Torus of Revolution Generated by Curves of Eight

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Abstract. Among the geometric bodies of revolution we find the torus of revolution generated from a circumference that rotates around an axis. Given the classic definition used in Mathematics, interest arises in finding other curves that generate the torus of revolution when rotating around an axis. There is already work done, about the construction of toruses of revolution, using a lemniscatic curve. In this article, making the respective analysis and the necessary programming using the Mathematica 11.1 software, allowed us to carry out the necessary calculations and geometric visualizations of the mathematical object: So a torus of revolution was built from the curve of eight in its parametric form and even the equation of the torus in its Cartesian form. The study was extended and the torus of revolution was generated from rational and irrational curves that rotate around an axis. Curves were determined that were on the torus generated by a curve of eight, which when properly projected to planes, curves that have symmetries were obtained. When points on these curves are properly taken, special irregular polygons are obtained. By obtaining these results, a satisfactory answer to the research question was obtained, as well as a way to define it. In addition, it has shown us a wide path of research on the different curves that can generate a torus of revolution.

Keywords: Torus · Turn of eight · Mathematica

1 Introduction

In the geometry of space we speak of geometric bodies of revolution, when they are generated by turning a curve around an axis. Among this group is the bull of revolution. If we look for real examples where toroids are used, we can mention a wide range of electronic circuits: inductors and toroidal transformers. Since its use has increased enormously due to its symmetry, it has superior electrical performance and low leakage of magnetic flux escaping from the core. Another example is the

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particle accelerator, a machine that uses electromagnetic fields to propel charged particles at very high speeds and energies that travel in a toroidal-shaped tube contained in well-defined beams.

In Mathematics, according to [1] if we consider the circumference of radius "b" in the plane $y = 0$ whose center is at the point (a,0,0) with $a > b$. If "a" is on the circumference we rotate it around the z axis, then we generate a surface called Torus of Revolution.

Given the definition of a torus of revolution, the research question arises: Can a torus of revolution be generated from the figure eight curve as it is rotated around a straight line [2]. Once the research question has been formulated, the respective analysis is carried out to start the programming. The objectives set for this research are:

1. Generate a torus of revolution from the figure eight curve.
2. Define curves in the torus of eight.
3. Polygons whose vertices belong to a given parametric curve.

2 Torus of Revolution Generated by Curves of Eight Programmed in Mathematica

Once the research question has been formulated, the respective analysis is carried out to start the programming.

2.1 The Figure Eight Curve

To generate the parametric equation of the torus of revolution with figure eight curves, we consider the flat curve:

$$Ocho[t,(A,B)] = (A, Bsin(t), Bsin(t)cos(t)) \tag{1}$$

which is resting on the plane whose formulation $x = A$. With Mathematica we define it as follows:

$$Ocho[t_,\{A_,B_\}] := \{A, BSin(t), BSin(t)Cos(t)\}$$

And we graph with Mathematica, according to Eq. (1) we have that (Fig. 1):

$$g1 = ParametricPlot3D\,[Ocho[t,\{1,1\}],\{t,0,2\pi\}, PlotStyle \rightarrow Hue[0.75]]$$

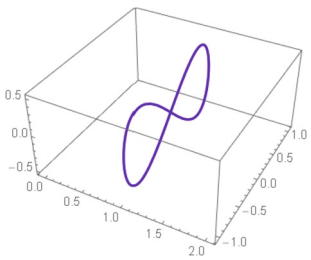

Fig. 1. Curve eight

2.2 Plane Where the Figure Eight Curve Rests

For the case that we are giving without a doubt is the equation x = 1. We corroborate this by graphing the equation of the plane by (Fig. 2).

$$g2 = ParametricPlot3D[\{1, u, v\}, \{u, -1, 1\}, \{v, -1, 1\}, Mesh \to False, PlotStyle \to Yellow]$$

