

A Galerkin Approach for Fractional Delay Differential Equations Using Hybrid Chelyshkov Basis Functions

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Abstract. This study proposes a numerical technique based on a hybrid of block-pulse functions and Chelyshkov polynomials to solve fractional delay differential equations. The Galerkin approach transforms the solution of fractional delay differential equations into a system of algebraic equations using the fractional operational matrix of integration for these hybrid functions. The suggested method's accuracy and efficiency are demonstrated using numerical examples.

Keywords: Chelyshkov polynomials · Block-pulse functions · Hybrid functions · Fractional delay differential equations · Operational matrix

1 Introduction

Fractional calculus is a fascinating topic that piques the interest of a significant number of researchers. G.W. Leibniz (1695, 1697) and Leonhard Euler (1730) initiated the construction of the fractional calculus, which has been carried on up until the present day [1].

The applications of fractional calculus in fields such as physics, engineering, and others have piqued the interest of scientists in recent decades [2–12]. Nowadays, more proceedings and special issues in journals refer to fractional calculus applications in numerous scientific domains, such as special functions, control theory, chemical physics, stochastic processes, anomalous diffusion, and rheology, etc. See [13].

Fractional derivatives and integrals have been practical tools for characterizing memory and hereditary properties in various materials and processes. For instance, fractional differential equations are employed to explain a range of natural phenomena in physics, chemistry, fluid mechanics, and mathematics [14].

In contradiction to Ordinary Differential Equations (ODEs), Delay Differential Equations (DDEs) incorporate addition derivatives from earlier-time, making the mathematical model closer to the real-world occurrence. Biology, economics, medicine, chemistry, control, and electrodynamics are only a few of the topics where delay differential equations have been widely used for analysis

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and prediction [15,16]. As a generalization, Fractional Delay Differential Equations (FDDEs) deal with differential equations governed by fractional differential operators. In the last two decades, numerical approaches based on orthogonal polynomials have been commonly used to approximate the solution of fractional differential and integral equations. The primary characteristic of these methods is that they convert fractional problems to a system of algebraic equations that can be solved more conveniently using the orthogonal properties of polynomials and typical spectral methods.

Several numerical methods have been applied to solve DDEs. For instance, Chebyshev polynomials [17], Bernoulli polynomials [18], variational iteration method [19], one-leg θ-method [20], Adomian decomposition method [21], hybrid of block-pulse functions and Taylor series [22], Legendre wavelet [23], etc. However, not many studies focus on finding numerical solutions to FDDEs. Some of these works are Hermite wavelet method [24], spectral-collocation method [25], Legendre pseudospectral method [26], Adams-Bashforth-Moulton method, and the linear interpolation method [27], finite difference method [28] and Bernoulli wavelets method [36].

This research came up with a novel idea for an efficient numerical method to approximate the solution to the FDDEs presented below, using a combination of block-pulse functions and Chelyshkov polynomials. Consider

$$\begin{cases} D^{\alpha}q(t) = p(t, q(t), q(t-\tau)), & 0 \le t \le 1, r-1 < \alpha \le r, 0 < \tau < 1, \\ q^{(i)}(0) = \lambda_i, & i = 0, 1, ..., r-1, \\ q(t) = \xi(t), & t < 0. \end{cases} \tag{1}$$

$$\begin{cases} D^{\alpha}q(t) = p(t, q(t), q(t\tau)), & 0 \le t \le 1, r-1 < \alpha \le r, 0 < \tau < 1, \\ q^{(i)}(0) = \mu_i, & i = 0, 1, ..., r-1, \end{cases} \tag{2}$$

where p is an analytical function, τ is delay, λ_i, μ_i, $i = 0, 1, ..., r-1$, are real constants, $\xi(t)$ is an arbitrary known function, q is the solution to be determined, and $D^{\alpha}, (r-1 < \alpha \le r)$ is the fractional derivative in the Caputo sense [36,38]. The goal of this work is to get a direct numerical method that is based on hybrid Chelyshkov orthogonal polynomials for solving a system of FDDEs (1) and (2).

