Interval Approximation of the Discrete Helmholtz Propagator for the Radio-Wave Propagation Along the Earth's Surface

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Abstract. A new finite-difference approximation of the two-dimensional parabolic equation is proposed in this paper. The specifics of the tropospheric radio-wave propagation problem are taken into account. Rational approximation of the discrete in both dimensions propagation operator is considered. The method of rational interpolation is used instead of local Padé approximation. The results of numerical modeling confirm the advantages of the proposed approach.

Keywords: Parabolic equation · Helmholtz equation · Rational interpolation · Radio-wave propagation · Finite-difference methods

1 Introduction

Starting with the research of Leontovich and Fock [8], the problem of tropospheric radio-wave propagation modeling by the parabolic equation (PE) method remains relevant to this day [4,10,20,22,23]. Bearing in mind the works on the generalization of the PE method [6], further in this paper PE and one-way Helmholtz equation are considered as synonyms.

There are two effective numerical methods for solving PE: split-step Fourier (SSF) method [18] and finite-difference (FD) approximations [13]. SSF method is somewhat more efficient in terms of computational speed, but it has problems with lower [21] and upper [16] boundary conditions modeling. FD methods enable efficient and reliable modeling of boundary conditions but require more computational resources. However, as will be shown in this paper, their potential has not yet been exhausted.

Keeping in mind the steady increase in the operating frequency of the modern radio systems, the problem of improving the computational speed of both PE numerical methods is urgent. This can be achieved primarily by using more efficient approximations. In particular, the method of wavelet transformations as a kind of generalization of the SSF method is rapidly developing [4,24].

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It was previously shown [15] that the semi-discrete propagation operator as a function of the vertical diffraction operator can be effectively approximated using an interval approximation, rather than a local one. This made it possible to increase the grid step along the longitudinal coordinate, but a dense grid is still required along the transversal coordinate. To deal with this problem, it was suggested to use a local Padé approximation of the fully discrete propagation operator [5,14]. In this paper, it is proposed to use interval approximation of the fully discrete propagation operator which will allow using an even more sparse grid and obtaining faster numerical solutions.

2 Approximation of the Propagator

Step-by-step solution of the one-way Helmholtz equation is written as follows [15]

$$u^{n+1} = Pu^n, \tag{1}$$

$$P = \exp\left(ik\Delta x\left(\sqrt{1+\frac{1}{k^2}\frac{\partial^2 u}{\partial z^2}+(m^2-1)}-1\right)\right),$$

where $u(x,z)$ is the complex-valued electromagnetic wave component, $u_j^n = u(n\Delta x, j\Delta z)$ is its discrete analogue, P is the semi-discrete propagation operator, $k = 2\pi/\lambda$ is the wavenumber, λ is the wavelength, $m^2(x,z)$ is the tropospheric refractive index. The schematic description of the considered problem is depicted in Fig. 1.

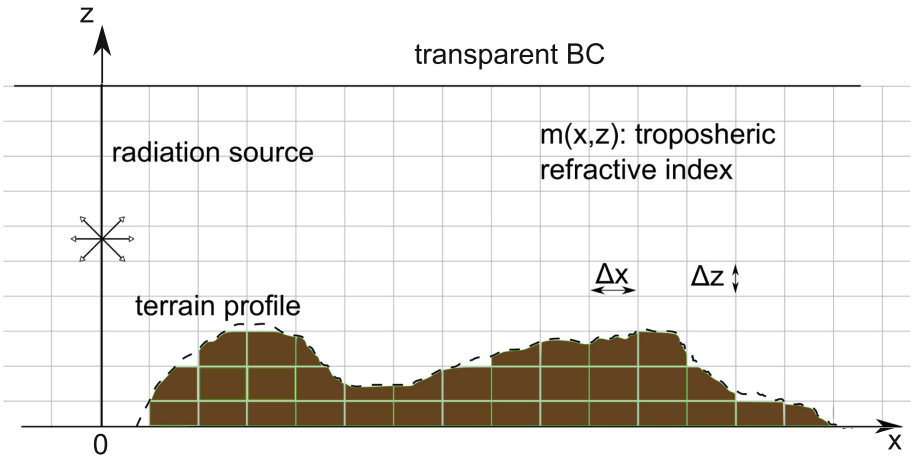

Fig. 1. Schematic description of the considered problem.