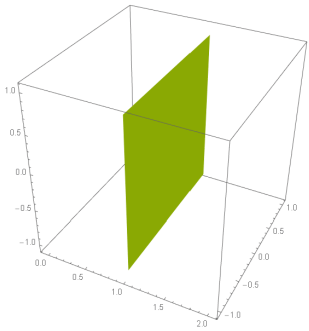

Fig. 2. Plane x = 1, where the figure eight curve is

Using the Show command we paste the figures g1 and g2: $Show[g1, g2]$

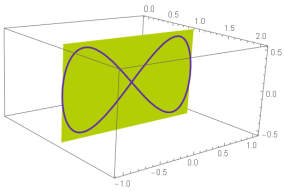

Fig. 3. The plane with the curve of eight

Now we calculate the maximum distance from the origin of coordinates to the points of the figure eight curve, for this we define the function squared distance (Fig. 3).

$$d[t_] := (Ocho[t, \{A, B\}] - \{A, 0, 0\}).(Ocho[t, \{A, B\}] - \{A, 0, 0\})$$

We calculate the critical point:

$$Solve[d'[t] == 0, t]$$

We take the critical point $t_0 = \frac{\pi}{2}$ and evaluate the second and third drifts at that point.

$$(d''[t]/.t \to \frac{\pi}{2}) == (d'''[t]/.t \to \frac{\pi}{2}) == 0$$

But in the fourth derivative and evaluated at that point, we get:

$$d''''[t]/.t \to \frac{\pi}{2}$$
$$24B^2$$

Thus, the distance function has a maximum at $t_0 = \frac{\pi}{2}$. It is simply calculated

$$\sqrt{d[t]}/.t \to \frac{\pi}{2}//PowerExpand$$
$$B$$

Then we define the circle circumscribed to the figure eight curve (Fig. 4).

$$Cccocho[t_], \{A_, B_\} := \{A, BCos[t], BSin[t]\}$$

With this we have:

$$\text{g3} = \text{ParametricPlot3D}[\text{Cccocho}[t, \{1, 1\}], \{t, 0, 2\pi\}, \text{PlotStyle} \to \text{Red}]$$

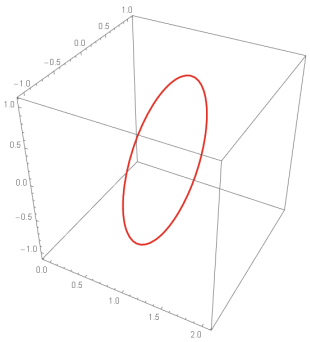

Fig. 4. Circumference

All this is reflected in the following (Fig. 5):

$$\text{Show}[\text{g1}, \text{g2}, \text{g3}, \text{PlotRange} \to \text{All}]$$

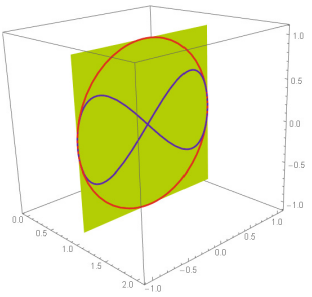

Fig. 5. Circle where the figure eight curve is inscribed

2.3 Generation of the Bull with the Curve of the Eight

If we have the Eq. (1), we will rotate it around the "Z" axis and we will obtain the torus of the curve of eight. We obtain the equation using the Mathematica software.

$$ToroOcho[\{u_, v_\}, \{A_, B_\}] := \{\{Cos[v], -Sin[v], 0\}, \{Sin[v], Cos[v], 0\}, \{0, 0, 1\}\}.$$
$$Ocho[u, \{A, B\}]$$

We run generically

$$ToroOcho[\{u, v\}, \{A, B\}]$$When executing the command we get:
$$\{ACos[v] - BSin[u]Sin[v], BCos[v]Sin[u] + Asin[v], BCos[u]Sin[u]\}$$

To obtain the graph with Mathematica [3] we type

$$\text{tor} = \text{ParametricPlot3D}[\text{ToroOcho}[\{u, v\}, \{1, 4\}], \{u, 0, \pi\}, \{v, 0, 2\pi\},$$
$$\text{PlotStyle} \rightarrow \text{Directive}[\text{Opacity}[0.7], \text{Yellow}, \text{Specularity}[\text{White}, 50]],$$
$$\text{PlotPoints} \rightarrow 60]$$

