced the three space intervals by taking, $x_i = a_x + i\Delta x$, $y_i = a_y + i\Delta y$, $z_i = a_z + i\Delta z$ with $\Delta x = \frac{(b_x - a_x)}{m}$, $\Delta y = \frac{(b_y - a_y)}{m}$, $\Delta z = \frac{(b_z - a_z)}{m}$ for $i = 0, \ldots, m$, and we apply the finite difference method to the space partial derivatives.

At each grid point x_k, y_i, z_j, the solution $u_{k,i,j}(t)$ verifies the following non-linear ODE

$$u_{k,i,j}(t)' = \mathcal{DL}(u_{k,i,j}) + (1-\alpha)\big(LGD_1 y_i u_{k,i,j}(t)^+ - LGD_2 z_j u_{k,i,j}(t)^-\big) \quad (7)$$

for $k, i, j = 0, \ldots, m$, and $\mathcal{DL}(u_{k,i,j})$ is the discretized operator of $\mathcal{L}(u)$.

In matrix form

$$\bar{u}(t)' = \mathbf{A}(\bar{x}, \bar{y}, \bar{z})\bar{u}(t) + (1-\alpha)\big(LGD_1 \bar{y}\bar{u}(t)^+ - LGD_2 \bar{z}\bar{u}(t)^-\big), \quad (8)$$

where $\mathbf{u}(t)', \mathbf{u}(t)$, and $\mathbf{A}(\bar{x}, \bar{y}, \bar{z})$ are a 3-dimensional tensor respectively, and $\bar{x}, \bar{y}, \bar{z}$ are vectors in $\mathbb{R}^{m+1}$ given by

$$\bar{x} = (a_x, x_1, \ldots, x_{m-1}, b_x), \quad \bar{y} = (a_y, y_1, \ldots, y_{m-1}, b_y), \quad \bar{z} = (a_z, z_1, \ldots, z_{m-1}, b_z)$$

and the final condition $u(T, \bar{x}, \bar{y}, \bar{z}) = \bar{\Phi}(x)$ holds, where
$\bar{\Phi}(x) = \big(\Phi_0(x), \Phi_1(x), \ldots, \Phi_m(x)\big)$.

To solve the system (8), we use the Euler scheme with N ($0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$) time sub-intervals of uniform length $\Delta t = t_{i+1} - t_i$ for $i = 0, \ldots, N-1$, so we get

$$\begin{aligned} &\bar{u}(t_i) = \bar{u}(t_{i+1}) - \Delta t\big(\mathbf{A}(\bar{x}, \bar{y}, \bar{z})\bar{u}(t_{i+1}) + (1-\alpha)\big(LGD_1 \bar{y}\bar{u}(t_{i+1})^+ - LGD_2 \bar{z}\bar{u}(t_{i+1})^-\big)\big) \\ &i = 0, \ldots, N-1. \end{aligned} \quad (9)$$

We denote the piecewise approximation of $u(t, x, y, z)$ by $u_{k,i,j}(t)$ for $x \in [x_k, x_{k+1})$, $y \in [y_i, y_{i+1})$, $z \in [z_j, z_{j+1})$ with $i, j, k = 0, \ldots, m-1$.

For $t = 0$, we are interested in computing the value $u(0, x, y, z)$ for given x, y, z. To do so, we simply choose the closest points of the grid such that $x_k \approx x$, $y_i \approx y$, $z_j \approx z$ for some $k, i, j = 0, \ldots, m$ and we approximate the solution value by $u(0, x_k, y_i, z_j)$.

5 Numerical Results

In this section, we present some numerical results of our method. We study a simplified case when the intensities are deterministic, and we use as a benchmark the results obtained in Brigo et al.'s papers [7] by Monte Carlo simulations, with the same set of parameters. Since in [7] were not reported the machine time, we reproduced their simulations.

In this case, only one state variable, represented by underlying price, is present.

All the algorithms were implemented in MatLab(R2021a).

We consider a European call option with six months maturity, and strike price $K = 90$.

In Table 2 we report the results of our methods with the relative computational times.

Table 2. Prices of a European call with maturity 6 months and deterministic intensities.

Monte Carlo simulations			
N_t	Seconds	Price	by Brigo
10^3	616	16.4454	**16.4534**
Method of lines			
N_x	N_t	Seconds	Price
30	500	0.31	16.4453258
50	500	0.28	16.4272255
90	1000	0.57	16.4643242
150	5000	1.93	16.4555334
200	5000	1.52	16.4568071
300	10000	3.7	16.4574087
500	50000	18.48	16.4574889

As shown in the Table 2, Monte Carlo simulations give 16.4454 in about 10 minutes (fairly close to 16.4543 in [7]), while with 30 nodes, we get a result close to the benchmark with almost nihil computational time. If we want better accuracy with Monte Carlo simulations, we have to increase the number of sample paths, but it implies a high increase in computational time. Differently our method, one might increase the number of spatial and time nodes getting results in less than 20 s. We achieve better performance and comparable accuracy also with respect to [3], where the computational time is about 25 s. By increasing the number of spatial and temporal nodes, the third and fourth digits stabilize, showing the convergence of the method. The first two decimal digits coincide with those obtained by Brigo et al. [7], and we probably achieve better accuracy.