Operator P can be approximately split into two [9]

$$P \approx P_1 P_2, \tag{2}$$

$$P_1 = \exp\left(ik\Delta x\left(n(x,z)-1\right)\right),$$

$$P_2 = \exp\left(ik\Delta x\left(\sqrt{1+\frac{1}{k^2}\frac{\partial^2 u}{\partial z^2}}-1\right)\right). \tag{3}$$

Operator P_1 corresponds to refraction, P_2 corresponds to propagation and diffraction. In the case of a homogeneous medium ($m \equiv 1$) operator P_1 turns into identical and approximate equality turns into exact one. It should be noted that such an approximation is also used in the derivation of the SSF method, and thus its adequacy has been confirmed by a huge number of experiments.

Action of operator P_1 is calculated directly, thus, it remains to construct a numerical approximation of P_2. Express the second differential operator in terms of the second difference operator δ^2 [7]

$$\frac{\partial^2 u}{\partial z^2} = g(\delta^2)u, \tag{4}$$

$$g(\xi) = -\frac{1}{\Delta z^2}\ln^2\left(1+\frac{\xi}{2}+\sqrt{\left(1+\frac{\xi}{2}\right)^2-1}\right),$$

$$\delta^2 u = u_{j-1} - 2u_j + u_{j+1}.$$

Substitution (4) into (3) gives a fully discrete version of operator P_2 as a function of δ^2

$$P_2\left(\delta^2\right) = f\left(\delta^2\right) = \exp\left(ik\Delta x\left(\sqrt{1+\frac{1}{k^2}g\left(\delta^2\right)}-1\right)\right).$$

We are seeking an approximation of function $f\left(\delta^2\right)$ in the form

$$f(\xi)\,u \approx \prod_{l=1}^{p}\frac{1+a_l\xi}{1+b_l\xi}u, \tag{5}$$

where a_l and b_l are some coefficients to be determined. Such an approximation reduces the action of the propagation operator to a sequential solution of one-dimensional differential equations [5,14].

To choose an appropriate approximation, it is first necessary to determine what is meant by an operator function. Consider the definition, based on Cauchy's integral formula

$$f\left(\delta^2\right) = \frac{1}{2\pi i}\oint_{\Gamma} f(\xi)\left(\xi I - \delta^2\right)^{-1} d\xi,$$

where Γ encloses all eigenvalues of operator δ^2. The solution of the eigenvalue problem $\delta^2 u = \omega u$ is written as follows

$$\omega = -4\sin^2\left(\frac{k_z \Delta z}{2}\right),\ u(z) = \exp(ik_z z).$$

Function u has the physical meaning of a plane wave, where $k_z = k\sin\theta$ is a vertical wavenumber, θ is a propagation angle. Thus, we have established the relationship between argument ξ, transversal grid step Δz and propagation angle θ. A good coincidence between function f and its approximation (5) is required only for the desired propagation angles. In is common to use the Padé approximation [3,5] at point $\xi = 0$ to calculate the coefficients of rational approximation (5). Padé approximation is local, i.e. it gives excellent accuracy for small propagation angles, but its accuracy monotonically decreases with increasing ξ.

However, in some cases, it is necessary to provide a certain accuracy for a pre-known range of propagation angles [15]. For example, it is a propagation over irregular terrain, where the maximum propagation angle can be estimated from the geometry of the problem. Instead of using a local approximation, in this paper it is proposed to use a rational approximation on a segment. Rational interpolation in Chebyshev roots [19] is used as an approximation method in this research.

The desired approximation interval for function f is expressed as follows

$$\xi \in \left[-4\sin^2\left(\frac{k\sin\theta_{max}\Delta z}{2}\right), 0\right],$$

where θ_{max} is the maximum propagation angle of interest.

