Multiple-Precision Arithmetic of Biot-Savart Integrals for Reconnections of Vortex Filaments

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Abstract. In this paper, we show an efficient application of multiple-precision arithmetic to numerical computation of the Biot-Savart integral, which is a mathematical model of motion of vortex filaments. Since it is a non-linear integro-differential equation, numerical methods play a significant role in analysis. Hence reliable schemes are desired even though their computational costs are high. Multiple-precision arithmetic enables us to estimate rounding errors quantitatively, and comparing various precision arithmetic. Thus we conclude reliability of numerical results. In particular, reconnection of vortex filaments is investigated, and we meet oscillation of numerical solutions due to singularity. The proposed method clarifies that the divergence immediately after reconnection is still reliable in terms of rounding errors.

Keywords: Computational fluid dynamics · Vortex reconnection · Numerical instability and reliability · Multiple-precision arithmetic

1 Introduction

We consider dynamics of incompressible fluid occupying whole $\mathbb{R}^3$, whose velocity is denoted by $\vec{v} = \vec{v}(\vec{x})$ at $\vec{x} \in \mathbb{R}^3$. Its vorticity is defined by $\vec{\omega} = \mathrm{rot}\,\vec{v}$, and conversely, the Biot-Savart law gives $\vec{v}$ from $\vec{\omega}$ under the hypothesis of incompressibility. Let $C \subset \mathbb{R}^3$ be a non self-intersecting smooth curve, and suppose that vorticity concentrates on C with constant circulation. If $\vec{\omega}$ tangents to C at every point, then it is called a vortex filament. Let us denote the vortex filament at time t by $C = \{\vec{x}(t,s)\ ;\ t > 0, s \in I(t)\}$ with the arc length parameter s belonging to an interval $I(t) \subset \mathbb{R}$. It is then well known that C has self-induced velocity which is described by the Biot-Savart law

$$\frac{\partial \vec{x}}{\partial t}(t,s) = \frac{\Gamma}{4\pi} \int_{I(t)} \frac{(\partial \vec{x}/\partial s)(t,s') \times \left(\vec{x}(t,s) - \vec{x}(t,s')\right)}{\left|\vec{x}(t,s) - \vec{x}(t,s')\right|^3} ds', \quad t > 0, s \in I(t) \quad (1)$$

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where Γ is circulation and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$. Since the Biot-Savart law (1) is a non-linear integro-differential equation with a singular integral kernel, it is hard not only to prove well-posedness, i.e. unique existence of the solution and stability in a suitable normed space, but also to analyze the behavior of vortex filaments analytically. Therefore approximation methods have been developed so far (see [13,14,18] and references therein), and several computational approaches have been presented in recent years with the help of rapid growth of computational resources. The local induction approximation [5,9] and its correction [3,10,19], and regularization methods [15–17] are known as conventional methods, including treatments of singularities. Although numerical methods are indispensable in the analysis of the motion of vortex filaments, their reliability have not been investigated as the authors' best knowledge.

Recently Kimura and Moffatt have numerically observed the occurrence of singularities in C, where it is self-intersecting in their dynamics obeyed by (1) starting from non-self-intersecting smooth curves as initial conditions[11,12]. This is called *reconnection* of vortex filaments, which attracts a lot of interest in connection with smoothness of solutions of fluid dynamics.

In this paper, we discuss reliability of reconnection of vortex filaments from the standpoints of numerical computation. Particularly we concentrate ourselves on estimation of behavior of rounding errors, which give serious influences in unstable numerical processes. In time evolution problems, stability of numerical schemes indicates that the accumulation of computational errors is not crucial, hence it is one of fundamental concepts for reliability of numerical solutions. Since it is hard to prove stability of schemes theoretically in most cases, we develop a practical and efficient strategy to estimate accumulation of rounding errors.

The rest of this paper is arranged as follows. Section 2 briefly describes the numerical scheme proposed by Kimura and Moffatt. Our proposed methodology is presented in Sect. 3, and we show its efficiency and feasibility in Sect. 4 by case studies.

It should be noted that *stability* depends on a choice of topology where we discuss numerical schemes and corresponding solutions. Although terminology of *stability* with absence of function spaces is abuse, we use it for clarity.

2 Numerical Schemes for Motion of Vortex Filaments

In this section the numerical method for (1) is presented which is used in this paper. The method has been proposed by Kimura and Moffatt [11,12], and is efficient to avoid the singularities of the integral kernel in spite of its simplicity.