Initially, to solve FDDEs, we shall discuss hybrid Chelyshkov polynomials and the properties of these polynomials. After that, the Galerkin method was applied to solve FDDEs utilizing the fractional integration operational matrix of the hybrid Chelyshkov polynomials. The most notable benefit of using hybrid Chelyshkov orthogonal polynomials and the Galerkin technique for the problem of solving FDDEs is the ability to convert the problem to a set of algebraic equations with unknown coefficients. Numerical examples demonstrate that the provided numerical method is accurate and efficient.

The components of this work are organized as follows: In Sect. 2, we discuss the definitions of fractional derivatives and integrals and their properties. In Sect. 3, an overview of the hybrid Chelyshkov polynomial is presented, including a description of the polynomial and its history. Three subsections make up Sect. 3. In the first, we are given an overview of the definition of Chelyshkov polynomials. The second is linked to the hybrid Chelyshkov functions (HCFs)

and the transformation and fractional operational matrices for HCFs. A numerical method for solving FDDEs will be presented in Sect. 4, and it will be based on HCFs and the operational matrices of the problem. In Sect. 5, the approach based on HCFs that was proposed is applied to the resolution of specific numerical cases. In Sect. 6, we will provide some concluding remarks.

2 Fractional Calculus: Definition and Properties

Fractional calculus has been acknowledged as an indispensable tool for dealing with various processes. This section will cover the principles of fractional calculus shortly and concisely. Multiple definitions exist for variable-order fractional derivatives, including the Riemann-Liouville sense and the Caputo sense. In this paper, the definition in Caputo's sense is examined. For additional information, please see [1,38].

Definition 1. *The formula that defines the Caputo fractional derivative of order α is as follows:*

$$\mathcal{D}^{\alpha}p(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-u)^{n-\alpha-1}\frac{d^n}{du^n}p(u)du, \quad n-1 < \alpha \leq n,\, t > 0, \tag{3}$$

where n is the upper limit of α.

This operator $\mathcal{D}^{\alpha}$ is defined to have the following properties:

$$\mathcal{D}^{\alpha}c = 0, \tag{4}$$

where c is a constant.

$$\mathcal{D}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha} \tag{5}$$

and

$$\mathcal{D}^{\alpha}(\lambda p(t) + \mu q(t)) = \lambda\mathcal{D}^{\alpha}p(t) + \mu\mathcal{D}^{\alpha}q(t). \tag{6}$$

Definition 2. *According to Riemann-Liouville, the formula for the fractional integral of order $\alpha \geq 0$ can be written as follows:*

$$\mathcal{I}^{\alpha}p(t) = \frac{1}{\Gamma(\alpha)}\int_0^x (t-u)^{\alpha-1}p(u)du, \qquad \alpha > 0,\, t > 0. \tag{7}$$

This form of the fractional integral possesses the following fundamental characteristics:

$$\mathcal{I}^{0}p(t) = p(t), \tag{8}$$

$$\mathcal{I}^{\alpha}\mathcal{I}^{\delta}p(t) = \mathcal{I}^{\alpha+\delta}p(t), \tag{9}$$

$$\mathcal{I}^{\alpha}\mathcal{I}^{\delta}p(t) = \mathcal{I}^{\delta}\mathcal{I}^{\alpha}p(t), \tag{10}$$

Riemann-Liouville integrals also have additional properties:

$$\mathcal{I}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}t^{\beta+\alpha}. \tag{11}$$

Here, by Definitions 1 and 2, we can get the following formula:

$$\mathcal{I}^{\alpha}\mathcal{D}^{\alpha}p(t) = p(t) - \sum_{t=0}^{m-1} p^{(i)}(0)\frac{t^i}{i!}. \tag{12}$$

3 Chelyshkov Functions

Vladimir. S. Chelyshkov [39] developed one of the most recent classes of orthogonal polynomials, the Chelyshkov polynomials. These polynomials are defined explicitly by

$$\rho_{N,k}(t) = \sum_{j=k}^{N} \mathbf{z}_{k,j}^{N} t^{j}, \quad k = 0, 1, ...N, \qquad \mathbf{z}_{k,j}^{N} = (-1)^{j-k}\binom{N-k}{j-n}\binom{N+j+1}{N-k} \tag{13}$$

Orthogonality

The Chelyshkov polynomials satisfy the orthogonality condition in terms of the weight function $w(t) = 1$ in the range $[0, 1]$, and their orthogonality requirement is:

$$\int_0^1 \rho_{N,i}(t)\rho_{N,j}(t) = \frac{\delta_{ij}}{i+j+1}. \tag{14}$$

3.1 Hybrid of the Chelyshkov Polynomials and Block-Pulse Functions

This section will focus on the fundamental definition of HCFs and some of the aspects associated with them. The Chelyshkov orthogonal polynomials are very significant in the performance of HCFs. Following the introduction of the Chelyshkov orthogonal polynomials, HCFs will be built by mixing these orthogonal polynomials with block-pulse functions [40].