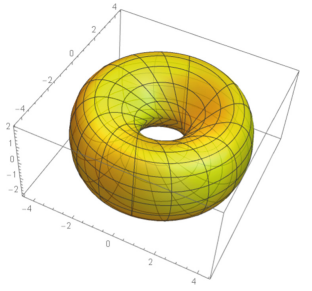

Fig. 6. Graph of the ToroOcho$[\{u, v\}, \{1, 4\}]$

Next, the torus of eight, the curve of eight, the circle circumscribed to the curve of eight and the plane that contains it to the curve that generates the torus of eight is shown (Figs. 6 and 7).

$$\text{Show}[\text{g1}, \text{g2}, \text{g3}, \text{tor}, \text{PlotRange} \rightarrow \text{All}]$$

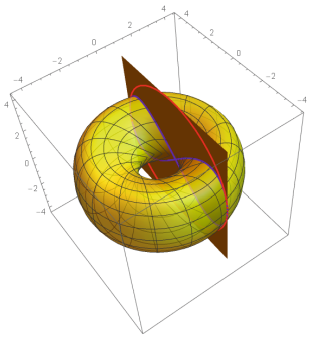

Fig. 7. Graph of the ToroOcho[$\{u, v\}, \{1, 4\}$], plane x=1, circumference and curve of the eight.

2.4 The Cartesian Equation of the Torus with the Curve of the Eight

It is natural to take into account the parametric equations of the torus of the curve of eight to obtain its Cartesian equation. We choose the abscissa component:

$$x = ToroOcho[\{u, v\}, \{A, B\}][[1]]$$
$$ACos[v] - BSin[u]Sin[v]$$

Then the ordered:

$$y = ToroOcho[\{u, v\}, \{A, B\}][[2]]$$
$$BCos[v]Sin[u] + ASin[v]$$

Finally, regarding the variable z:

$$z = ToroOcho[\{u, v\}, \{A, B\}][[3]]$$
$$BCos[u]Sin[u]$$

Later:

$$(x^2 + y^2 - A^2)(B^2 + A^2 - x^2 - y^2)//Simplify$$
$$B^4Cos[u]^2Sin[u]^2$$

With all this it is obtained that:

$$(x^2 + y^2 - A^2)(B^2 + A^2 - x^2 - y^2) = z^2B^2//Simplify$$

Consequently, the Cartesian equation of the torus of eight is:

$$(x^2 + y^2 - A^2)(B^2 + A^2 - x^2 - y^2) = z^2 B^2$$

Now it is graphed with Mathematica Software. Assuming the values: A=1; B=4 and considering (Fig. 8):

$$tor0 = ContourPlot3D[(x^2 + y^2 - A^2)(B^2 + A^2 - x^2 - y^2) == B^2 z^2,$$
$$\{x, -4.2, 4.2\}, \{y, -4.2, 4.5\}, \{z, -4, 4\}, Mesh \to False,$$
$$PlotPoints \to 75, ContourStyle \to Directive[Yellow, Opacity[0.8]]]$$

We will graph the torus of revolution together with the figure eight curve

$$Show[g1, tor0, PlotRange \to All]$$

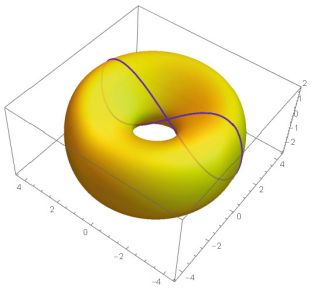

Fig. 8. Graph of the EightTorus[u,v,1,4] and curve of the eight that generates it.