Figure 2 demonstrates the dependence of the approximation error on the value of ξ. It is clearly seen that Padé approximation gives excessive accuracy for small values of ξ, but the accuracy monotonically decreases with increasing ξ. At the same time, the rational interpolation calculated based on the maximum propagation angle of 22° gives a more uniform distribution of the error on the segment of interest.

3 Analysis of the Numerical Results

This section presents the numerical results, obtained by several approaches. The following methods are used: the proposed method of rational interpolation of the discrete propagation operator, Padé approximation of the discrete propagation operator [14], Padé approximation of the semi-discrete propagation operator with 2nd [16] and 4th [17] order approximation of the transversal operator, and SSF method. The implementation of the rational interpolation method from Chebfun library [1] (*ratinterp* function) was used. The proposed method is implemented as a Python 3 library and is freely available [11].

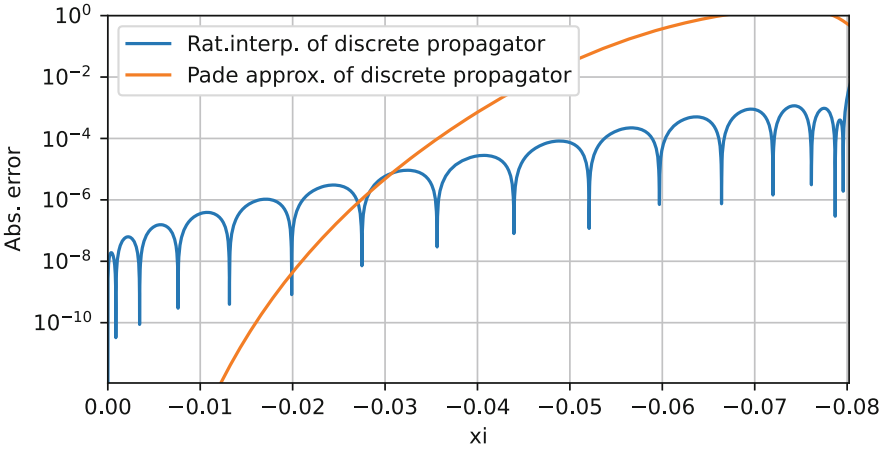

Fig. 2. Dependence of the approximation error of the discrete propagator on the value of ξ for Padé approximation and rational interpolation. $\Delta x = 50\lambda$, $\Delta z = 1.1\lambda$, rational approximation order is equal to [7/8].

3.1 Wedge Diffraction

The advantages of the proposed method for propagation over irregular terrain will be shown on the wedge diffraction problem. The radiation source is located at an altitude of 100 m and emits a harmonic signal at a frequency of 3 GHz. The wedge is located at a distance of 2 km from the source. Rational interpolation is calculated based on the maximum propagation angle of 22°. The order of rational approximation is equal to [7/8], $\Delta x = 50\lambda$, $\Delta z = 1.1\lambda$. The wedge is modeled using a staircase approximation.

Figure 3 demonstrates the dependence of the absolute error of the discrete horizontal wavenumber $\tilde{k}_x$ [12] on the propagation angle for FD approximations under consideration. It is clearly seen that the proposed method gives the best approximation on the considered interval. For the stability of the numerical scheme, it is required that $\mathrm{Im}\left(\tilde{k}_x\right) > 0$ outside the desired interval. This will guarantee the attenuation of waves propagating at irrelevant propagation angles and evanescent waves. It is clearly seen from Fig. 4 that this condition is satisfied for all FD schemes under consideration.

Figure 5 depicts the two-dimensional distribution of the electromagnetic field amplitude, computed by the proposed method. We verify the result with the SSF method. The two-dimensional distribution of the error between the results, obtained by the proposed method and SSF, is depicted in Fig. 6. Both methods used the same computational grid. It can be seen that both methods yield indistinguishable results for the desired range of propagation angles. In particular, the proposed method with the selected computational parameters correctly calculates the field in the diffraction zone behind the obstacle.