The hybrid of Chelyshkov polynomials and block pulse functions, $\phi_{nm}(t)$, $n = 1, 2, ..., N$, $m = 0, 1, 2, ..., M$ are defined on the interval $[0, 1]$ as

$$\phi_{nm}(t) = \begin{cases} \sqrt{N}\rho_{N,m}(Nt - n + 1) & t \in [\frac{n-1}{N}, \frac{n}{N}] \\ 0 & otherwise, \end{cases} \tag{15}$$

where $\rho_{N,m}$ denotes the Chelyshkov polynomials of degree m defined in (13).

Function Approximation

The set of $\{\phi_{nm}(t), n = 1, 2, ..., N, m = 0, 1, 2, ..., M\}$ constitutes an orthogonal basis over $[0, 1)$ and every square inegrable function $p(t)$ in interval $[0, 1)$ may be enlarged as

$$p(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M} c_{nm} \phi_{nm}(t) = C^T \Phi(t) \tag{16}$$

where

$$\Phi(t) = \big[\phi_{10}(t), \phi_{20}(t), ..., \phi_{N0}(t), \phi_{11}(t), \phi_{21}(t), ..., \phi_{N1}(t), ..., \phi_{1M}(t), \phi_{2M}(t), ..., \phi_{NM}(t)\big]^T, \tag{17}$$

$$C = [c_{10}, c_{20}, ..., c_{N0}, c_{11}, c_{21}, ..., c_{N1}, ..., c_{1M}, c_{2M}, ..., c_{NM}]^T. \tag{18}$$

The vectors $\Phi(t)$ and C in series (16) can be rewritten as

$$p(t) \simeq \sum_{i=1}^{\hat{m}} c_i \phi_{nm}(t) = C^T \Phi(t), \tag{19}$$

where $\Phi(t) = \big[\phi_1(t), \phi_2(t), ..., \phi_{\hat{m}}(t)\big]$, and $C = [c_1, c_2, ..., c_{\hat{m}}]$. Also $c_i = c_{nm}$, $\phi_i(t) = \phi_{nm}(t)$, $i = (n-1)(M+1) + m + 1$, $\hat{m} = (M+1)N$.

3.2 Operational Matrix

The solution of variable-order fractional differential or integral equations is accomplished using an operational matrix technique. The primary idea behind this part is to generate the operational integration matrix using hybrid Chelyshkov functions.

Theorem 1. *[29–31] Given that $\Psi(t)$ represents the Chelshkov polynomial vectors, the Riemann-Liouville fractional integral of order α of the Chelyshkov polynomial vector can be expressed as:*

$$\mathcal{I}^{\alpha} \Psi(t) = \Theta^{(\alpha)} \Psi(t), \tag{20}$$

where $\Theta^{(\alpha)}$ indicates the fractional integral matrix of order α and has the dimensions $(M+1) \times (M+1)$. The (i, j)-th element of this matrix is generated by the following operations:

$$\Theta_{i,j}^{(\alpha)} = \sum_{j=i}^{M} \sum_{s=j}^{M} \frac{\mathbf{z}_{j,s} \mathbf{z}_{i,j} \Gamma(j+1)(2j+1)}{\Gamma(j+\nu+1)(j+s+1)}, \qquad i, j = 1, 2, ..., M+1.$$

Lemma 1. *[29–31] The $\hat{m}$ HCFs vector $\Phi(t)$ can be convert into the $(M+1)$ Chelyshkov polynomials vector $\Psi(t)$ by expanding as follows:*

$$\Phi(t) = \Lambda\Psi(t), \tag{21}$$

where Λ is a $\hat{m} \times (M+1)$ matrix,

$$\Lambda_{ij} = (2j+1)\sum_{r=j}^{M}\sum_{s=m}^{M}\sum_{k=0}^{r}\frac{\binom{r}{k}(n-1)^{r-k}N^{-\frac{1}{2}-r}\mathbf{z}_{r,j}\mathbf{z}_{s,m}}{r+s+1},$$

and $i = 1, ..., \hat{m},\ j = 1, ..., M+1$.