Firstly we assume that initial shape of the vortex filament is a closed curve, and has parameterization as $C_0 = \{\vec{x}(0,\theta)\,;\, 0 \le \theta < 2\pi\}$. Let Δt be a positive number, N be a power of two, and $\theta_j = 2\pi j/N$ for $0 \le j < N$. Then we allocate N nodes $\vec{x}(0,\theta_j)$ on C_0, and for each time step $t_k = k\Delta t$, we calculate

$$\Delta s_j = \frac{\left|\vec{x}_{j+1} - \vec{x}_j\right| + \left|\vec{x}_j - \vec{x}_{j-1}\right|}{2}, \quad 0 \le j < N, \tag{2}$$

where $\vec{x}_j = \vec{x}(t_k, \theta_j)$ with identifying $\vec{x}_{-1} = \vec{x}_{N-1}$ and $\vec{x}_N = \vec{x}_0$ for periodicity. Then the Biot-Savart integral (1) is discretized as

$$\vec{v}(\vec{x}_i) = \frac{\Gamma}{4\pi} \sum_{\substack{0 \le j < N \\ j \ne i}} \frac{\vec{t}_j \times (\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} \Delta s_j, \quad 0 \le i < N.$$

Particularly singularity in integration is approximated with ignoring the influence from neighborhood of $\vec{x}_i$. The unit tangent vector $\vec{t}_j$ is calculated by the fast Fourier transform (FFT), and the classical Runge-Kutta method is adopted for time-stepping. Then we can find $\{\vec{x}(t_k, s_j)\,;\, 0 \le j < N\}$ which approximates C at $t = t_k$.

Secondly, we consider the case when the initial condition $C_0 = \{\vec{x}(0, p)\,;\, p \in \mathbb{R}\}$ is an unbounded curve. Let N be a positive integer, $h = \pi/(2N)$, and we introduce $p_j = \sinh\left(\frac{\pi}{2}\sinh(jh)\right)$ for $-N \le j \le N$. Then C is approximated by $2N+1$ nodes $\vec{x}(p_j)$, and the double exponential rule [20] imply an approximation of (1) by

$$\vec{v}(\vec{x}_i) = \frac{\Gamma}{4\pi} \sum_{\substack{-N \le j \le N \\ j \ne i}} \frac{\vec{t}_j \times (\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} \Delta p_j, \quad -N \le i < N$$

where

$$\Delta p_j = \frac{\pi}{2} h \cosh(jh) \cosh\left(\frac{\pi}{2}\sinh(jh)\right)$$

and $\vec{t}_j$ is calculated from the five-point finite difference. Finally, the classical Runge-Kutta method is also employed for time-stepping.

3 Quantitative Estimate of Rounding Errors by Multiple-Precision Arithmetic

In scientific numerical computations, all real numbers and their arithmetic are approximated by floating-point arithmetic. In the standard numerical computation environments, it is hard to estimate the influence of them. We propose the use of multiple-precision arithmetic to estimate rounding errors introduced by this approximation.

On digital computations, a floating-point number consists of sign, exponent, and fractional parts. Since each part has finite information, rounding errors are inevitable in number representations and their arithmetic. Though rounding errors are thought to be sufficiently smaller than discretization errors so that they are negligible in scientific computation, they may accumulate rapidly in unstable numerical processes and give fatal influences. On the other hand, rounding errors depend on various issues, such as expressions in user programs and execution orders of arithmetic. Thus it is infeasible to estimate them theoretically in advance.

The double precision arithmetic defined in IEEE754 standard [2] is widely used in scientific computation. IEEE754 double precision has 53-bits in its fractional part which corresponds to approximately 16 decimal digits. However, in some advanced scientific and engineering computations, influence of rounding errors is serious. In such cases, particular algorithms must be designed so as to reduce rounding errors depending on problems or parameters in computations.

One of efficient strategy to reduce rounding errors is extending precision of the fractional part of floating-point numbers. Recently a quadruple precision arithmetic environment *quadmath* [7] has been implemented in the de facto standard compiler GNU Compilers Collection (GCC) on AMD64 and Intel64, which are popular processor architectures for PC. Furthermore, one of the authors has proposed a multiple-precision arithmetic environment *exflib* [6] which is suitable for scientific and engineering computation on AMD64 and Intel64 architectures. Exflib enables us to specify arbitrarily computational precision in advance, and provides user-friendly interfaces so as to use same expressions as the built-in types in the programming language C++. Table 1 shows bits length and precision of floating-point types used in the present research.