2.5 Curves in the Torus with the Curve of the Eight

To define curves in the torus of eight, the parametric equation [4] of the torus of eight is used

$$\alpha[\text{t_}, \alpha_1 : \{_, _\}, \{\text{A_}, \text{B_}\}] := \text{ToroOcho}[\{u, v\}, \{A, B\}]/.\{u \to \alpha[[1]], v \to \alpha[[2]]\}$$

For example:

$$\alpha[u, \{ArcCos[u], ArcSin[u]\}, \{1, 4\}]//\text{TrigExpand}$$

Which is an irrational curve and also generates the Bull of eight. which is graphically obtained by (Fig. 9).

$$cur1 = ParametricPlot3D[\{\sqrt{1 - u^2} - 4u\sqrt{1 - u^2}, -4u^2 + u + 4, 4u\sqrt{1 - u^2}\},$$
$$\{u, -1, 1\}, PlotStyle \to Hue[0.]]$$

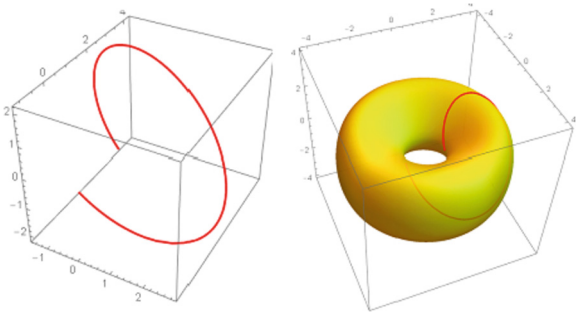

Fig. 9. Graph of the irrational curve that is in the torus of eight.

Considering the expansion of the function Cosine and Sine in Taylor series with a good approximation, as we show

$$cos[u_] := 1 - \frac{u^2}{2} + \frac{u^4}{24} - \frac{u^6}{720} + \frac{u^8}{40320} - \frac{u^{10}}{3628800}$$
$$sin[u_] := u - \frac{u^3}{6} + \frac{u^5}{120} - \frac{u^7}{5040} + \frac{u^9}{362880}$$

And we define the following matrix:

$$A1[u_] := \{\{cos[u], -sin[u], 0\}, \{sin[u], cos[u], 0\}, \{0, 0, 1\}\}$$

Thus we have that the figure eight curve would be as follows

$$NewOcho[u_, \{A_, B_\}] := \{A, Bsin[u], Bsin[u]cos[u]\}//Expand$$

In general, the figure eight curve would be:

$$NewOcho[u, \{A, B\}]$$
$$\{A, Bu - \frac{Bu^3}{6} + \frac{Bu^5}{120} - \frac{Bu^7}{5040} + \frac{Bu^9}{362880}, Bu - \frac{2Bu^3}{3} + \frac{2Bu^5}{15} - \frac{4Bu^7}{315} + \frac{2Bu^9}{2835}$$
$$- \frac{31Bu^{11}}{1209600} + \frac{Bu^{13}}{1555200} - \frac{101Bu^{15}}{9144576000} + \frac{Bu^{17}}{8128512000} + \frac{Bu^{19}}{1316818944000}\}$$

We define a rational algebraic version of the torus with the figure eight curve:

$$TorOchoR[\{u_, v_\}\{A_, B_\}] := A1[u].NewOcho[v, \{A, B\}]//Expand$$

Executing in general form by means of the command: $TorOchoR[\{u, v\}, \{1, 4\}]$ We will assume that A=1 and B=4 in the above equation. To graph it, use the command (Fig. 10):

$$TorochoR = ParametricPlot3D[TorOchoR[\{u, v\}, \{1, 4\}]//Evaluate$$
$$\{u, -\pi, \pi\}, \{v, 0, \pi\}]$$

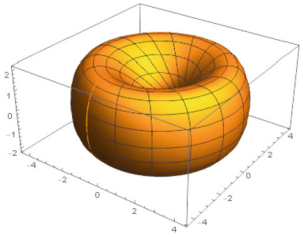

Fig. 10. Graph of the rational algebraic version of the torus with the curve of eight.