Theorem 2. *[29–31] Consider the $(M+1)$ Chelyshkov polynomials vector to be denoted by $Psi(t)$. Then, the HCFs representation of $Psi(x)$ can be developed as follows:*

$$\Psi(t) = \Pi\Phi(t), \tag{22}$$

where Π is $(M+1) \times \hat{m}$ matrix and

$$\Pi_{i,j} = \sum_{r=i-1}^{M}\sum_{s=m}^{M}\sum_{k=0}^{r}\frac{\mathbf{z}_{r,i-1}\mathbf{z}_{s,m}N^{-\frac{1}{2}-r}\binom{r}{k}(n-1)^{r-k}}{k+s+1}.$$

Theorem 3. *[29–31] Using $\Phi(t)$ as the HCFs vector, the fractional integration of order α for this vector can be written as:*

$$\mathcal{I}^{\alpha}\Phi(t) = \mathcal{P}^{(\alpha)}\Phi(t), \tag{23}$$

where $\mathcal{P}^{(\alpha)} = \Lambda\Theta^{(\alpha)}\Pi$ is a $\hat{m} \times \hat{m}$ matrix, and Λ and Π are transformation matrices derived in Eqs. (21) and (22), respectively. In addition, Θ^{ν} is operational matrix of fractional integration for HCFs vector $\Phi(t)$.

4 Problem Statement and Approximation Scheme

In this part, we solve FDDEs (1) and (2) by employing the Galerkin method and the fractional integration operational matrices of the HCFs.

Problem(1)

Consider the FDDEs (1). To solve this problem, we approximate $D^{\alpha}q(t)$ via HCFs basis $\phi_i(t), i = 0, 1, 2, ..., \hat{m}$, as

$$D^{\alpha}q(t) = C^{T}\Phi(t), \tag{24}$$

where C is an unknown vector that can be acquired. By utilizing (20), we obtain:

$$q(t) \simeq I^{\alpha}\left(C^{T}\Phi(t)\right) + \sum_{i=0}^{r-1} \frac{\lambda_i}{i!} t^i = C^{T}\mathcal{P}^{(\alpha)}\Phi(t) + d^{T}\Phi(t). \tag{25}$$

In other words, for the problem (1), we get:

$$q\left(t-\tau\right) = \begin{cases} \xi\left(t-\tau\right), & 0 \le t \le \tau \\ C^{T}\mathcal{P}^{(\alpha)}\Phi\left(t-\tau\right) + d^{T}\Phi\left(t-\tau\right), & \tau \le t \le 1 \end{cases} = C_{\tau}^{T}\Phi\left(t\right). \tag{26}$$

We can obtain the following residual function by substituting Eqs. (24)–(26) for the original equations in problem (1).

$$R(t) = C^{T}\Phi(t) - p\left(t, C^{T}\mathcal{P}^{(\alpha)}\Phi(t) + d^{T}\Phi(t), C_{\tau}^{T}\Phi\left(t\right)\right). \tag{27}$$

To obtain the answer $q(t)$, we must first identify the residual $R(t)$ at the $\hat{m}+1$ points. As demonstrated below, we use the roots of shifted Chebyshev polynomials to determine appropriate collocates.

$$R(t_j) = 0, j = 1, 2, ...\hat{m}+1. \tag{28}$$

It is possible to find the solution to this problem using the vector C coefficients that are unknown. As a direct consequence of this, we can obtain the numerical solution by substituting the resulting vector C into the Eq. (25).

Problem(2)

We use a method that was employed for the previous problem (1) to solve the FDDEs (2). Replacing Eqs. (24) and (25) in (2), we have the following residual function:

$$R(t) = C^{T}\Phi(t) - p\left(t, C^{T}\mathcal{P}^{(\alpha)}\Phi(t) + d^{T}\Phi(t), C^{T}\mathcal{P}^{(\alpha)}\Phi(\tau t) + d^{T}\Phi(\tau t)\right). \tag{29}$$

As shown follows, we then compute the residual value $R(t)$ at the $\hat{m}+1$ roots of the shifted Chebyshev polynomials.

$$R(t_j) = 0, \quad j = 1, 2, ...\hat{m}+1. \tag{30}$$

These equations are then solved for unknown coefficients of the vector C. As a result, by solving these systems by determining C, we obtain the numerical solution to this problem (2).