The Fig. 1 shows the surface of value for the European call option in the function of time and stock price.

Here, we show another case when one intensity is diffusive, whether the other is deterministic. We compared these results with the constant intensity case using the same number of nodes and time steps with different maturities.

The Table 3 shows that randomness of the intensity effect that we notice up to the first decimal digit, when maturity increases, confirming it might be significant to consider stochastic intensity models. Moreover, the computational time increased, but it is yet competitive with respect to Monte Carlo simulations. The Fig. 2 show the surfaces of value for the European call option to fixed time $t = 0$, intensity $y = y_0$ and stock price $S = S_0$ respectively.

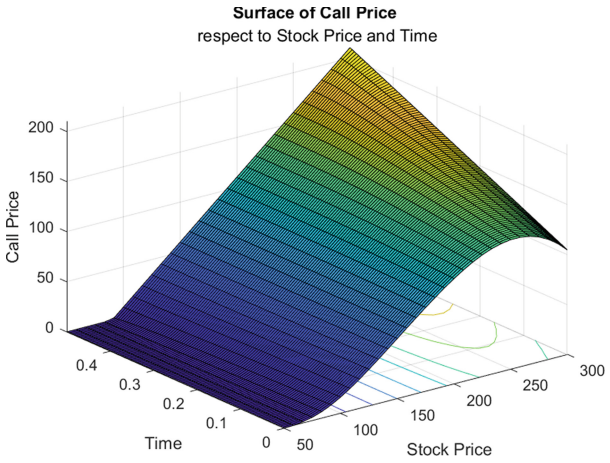


Fig. 1. Surface for a European call option with input arguments: $S_0 \in [50, 300]$, $K = 90$, $r = 0.005$ $\sigma = 0.4$, $\lambda_1 = 0.04$, $LGD_1 = 0.6$, $\lambda_2 = 0.02$, $LGD_2 = 0.6$, $c = 0.002$, $f = r$, $T = 0.5$.

Table 3. Price a European call with different maturities in the deterministic and stochastic case

Maturity	Deterministic case		One stochastic case	
	Seconds	Price	Seconds	Price
6 months	1.73	16.456	3.38	16.445
9 months	2.38	18.675	4.88	18.513
1 year	3.08	20.459	6.43	19.872

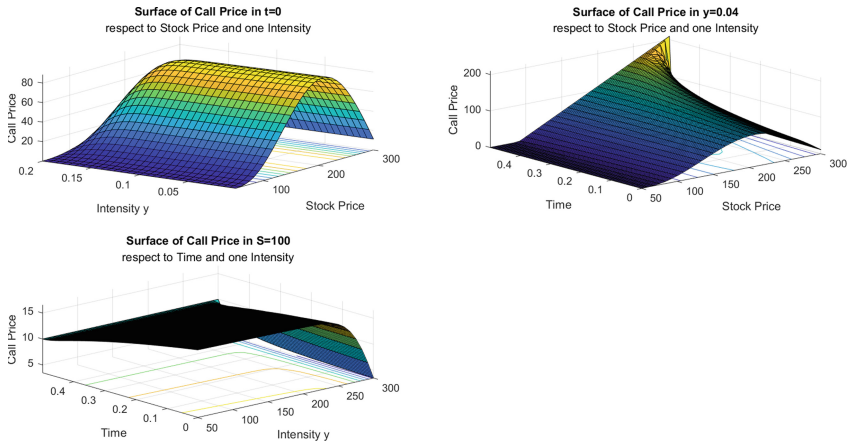


Fig. 2. Surface for a European call option to fixed time $t = t_0$, $y = y_0$ and $S = S_0$ respectively.

6 Conclusion

In this work, we developed a simple approximation procedure for the adjusted value of a derivative contract subject to counterparty risk, collateralization and founding costs, assuming a diffusion model for the default intensities and close-out values as a portion of the adjusted price itself. This generated a non-linear BSDE, with an associated non-linear PDE characterizing the price.

By the simple method of lines applied to this PDE with deterministic intensities, we showed that accurate approximations could be achieved in very manageable computational times, differently from what happens when employing Monte Carlo simulations. Future work will concern the numerical study of the complete model, i.e. when intensities are diffusive processes. As seen in the theoretical part of the work we will obtain a 3-dimensional non-linear PDE. We expect that the addition of stochastic factors in the model will lead to an increase in computational times but it will still be highly competitive with the Monte Carlo simulations machine times, which will be prohibitive. Moreover, we will study a sensitivity analysis and theoretical convergence in the complete case and finally will also regard the implementation of the solver in high-performance computing environments.

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