Now let's analyze the results obtained using Padé approximation for the same computational grid and approximation order. Figures 7 and 8 depict the spatial distribution of the error for two variants of Padé approximation. As it was predicted theoretically using discrete wavenumber, Padé approximation, all other things being equal, allows for smaller propagation angles than the rational interpolation method. In this example, both variants of Padé approximation yield a significant underestimation of the filed in the diffraction zone. Of course, reducing the size of the computational grid cells and increasing the approximation order allows one to increase the accuracy of Padé approximation and expand the maximum propagation angle, but at the same time, it will increase the computational costs. Namely, Fig. 9 depicts the spatial distribution of the error for the Padé approximation of the fully-discrete propagator with the reduced longitudinal grid step. One can see that Padé method requires a three times thicker computational grid for achieving an acceptable accuracy. Thus, the proposed method is three times more computationally effective in this example.

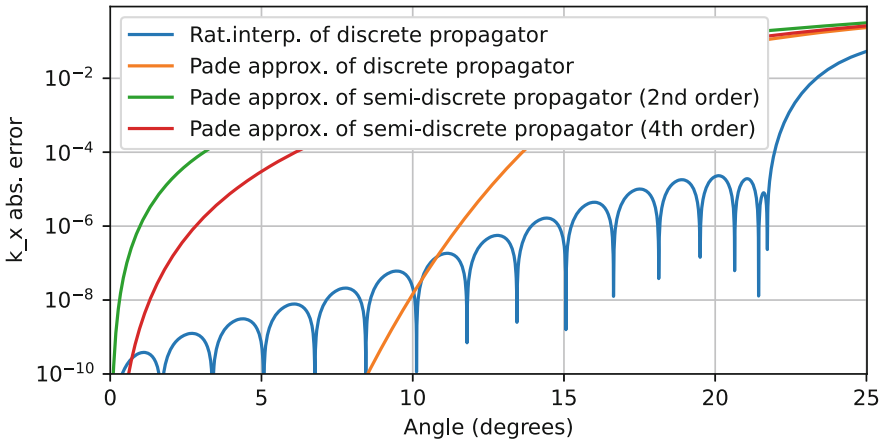

Fig. 3. The dependence of the absolute error of the discrete horizontal wavenumber on the propagation angle. In all cases $\Delta x = 50\lambda$, $\Delta z = 1.1\lambda$, rational approximation order is equal to [7/8].

3.2 Propagation in a Surface Duct

The purpose of this example is to demonstrate the possibility of inhomogeneous tropospheric refraction index modeling within the proposed method. Propagation in a 150 m high surface waveguide [9] is considered. The source is located at an altitude of 200 m and emits a 3 GHz harmonic signal. Figure 10 depicts the spatial distribution of the field amplitude, computed by the proposed method. Since there are no horizontal obstacles in this example and propagation occurs at small angles to the horizon, it is advisable to reduce the desired angle range to $[0, 5°]$. This, in turn, allows us to increase the grid step along the longitudinal coordinate to 300λ. The comparison is carried out with Padé approximation

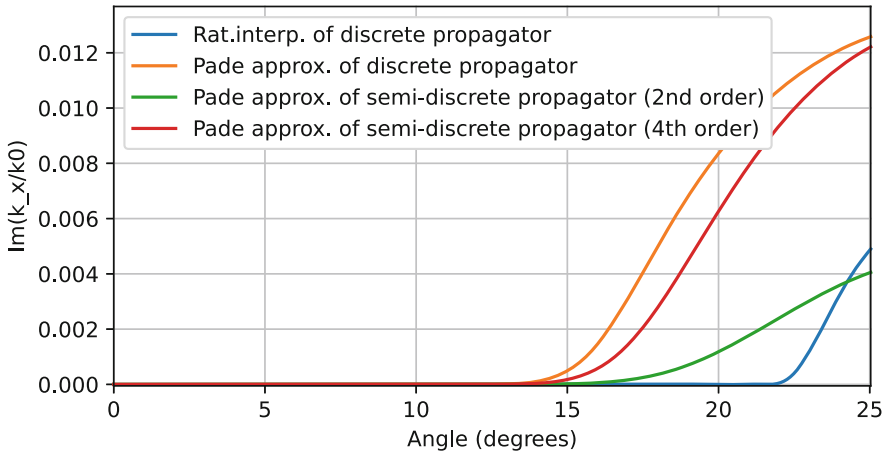

Fig. 4. The dependence of $\mathrm{Im}\left(\tilde{k}_x\right)$ on propagation angle. In all cases $\Delta x = 50\lambda$, $\Delta z = 1.1\lambda$, rational approximation order is equal to [7/8].