5 Numerical Results

This section will examine three different example scenarios to highlight the usefulness and advantages of the proposed technique.

Example 1. Consider the FDDEs

$$\begin{cases} D^{\alpha}q(t) = q(t-\tau) - q(t) + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 2t\tau - \tau^2 - \tau, & 0 \le t \le 1,\ 0 < \alpha \le 1, \\ q(t) = 0, & t \le 0. \end{cases}$$

In the case where $\alpha = 1$, the exact solution to this system is $q(t) = t^2 - t$. The approximate solutions for several different values of α and the absolute error for $\alpha = 1$ are displayed in Fig. 1 for the case in which $M = 2$, $N = 2$, and $\tau = 0.01$ are the input values. When utilizing the current approach with $\hat{m} = 6$, the RMSE for various values of t is displayed in the table referenced by the Table 1. As anticipated, the outcomes of this example get closer and closer to the exact solution as the fractional order α approaches 1.

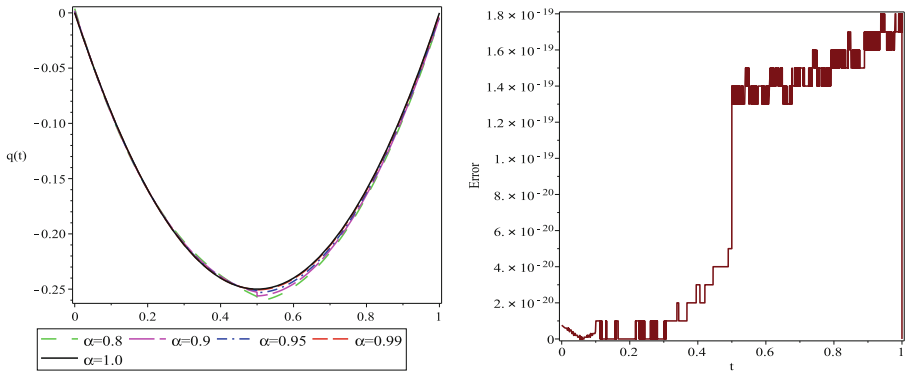

Fig. 1. The approximate solution for various values of α (Left) and absolute error for $\alpha = 1$ (Right).

Table 1. The RMSE ($\|q - \tilde{q}\|_2$) for different values of t (Example 1).

Methods	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
[32]	3.46×10^{-16}	2.35×10^{-16}	3.46×10^{-16}	9.69×10^{-17}
[33]	1.19×10^{-14}	1.63×10^{-15}	7.29×10^{-15}	4.64×10^{-15}
[34]	7.58×10^{-14}	3.90×10^{-14}	1.45×10^{-14}	7.96×10^{-14}
[36]	0	1.11×10^{-16}	3.15×10^{-14}	3.23×10^{-14}
HCFs	0	2.00×10^{-16}	1.00×10^{-16}	0

Example 2. Consider the following FDDEs

$$\begin{cases} D^{\alpha}q(t) = -q(t-0.3) - q(t) + e^{-t+0.3}, \quad 0 \le t \le 1, \quad 2 < \alpha \le 3 \\ q(0) = 1, \quad q'(0) = -1, \quad q''(0) = 1, \\ q(t) = e^{-t}, t < 0. \end{cases}$$

$q(t) = e^{-t}$ that provides an exact solution for the present test problem for $\alpha = 3$. The approximate solutions for several different values of α and the absolute error for $\alpha = 3$ are displayed in Fig. 2, when $M = 8, N = 2$, and $\tau = 0.3$ are the input variables. The approximate solutions obtained by applying the current approach with $\hat{m} = 14$ are displayed in the Table 2. As the fractional-order α approaches 3, these results demonstrate that the approximate solutions converge to the precise solution.

Table 2. The approximate solutions for different values of t (Example 2).

