A Non Parametric Test for Decreasing Uncertainty Residual Life Distribution ($DURL$)

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Abstract. A new statistical test is presented to detect the monotonicity of uncertainty (F is $DURL$) based on derivative criteria and the histogram method. Some properties such as consistency and asymptotic normality are discussed. Simulation by the Monte-Carlo method gives the critical values of the statistics. We compare the power estimate with that of the test based on the criteria of monotonicity of residual entropy. Finally, we show on real survival data that the two tests lead to the same conclusions.

Keywords: Reliability · Information measures · Uncertainty · Lifetime distribution · Non Parametric Test · Decreasing Uncertainty of Residual Lifetime ($DURL$)

1 Introduction

Uncertainty measures are useful in a variety of scientific activities, including Artificial Intelligence (Expert Systems) [19], Wireless and Sensor Networks [16]... Measuring uncertainty can be provided by several methodologies: probability, entropy, possibility, fuzzy sets, belief functions, etc. A random variable's differential entropy $H(f) = -E[logf(X)]$ can be thought of as a measure of its information or uncertainty. However, if X represents the lifetime of a component (or system), this measure doesn't give any information about the remaining lifetime of the system. A direct approach to measuring uncertainty in the residual lifetime distribution has been provided in [11].

Let X be a nonnegative random variable representing the lifetime of a physical or biological component (or system) with cumulative distribution function $F(x) = P(X < x)$ and density $f(x)$. As a result, the probability distribution function for reliability (or survival) $\bar{F}(x) = P(X > x)$ i.e. the probability that the component (or system) survives the time interval $(0, x)$. A distribution has decreasing (increasing) uncertainty of residual life $DURL(IURL)$ if the function

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O. Gervasi et al. (Eds.): ICCSA 2022, LNCS 13376, pp. 188–196, 2022.
https://doi.org/10.1007/978-3-031-10450-3_14

$H(f;t) = -E[log f_t(X_t)]$ is decreasing (increasing) in t. Here $f_t(x)$ is the density of the random variable $X_t = X - t/X > 0$, the remaining lifetime of a system of age t and $H(f,t)$ is the dynamic residual entropy which measures the expected uncertainty contained in the conditional density of X_t about the predictability of the residual lifetime of the system.

Considerable attention has been given to testing the membership of a probability distribution to a non parametric class such as IFR(Increasing Failure Rate) or $DMRL$ (Decreasing Mean Residual Life). The objective of this paper is to present a new statistic to test the membership to $DURL$ class.

Section 2 describes classical non parametric reliability (aging) distributions. Section 3 focuses on the non parametric $DURL$ ($IURL$) class and some of its properties, specifically the relationship with classical non-parametric reliability distributions. Section 4 introduces a new method for determining whether a probability distribution belongs to the $DURL$ class. The test is based on derivative criteria and histogram method. The following properties of such a test are discussed: mean and variance of the statistic, consistency,asymptotic normality, critical values and power of the test. Finally, in Sect. 5, an illustrative example is given on real lifetime data in order to compare this test with that based on criteria of monotonicity of residual entropy [9].

2 Non Parametric Reliability Distributions

Usual probability distributions are "parametric" in the sense that they depend on some parameters. For example, the exponential distribution depends on only one parameter (mean or failure rate); the normal distribution depends on two parameters (mean and variance). However, we can show examples in which the chi-square test leads to accept two different distributions, Gamma and Raighley, which are both characterized by Increasing Failure Rate (IFR).

Indeed, in practice, an engineer is not always interested by the concrete parametric probability distribution itself, but simply in some "aging property" of the system. The more simplest example is given by Reliability or Survival analysis studies , in which practitioners are interested only in a stage of aging or rejuvenation; for example "Increasing(Decreasing) Failure Rate" i.e. $IFR(DFR)$ [14]. So, we consider the class of all probability distributions with $IFR(DFR)$property. Because it contains a collection of probability distributions that share a common aging property, such a class is non-parametric rather than parametric.

Other stochastic models, such as queueing , insurance, medicine, or biological models, are also interested in the properties of such classes [15,17].

Now, in reliability, we are interested in the lifetime of the system. Let X be the system's failure time and denote its probability distribution function by $F(t) = P(X < t)$, $t \geq 0$, which is assumed to be absolutely continuous with density probability function $f(t)$.

The survival (or reliability) function, i.e. the probability of system survival over the period $(0,t)$, is represented by the function $\bar{F}(t) = 1 - F(t) = P(X > t)$. We are interested also about system's residual lifetime $X_t = X - t/X > t$ after it has survived until time t.