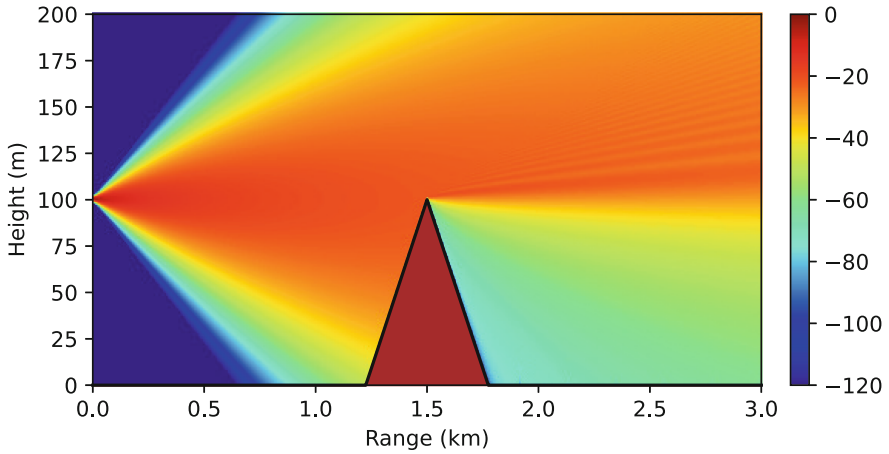

Fig. 5. Diffraction on the impenetrable wedge. Spatial distribution of the field amplitude $(20 \log |u(x, z)|)$, computed by the ratinterp-[7/8] approximation. $\Delta x = 50\lambda$, $\Delta z = 1.1\lambda$.

of the semi-discrete propagation operator, since it does not contain operator splitting approximation (2). It is clearly seen from Fig. 11, that both methods produce indistinguishable results.

Finally, we will demonstrate the advantages of the FD approximation over the SSF method associated with the upper transparent boundary condition modeling. Figure 12 demonstrates the effect of false reflections from the upper boundary of the computational domain. Errors are especially clearly visible in the shadow zones. Note that by increasing the height of the computational domain

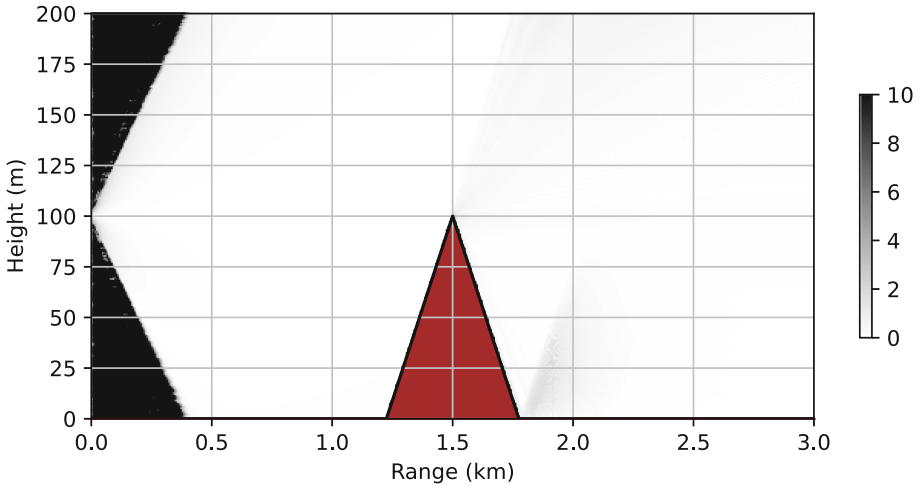

Fig. 6. Diffraction on the impenetrable wedge. Spatial distribution of the absolute error ($|20\log|u_{ri}| - 20\log|u_{ssf}||$) between the results, obtained by the ratinterp-[7/8] scheme (u_{ri}) and SSF method (u_{ssf}). $\Delta x = 50\lambda$, $\Delta z = 1.1\lambda$.