Methods	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
[35]	0.8187	0.6703	0.5488	0.4494
[36]	0.8187	0.6703	0.5488	0.4494
[37]	0.8187	0.6703	0.5488	0.4493
HCFs	0.8187	0.6703	0.5488	0.4494

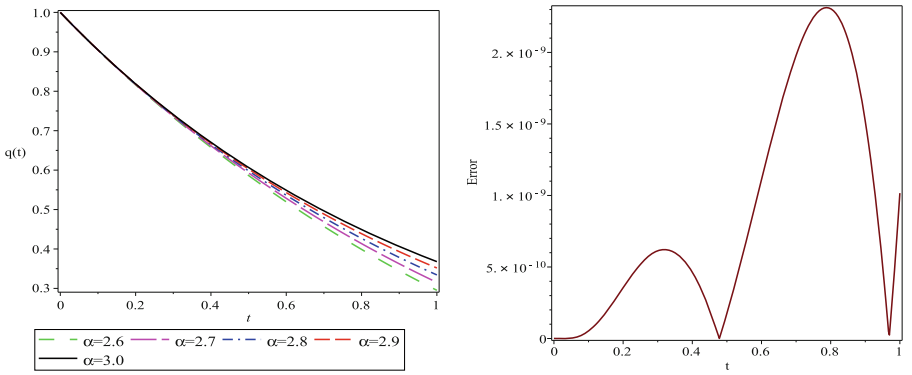

Fig. 2. The approximate solution for various values of α (Left) and absolute error for $\alpha = 3$ (Right).

Example 3. Consider the following FDDEs

$$\begin{cases} D^{\alpha}q(t) = 1 - 2\,q^2(\frac{1}{2}t), \quad 0 \le t \le 1, \quad 1 < \alpha \le 2, \\ q(0) = 1, \quad q'(0) = 0. \end{cases}$$

In this example, the exact solution is $q(t) = \cos(t)$ when $\alpha = 2$. The approximate solutions for several different values of α and the absolute error for $\alpha = 2$ are

displayed in Fig. 3 for the case when $M = 6$ and $N = 2$. The RMSE is shown for different values of t in Table 3, which uses the current approach and has $\hat{m} = 14$. Based on the information obtained, we can derive the following conclusion: the approximate solution will eventually converge to the exact solution as the fractional order α goes closer and closer to 2.

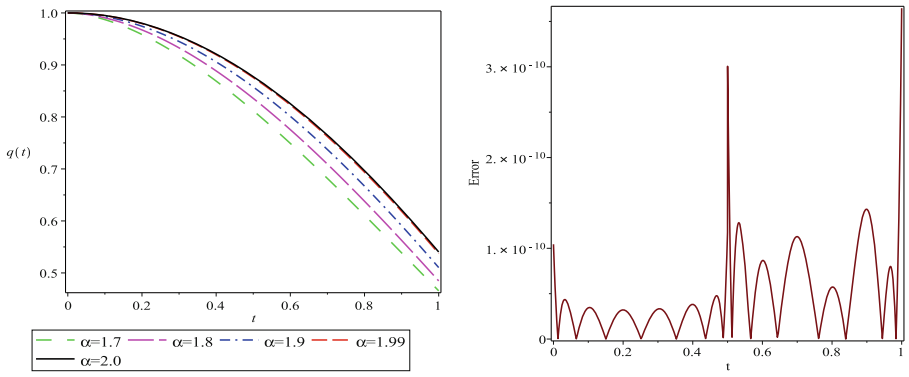

Fig. 3. The approximate solution for various values of α (Left) and absolute error for $\alpha = 2$ (Right).

Table 3. The RMSE ($\|q - \tilde{q}\|_2$) for different values of t and N (Example 3).

Methods	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
[36]	3.21×10^{-11}	3.81×10^{-11}	1.31×10^{-06}	1.82×10^{-06}
HCFs	3.20×10^{-11}	3.81×10^{-11}	8.90×10^{-11}	6.21×10^{-11}

6 Conclusion

This research presented a new and effective method for performing the numerical technique on hybrid Chelyshkov orthogonal polynomials. In the HCFs strategy, the operational matrix of fractional integration for HCFs is the first and most significant aspect of solving the system of FDDEs that we experience, as we have seen in Sect. 2, these matrices were derived. To solve FDDEs, these matrices, in addition to the Galerkin method, are utilized. As in the final step of the procedure, the robustness and efficiency of the suggested method are evaluated using some numerical examples of FFDEs.

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