We denote by $F_t(x) = P(X_t < x/X > t) = \frac{P(t \leq X < t+x)}{P(X>t)}$, the probability distribution of the system's residual lifetime; by $\bar{F}_t(x) = P(X_t > x) = \frac{\bar{F}(t+x)}{\bar{F}(t)}$ the reliability of a system of age t. The function $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$ represents the failure rate of the system at time t also called hazard or risk function. This is a local characteristic such as the density function and $\lambda_F(t)dt + o(dt)$ represents the probability of a failure during the interval of time $(t, t+dt)$ given no failure until time t.

We define now some usual non-parametric probability distributions [15,17]. In the following, by increasing (decreasing), we mean nondecreasing (nonincreasing).

Definition 1. *The survival function $\overline{F}$ has increasing (resp. decreasing) failure rate IFR (resp. DFR), if $\lambda_F(t)$ is increasing (resp. decreasing) in t on the interval $(0,t)$* [15,17].

This definition assumes that $F(.)$ is absolutely continuous. Otherwise, the definition remains valid by considering the relation between stochastic orderings and reliability theory. A non negative random variable X is $IFR(DFR)$ if and only if $X_t \geq_{st} (\leq_{st}) X_{t'}$ for all $t \leq t'$. So, the stochastic order $\leq_{st}$ characterizes the $IFR(DFR)$ probability distribution.

Similarly, the order $\leq_{icx}$ (increasing convex order) characterizes another usual aging notion in Reliability i.e. $DMRL$ ($IMRL$): Decreasing (Increasing) Mean Residual Life.

Definition 2. *The random variable X is $DMRL(IMRL)$ if $X_t = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}$ is decreasing (increasing) in $t \in [0,t]$ [15, 17].*

Definition 3. *A nonnegative random variable X is said to be $IFRA$ (Increasing Failure Rate in Average) if the average of its cumulative failure rate over $(0,t)$*

$$\frac{\Lambda(t)}{t} = \frac{1}{t}\int_0^t \lambda(t) = \frac{-log\overline{F}(t)}{t} \tag{1}$$

is increasing in $t \geq 0$ [15, 17].

Remark 1. There is some relations between these non-parametric distributions (classification). If F is IFR then it is also $IFRA$ and $DMRL$ in the sense that the class $IFR \subset IFRA$ and $IFR \subset DMRL$, the inclusion is strict. But there is no relations between $IFRA$ and $DMRL$. The same relations are valid for the dual classes DFR, $DFRA$ and $IMRL$.

3 Decreasing Uncertainty of Residual Life ($DURL$) Class

Ebrahimi and Pellerey [10] present a similar notion of dynamical entropy to Shannon information measure [18], by considering the age of the system S at time $t, t > 0$. More precisely, the residual entropy expresses the uncertainty

contained in the conditional density of the residual lifetime $X_t = X - t/X > t$ on the system's predictability of it's residual lifetime:

$$H(f;t) = -\int_t^{\infty} \frac{f(x)}{\overline{F}(t)} log \frac{f(x)}{\overline{F}(t)} dx \tag{2}$$

$H(f;t)$ can be seen as a dynamic measure of uncertainty about S associated to its lifetime distribution. In other words, $H(f;t)$ measures the expected uncertainty contained in conditional density of the residual lifetime X_t of a system of age t i.e. given $X > t$. In this sense $H(f;t)$ measures the concentration of conditional probability distributions. Note finally that the dynamic entropy of a new system (of age 0) $H(f;0) = H(f)$ is the ordinary Shannon's entropy [10] and that the function $H(f;t)$ uniquely determines the reliability function $\overline{F}$ or equivalently the failure probability distribution F.

We can now define an uncertainty ordering. The non negative random variable X have less uncertainty than Y ($X \leq_{LU} Y$), if $H(f;t) \leq H(g;t), t \geq 0$. If X and Y are the lifetimes of two systems S and S', respectively, and if $X \leq_{LU} Y$, then the expected uncertainty contained in the conditional density of X_t about the predictability of the residual lifetime of the first system S is less than expected uncertainty contained in the conditional density of Y_t about the remaining lifetime of the second system S'. Intuitively speaking, the better system is the system that lives longer and there is less uncertainty about its residual lifetime.

Ebrahimi [11] defines two nonparametric classes of lifetime distributions based on the measure of uncertainty $H(f;t)$.

Definition 4. *A survival function $\overline{F}$ has decreasing (resp. increasing) uncertainty of residual lifetime $DURL$ (resp. $IURL$), if $H(f;t)$ is decreasing (resp. increasing) in t* [11].

If a system has a survival probability distribution belonging to the class $DURL$, then as the system ages the conditional probability density function becomes more informative. So, if a system of age X has a $DURL$ distribution (resp. $IURL$), its residual lifetime can be predicted with more (resp. less) precision.