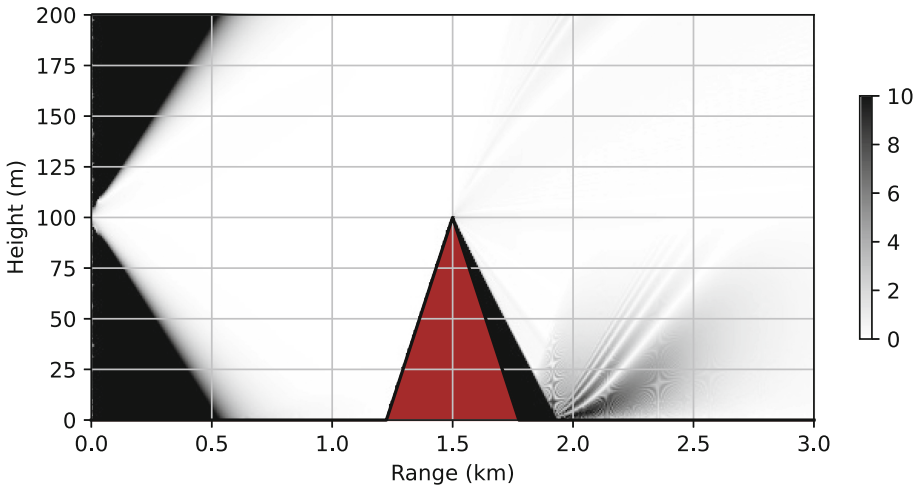

Fig. 7. Diffraction on the impenetrable wedge. Spatial distribution of the absolute error ($|20\log|u_{Pade}| - 20\log|u_{ssf}||$) between the results, obtained by Padé-[7/8] approximation of the semi-discrete propagator with with 4th order approximation of the vertical operator (u_{Pade}) and SSF method (u_{ssf}). $\Delta x = 50\lambda$, $\Delta z = 1.1\lambda$.

and the absorbing layer, it is always possible to achieve a coincidence of results between FD approximations and SSF method, but this will require an extra computational costs. In addition, there are no straightforward methods for getting rid of such errors within the SSF method, which somewhat complicates its

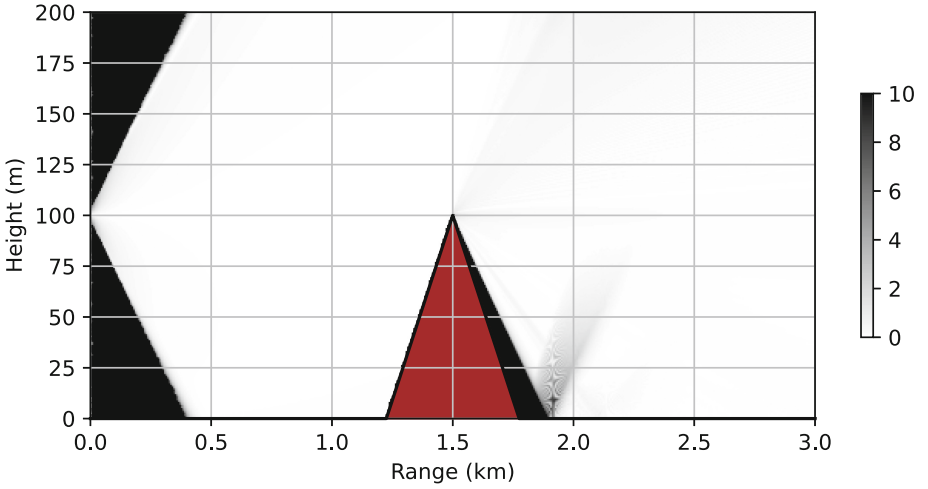

Fig. 8. Diffraction on the impenetrable wedge. Spatial distribution of the absolute error ($|20\log|u_{Pade}| - 20\log|u_{ssf}||$) between the results, obtained by Padé-[7/8] approximation of the fully discrete propagator (u_{Pade}) and SSF method (u_{ssf}). $\Delta x = 50\lambda$, $\Delta z = 1.1\lambda$.