A version of this torus involving irrational functions

$$TorOchoI[\{u_, v_\}, \{A_, B_\}] := A1[u].\{\sqrt{1-v^2} - 4v\sqrt{1-v^2},$$
$$4 + v - 4v^2, 4v\sqrt{1-v^2}\}$$

When we evaluate in the command above for A=1 and B=4, $TorOchoI[\{u, v\}, \{1, 4\}]$, we get the torus version of eight in irrational equations. To plot we will use the command

$$ParametricPlot3D[TorOchoI[\{u, v\}, \{1, 4\}]//Evaluate, \{u, -\pi, \pi\},$$
$$\{v, -1, 1\}, PlotPoints \to 75]$$

2.6 Special Polygon with 12 Sides

To present the dodecangle, we consider the following plane curve (Fig. 11):

β[t_]:=Delete[$\alpha[t, \{3t, t\}, \{1, 4\}], 3]$
When executing the command we get:
$\beta[t]$
$\{Cos[t] - 4Sin[t]Sin[3t], Sin[t] + 4Cos[t]Sin[3t]\}$

Its graph is achieved with

$$ParametricPlot[\beta[t], \{t, 0, 2\pi\}, PlotStyle \to Hue[0.65]]$$

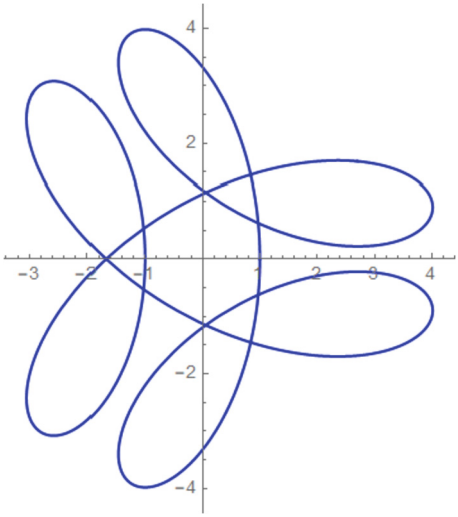

Fig. 11. Curve B(t)

Now, we construct with Mathematica a list in the domain of the curve that is $[0, 2\pi]$ with increments of $\frac{\pi}{24}$ eliminating the last term .

$$lis = Delete[Table[t, \{t, 0, 2\pi, \frac{\pi}{24}\}], -1]$$

With that list of dividing the interval, we evaluate the curve on all those elements.

$$pun = \beta/@lis$$

Then we consider a sublist

$$lis = \{pun[[36]], pun[[38]], pun[[12]], pun[[14]], pun[[4]], pun[[6]], pun[[28]], pun[[30]], pun[[20]], pun[[22]], pun[[44]], pun[[46]]\}$$

Considering the mini program:

$$Angulo[\{P_, Q_, R_\}] := ArcCos[\frac{(Q-P).(Q-R)}{Norm[Q-P]Norm[Q-R]}]$$

We calculate the interior angles of the dodecangle by:

$$Angulo/@Partition[Append[Append[lis1, pun[[36]]], pun[[38]], 3, 1]//N$$

It is obtained that the first two angle measures are equal and their approximate value is 2.87732. The third and fourth value are equal, Y its approximate value is of 2.35866. These two measures are repeated two by two alternately.

To get the length of the sides of the dodecangle we use:

$$Long[\{P_, Q_\}] := Norm[P-Q]$$
$$Long/@Partition[Append[lis1, pun[[36]]]], 2, 1]//N]$$

So we have six sides that measure 0.99942, and two triplets of sides that measure 1.01817 and 4.64618. The polygon that is generated is an irregular dodecangle. Whose graph we obtain with the command (Fig. 12):

$$G2 = Graphics[\{Hue[0.75], Line/@Partition[Apepend[lis1, pun[[36]]], 2, 1]\}]$$