Remark 2. 1. If F is $IFR(DFR)$, then it is $DURL(IURL)$. So that, $IFR \subset DURL$ and $DFR \subset IURL$.
2. If F is $DMRL$, then it is $DURL$. So that, $DMRL \subset DURL$.
3. IF F $IURL$, then it is $IMRL$. So that, $IURL \subset IMRL$.

Example 1. The exponential distribution is the only continuous distribution which is both $DURL$ and $IURL$ [11]. In other words, the class EXP of all exponential distributions is the boundary between $DURL$ and $IURL$ classes.

Example 2. Let X be a random variable having Weibull distribution with survival function $\overline{F}(t) = 1 - e^{\lambda t^{\alpha}}, t > 0$. Then X is $DURL$ ($IURL$) for $\alpha > 1$ (resp. $0 < \alpha < 1$) [11].

The following two counter-examples show that $DURL$ property does not imply IFR, nor does it imply $DMRL$.

Example 3. Let

$$\bar{F}(t) = (1 - exp(-\lambda_1 t))(1 - exp(-\lambda_2 t)), t \geq 0. \quad (3)$$

be the reliability function of a system with two components in series with independent reliabilities that are exponentially distributed with parameters λ_1 and λ_2 respectively. Then one can verify that $\bar{F}$ is IFR, but not $DURL$ [12].

Example 4. A common structure of redundancy is k-out-of-n. Such a system functions if and only if at least k components function. Two important particular cases are series ($k = 1$) and paralell ($k = n$) structures. The $IURL$ class is not preserved for such a structure. A counter-example is given by a parallel system with two components having exponential distributions. Obviously, each component is $IURL$. The lifetime of the parallel system is strictly increasing. As a result, the system is $DURL$, rather than $IURL$ [1].

An original example is given by Ebrahimi an Kermani [12] showing that $DMRL$ doest not imply $DURL$. The authors gave also another example for a renewal process with inter-failure time distribution F which is $DMRL$ and for which the limiting distributions of the forward waiting time from a particular time t to the next failure and the length of the renewal interval that contains the point t are $DURL$.

We can point out an interesting mathematical property of $DURL$ non-parametric probability distribution:

Lemma 1. *A survival function $\bar{F}$ is $DURL$, if and only if $H(f;x) - H(f;y) \geq 0$ for all $y \geq x$* [11].

The statistical test of [9] is based on the above lemma, while the present test is based on the following:

Lemma 2. *If $L(f;t) = -H'(f;t) \geq 0$, then the corresponding survival function $\bar{F}$ is $DURL(IURL)$. In other words,if we have negative (positive) local reduction of uncertainty, then $\bar{F}$ is $DURL(IURL)$* [11].

4 A Non Parametric Test for *DURL*

The main goal would be to test the hypothesis H_0: F is $DURL$ against the hypothesis H_1: F is not $DURL$. However, because $DURL$ is a non-parametric class of probability distributions, the statistical criteria under the null hypothesis H_0: $F \in EXP$ against H_1: $F \in DURL$ (and not exponential), where EXP is the class of all exponential probability distributions.

The alternative H_1 is based on uncertainty rather than aging.

4.1 Construction and Implementation of the Test

Under the hypothesis H_0, and according to Lemma 2, the function $H'(f;t) = 0$ and $\lambda_F(t) \neq 0$. As a result, the function $g(t) = H'(f;t) - 1 + \lambda_F(t) = 0$ under the assumption hypothesis H_0.

The test is built with the statistic $S(t) = E[g(t)] = \int_0^\infty g(t)f(t)dt$ which is estimated using an empirical probability density function and a histogram method based on a partition of the order statistics. According to the above-mentioned property, $g(t) = 0$ for H_0 and $g(t) < 0$ for H_1. The same logic applies to the empirical statistic $S_n(k)$: it is null under H_0 and negative under H_1. Small values of $S_n(k)$ favor H_1, whereas large values of $W_n(k) = -S_n(k)$ favor H_1.

So, at significance level α, we reject H_0 in favor of H_1 if $W_n(k) \geq C_{k,n}(\alpha)$, where $C_{k,n}(\alpha)$ is the critical point value determined by the $(1-\alpha)$-quantile of the distribution of $W_n(k)$ under exponentiality.

4.2 Mean and Variance of the Test $W_n(k)$

We can calculate the mean $E\big(W_n(k)\big)$ and the variance $Var\big(W_n(k)\big)$ using tedious algebra. Thus, we can prove the convergence and the asymptotic normality of the statistic $W_n(k)$.