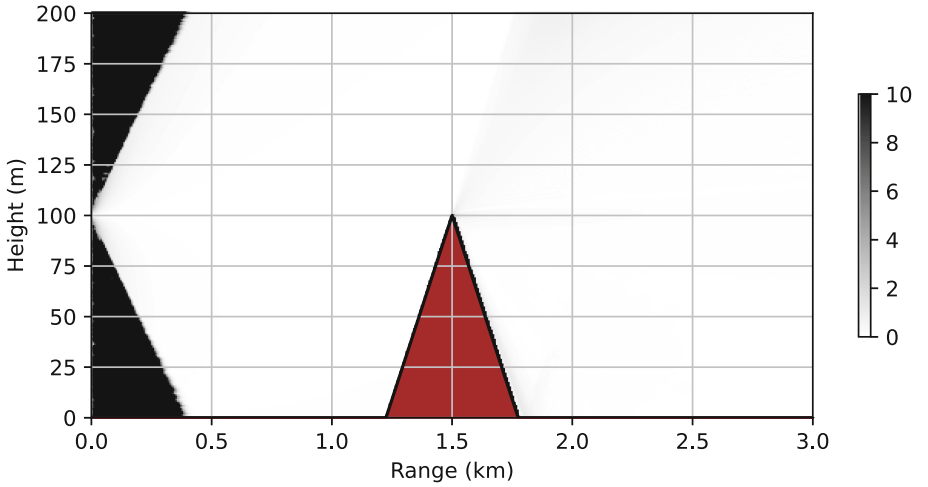

Fig. 9. Diffraction on the impenetrable wedge. Spatial distribution of the absolute error ($|20\log|u_{Pade}| - 20\log|u_{ssf}||$) between the results, obtained by Padé-[7/8] approximation of the fully discrete propagator (u_{Pade}) and SSF method (u_{ssf}). $\Delta x = 15\lambda$, $\Delta z = 1.1\lambda$.

practical use. For the FD approximation method, discrete nonlocal boundary conditions [2,16] were used in all the examples presented, allowing for accurate modeling of the transparent boundary.

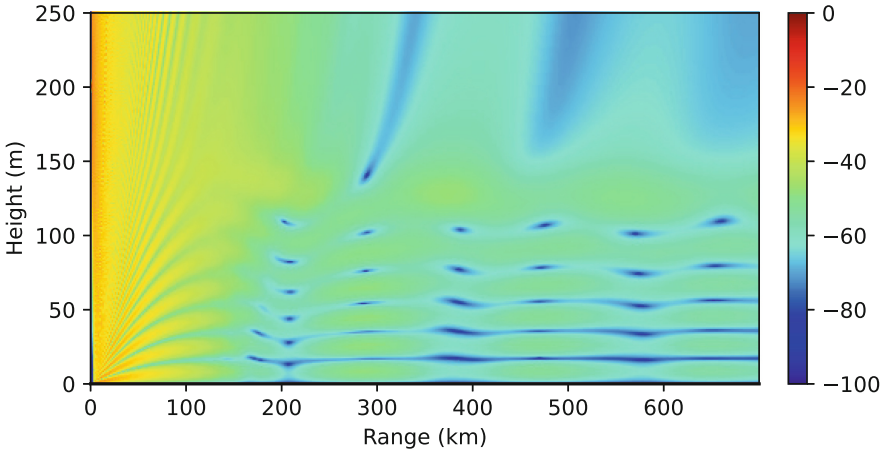

Fig. 10. Propagation in the duct. Spatial distribution of the field amplitude $(20 \log |u(x, z)|)$, computed by the ratinterp-[7/8] approximation. $\Delta x = 300\lambda$, $\Delta z = \lambda$.