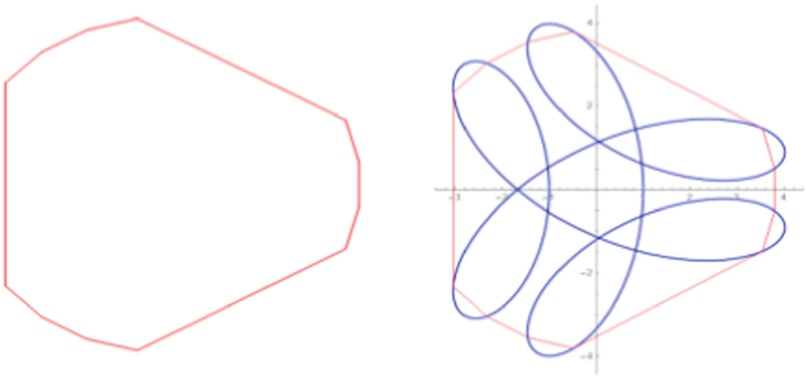

Fig. 12. Graph of the dodecangle and this together with the curve

2.7 Two Hexangles Inside the Irregular Dodecangle

From lis1 we extract the following sublist

$$lis11 = \{lis1[[1]], lis1[[3]], lis[[5]], lis1[[7]], lis1[[9]], lis[[11]]\};$$

And we calculate the measure of the interior angles.

$$Ang/@Partition[Append[Append[lis11, lis1[[1]], lis1[[3]], 3, 1]//FullSimplify$$

Obtaining 2pi 3 . We obtain the measure of the sides by:

$$Long/@Partition[Append[lis11, lis[[1]], 2, 1]//N$$

The approximate measure of the first sides is 2.0, 5.40083, which you alternate successively. Consequence of these two results, we conclude that it is a Hexangle. Análogamente de la sub lista

$$lis12 = \{lis1[[2]], lis1[[4]], lis[[6]], lis1[[8]], lis1[[10]], lis1[[12]]\};$$

It is also obtained that it is a hexaangle, and these two hexaangles are congruent and symmetric with respect to the X axis. The graphs show what is explained (Fig. 13):

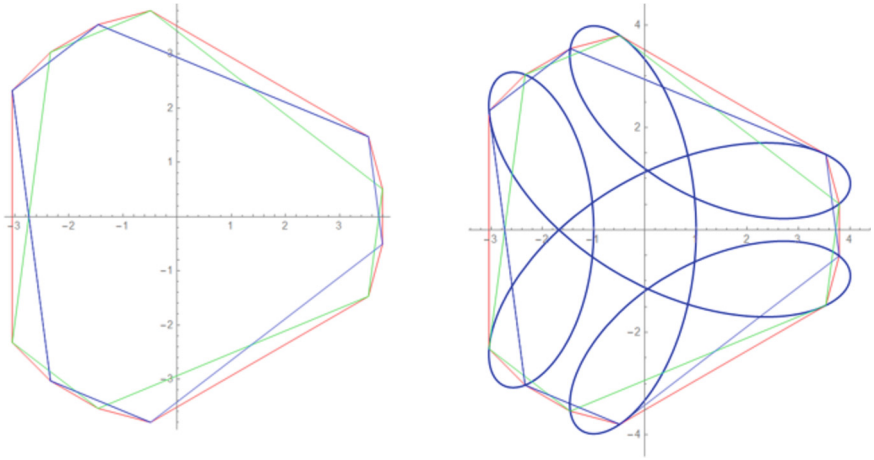

Fig. 13. Graph of the dodecangle together with the curve

2.8 Two Hexangles Inside the Irregular Dodecangle

To obtain the points of the triangles we create the following list

$$listris = Table[\{lis[[i]], lis1[[i+4]], lis1[[i+8]]\}, \{i, 1, 4\}]$$

Then we choose the first element of listris that ultimately gives us the vertices of the first triangle.