4.3 Consistency of the Test $W_n(k)$

We can prove the consistency of the test statistic $W_n(k)$ in the following sense

Theorem 1. *As $n \to \infty, k \to \infty$, and $\frac{k}{n} \to 0$, the statistic $W_n(k)$ converges to S.*

4.4 Normality of the Statistic $W_n(k)$

We can also prove the asymptotic normality of the statistic $W_n(k)$ as $n \to \infty$.

Theorem 2. *As $n \to \infty$, and $k \to \infty$, the normalized statistic $kW_n(k)$ converges to a standard normal distribution.*

Table 1. Critical values of $W_n(k)$

n	α			
	0.100	0.050	0.025	0.010
3	4.8006	5.5334	6.3069	7.0340
11	6.8219	7.5294	8.1805	8.7718
100	8.9363	9.5745	10.1214	10.7857

Table 2. Critical values of $-W_n(k)$

n	α			
	0.100	0.025	0.010	0.0001
3	2.3564	3.9297	5.3591	7.7634
10	–0.0337	1.2530	2.6525	4.1937
100	–3.4964	–2.5482	–1.6047	–0.4008

4.5 Critical Values of the Test $W_n(k)$.

Under the alternative hypothesis, the distribution of the statistic $W_n(k)$ is complicated. In this section, we determine critical values $C_{k,n}(\alpha)$ of the statistic $W_n(k)$ by means of Monte Carlo simulation. Simulations are provided for different confidence levels of $1-\alpha$: $0.900, 0950, 0.975$ and 0.990 and different values of sample size n. We generated 5000 samples of exponential distribution with mean 1. We computed for each sample the corresponding values of the empirical statistic distribution $W_n(k)$ (for large spacings of the observations) and $-W_n(k)$ (for little spacings). A function in Matlab gives the critical values for each α and n. They are summarized in Table 1 and Table 2.

4.6 Power Estimates

We can obtain power estimates by using, again, Monte Carlo simulation. We have provided several experiments under Weibull $(1, \theta)$ and Gamma $(\theta, 1)$ alternatives.

Table 3 and Table 4, show simulated powers for different sample sizes n. Simulated results are based on 10000 iterations. We observe that the presented test performs relatively well when the parameter θ of the distributions reaches 20.

Table 3. Power Estimates for Weibull $(\theta, 1)$ of $W_n(k)$

θ	n		
	20	10	5
20	0.9990	0.9862	0.8921
10	0.9288	0.7831	0.5686
5	0.5568	0.3750	0.2591
3	0.2506	0.1784	0.1321
1	0.0555	0.0522	0.0526

5 Illustrative Example

As an illustration, and for comparison purposes, we consider the data set from the US National Cancer Institute, resulting from a study by Bryson and Siddiqui [5]

Table 4. Power estimates of $J_n(k)$ for Weibull (θ,1) (Ebrahimi [9])

θ	n		
	20	10	5
10	1.0	0.9570	0.9255
5	0.9869	0.8543	0.8476
3	0.9110	0.7468	0.7379
1	0.0510	0.0460	0.0430

Table 5. The ordered survival times (in days from diagnosis) (Bryson and Siddiqui [5]).

7	429	579	968	1877
47	440	581	1077	1886
58	445	650	1109	2045
74	455	702	1314	2056
177	468	715	1334	2260
232	495	779	1367	2429
273	497	881	1534	2509
285	532	900	1712	
317	571	930	1784	

which provides a real sample of residual life time of size $n = 43$. Table 5 contains the data.

We have computed the observed value of the statistic $W_n(k)$ on the basis of this sample, which gives the following estimation of the statistic $W_n(k) = 30.4629$. When comparing the result with the critical values of the histogram method given in Table 1, we see that the statistic exceeds all percentiles.

As a result, we can conclude that the above sample provide from a $DURL$ distribution. That is, uncertainty about the residual lifetime of an individual from this population decreases over time. More simply, the uncertainty for the residual lifetime among patients tends to decrease for a long time and their remaining lifetimes can be predicted with more precision.

6 Conclusion

In this paper, we have presented a new statistical test for detecting monotonicity of uncertainty (F is $DURL$). This test is based on derivative criteria different from that of [9]. Some properties such as consistency, asymptotic normality, and power estimation have been discussed. Illustrations based on real survival data have been provided for comparison. A new test for decreasing uncertainty has been studied but based on the kernel estimation method, which leads to the same conclusions. We leave this question for a further paper.

Acknowledgement. The Ministry of Higher Education and Scientific Research ($DGRSDT$) is funding the project, in part, through PRFU N° $C00L07UN16042022$ 0004.

We appreciate the fruitful discussions with Professor Mounir Mesbah when the second author visited the LPSM laboratory, University of Jussieu-Sorbonne in 2018.

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