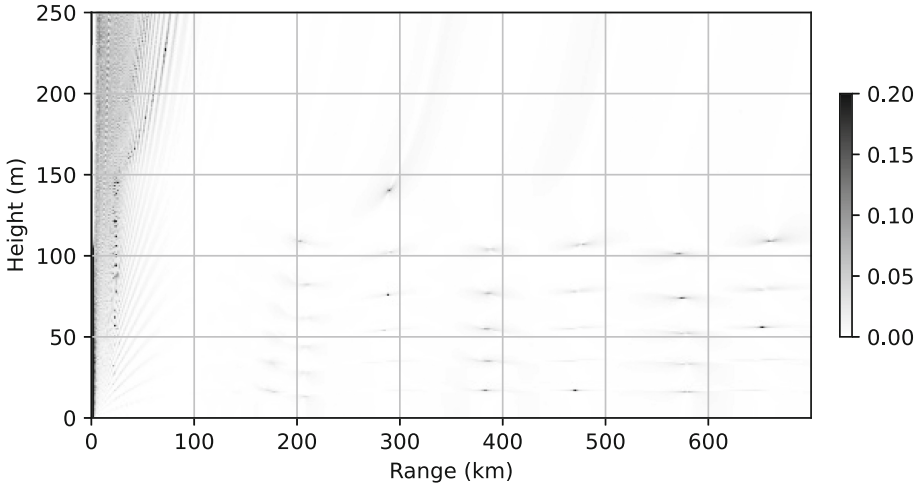

Fig. 11. Propagation in the duct. Spatial distribution of the absolute error $(|20 \log |u_{ri}| - 20 \log |u_{Pade}||)$ between the results, obtained by Padé-[7/8] scheme (u_{Pade}) and ratinterp-[7/8] scheme (u_{ri}). $\Delta x = 300\lambda$, $\Delta z = \lambda$.

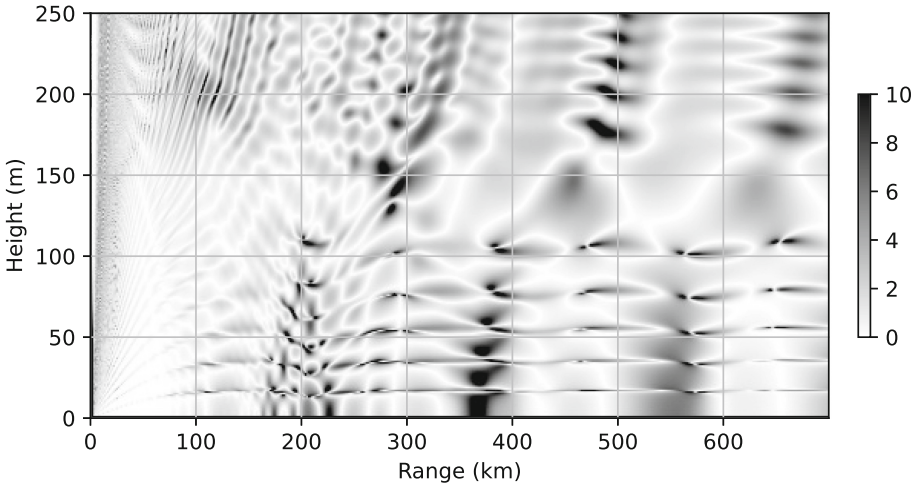

Fig. 12. Propagation in the duct. Spatial distribution of the absolute error ($|20\log|u_{ri}| - 20\log|u_{ssf}||$) between the results, obtained by ratinterp-[7/8] scheme (u_{ri}) and SSF method (u_{ssf}). $\Delta x = 300\lambda$, $\Delta z = \lambda$.

4 Conclusion

The proposed method does not require any significant changes in the existing program implementations of the FD methods, because it differs from the usual Padé method only in coefficients. In addition, the proposed method inherits the advantages of FD methods associated with efficient modeling of boundary conditions.

The paper shows a constructive way to test the stability of a numerical scheme for specific computational parameters. It is desired to establish a priori stability conditions in future works. It is also advisable to conduct a comparative analysis of the optimal computational grid parameters for all existing PE numerical methods in future studies.

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