$$listris[[1]]$$
$$\{\{4Cos[\frac{\pi}{24}]Cos[\frac{\pi}{8}] - Sin[\frac{\pi}{24}], -Cos[\frac{\pi}{24}] - 4Cos[\frac{\pi}{8}]Sin[\frac{\pi}{24}]\},$$
$$\{Cos[\frac{\pi}{8}] - 4Cos[\frac{\pi}{8}]Sin[\frac{\pi}{8}], 4Cos[\frac{\pi}{28}]^2 + Sin[\frac{\pi}{8}]\},$$
$$\{-Cos[\frac{5\pi}{24}] - 4Cos[\frac{\pi}{8}]Sin[\frac{5\pi}{24}], -4Cos[\frac{\pi}{28}]Cos[\frac{5\pi}{24}] + Sin[\frac{5\pi}{24}]\}\}$$

Calculate the measures of the interior angles

$$Ang/@Partition[Append[Append[listris[[1]], listris[[1]][[1]]], listriz[[1]][[2]]], 3, 1]//Simplify$$
$$\{\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\}$$

The length of the sides

$$Longitd/@Partition[Append[listris[[1]], listris[[1]][[1]]], 2, 1]//Simplify$$
$$\{\sqrt{3(9+4\sqrt{2})}, \sqrt{3(9+4\sqrt{2})}, \sqrt{3(9+4\sqrt{2})}\}$$

So we have that the points of listris[[1]] determine an equilateral triangle. To graph the four triangles involved, the following is used:

$$Triangulo[P : \{_, _\}, Q : \{_, _\}, R : \{_, _\}] :=$$
$$Graphics[\{Hue[Random[]], Line/@Partition[\{P, Q, R, P\}, 2, 1]\}]$$

And then we type (Fig. 14):

$$Show[Triangulo/@Table[\{list1[[i]], lis1[i+4]], lis1[[i+8]]\}, \{i, 1, 4\}]]$$

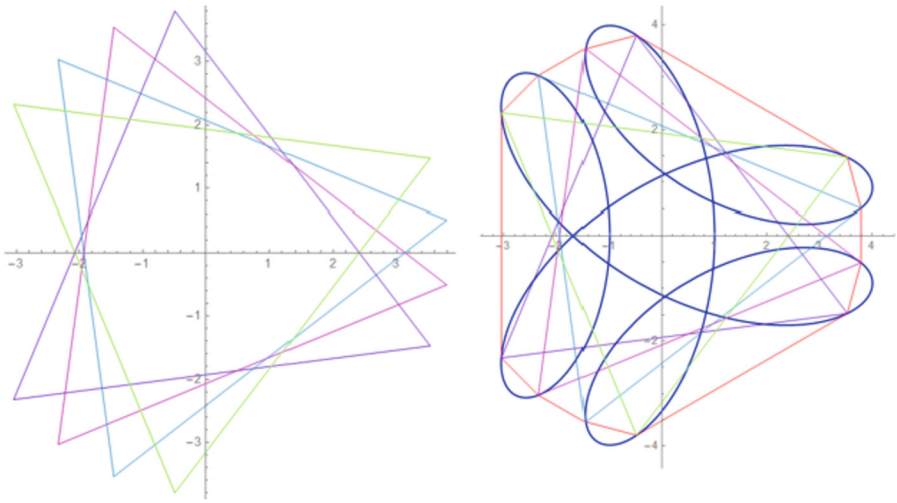

Fig. 14. Graph of the dodecangle along with the four equilateral triangles and curve

3 Conclusions

According to the results found, it is concluded that a torus of revolution can be constructed in its generic form from the curve of eight defined:

$$\{ACo[v] - BSin[u]Sin[v], BCos[v]Sin[u] + ASin[v], BCos[u]Sin[u]\}$$

The Cartesian equation of the torus generated by the figure eight curve was determined:

$$(x^2 + y^2 - A^2)(B^2 + A^2 - x^2 - y^2) = B^2 z^2$$

The generated torus can be defined from rational and irrational curves.

The existing spatial curves on the torus generated by the figure eight curve were analyzed, taking into account the projections to the XY plane, observing existing symmetries. Points were taken from the curves that would be the vertices of special polygons such as irregular dodecangles, irregular hexagons, and equilateral triangles.

It is recommended to deepen the study of the polygons that are generated with the projection of special curves that are on the torus generated by a figure eight curve.

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