Table 1. Precision of floating-point types used in experiments. In exflib, 50 decimal digits is specified as desired computational precision.

Type	Total (bits)	Sign (bits)	Exponent (bits)	Fraction	
				(bits)	(decimal digits)
IEEE754 double	64	1	11	53	(15.95)
quadmath	128	1	15	113	(34.02)
exflib, 50 digits	256	1	63	193	(58.10)

In order to estimate influences of rounding errors, we process computation with three different precisions and compare their results. Under the assumption that the results with exflib (50 digits) are most accurate among them, we are able to estimate rounding errors in results with IEEE754 double precision or quadruple precision arithmetic by using those with exflib (50 digits) as a reference.

4 Numerical Examples

This section is devoted to exhibiting efficiency and feasibility of the proposed method by case studies. In numerical experiments, the computational environment shown in Table 2 is used, and all computations are processed by the use of 32 threads parallel computation by OpenMP. Throughout the section, Γ is set to $4\pi/50$.

Table 2. Computational environment

CPU	Ryzen Threadripper 2990WX (3.0 GHz, 32 cores)
Memory	DDR4-1866 128GB
OS	CentOS Linux 7.9.2009
C++ Compiler	GCC Version 4.8.5
FFT	fftw Version 3.3.8
exflib	Version 20180620

4.1 Figure-of-Eight Vortex Filament

We consider as an initial condition

$$\begin{aligned} x(\theta) &= 0.5 \sin 2\theta, \\ y(\theta) &= 2.5 \sin \theta, \\ z(\theta) &= 0.05 \cos \theta, \end{aligned} \tag{3}$$

with $0 \le \theta < 2\pi$. In discretization, we allocate N nodes $\vec{x}(\theta_j)$ with $\theta_j = 2\pi j/N$, $0 \le i < N$ with $N = 8192$. Using the Runge-Kutta method with $\Delta t = 4 \times 10^{-5}$, numerical results at $t = 0.0024$ $(= 60\Delta t)$ are depicted in Fig. 1 (a) and (b), which are respectively obtained by the standard double precision arithmetic and 50 decimal digits computation by multiple-precision arithmetic. Oscillations appear in both results, while numerical results with a smaller $\Delta t = 2 \times 10^{-5}$ are smooth shown in Fig. 1 (c).

Adopting $\Delta t = 2 \times 10^{-5}$ enables us to meet numerical reconnection of the vortex filament around estimated reconnection time which is approximately $t \approx 0.3385$. Figure 2 (a) and (b) are respectively numerical results immediately before and after the reconnection by the standard double precision, where purple and green parts mean that they are in $z > 0$ and $z < 0$ in the initial condition (3). Numerical results in Fig. 2 (b) has disturbance similarly to Fig. 1 (a) and (b).

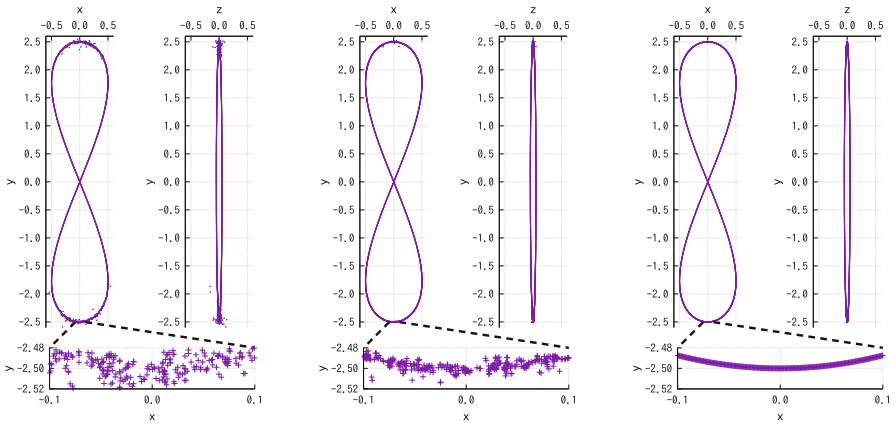
In order to evaluate rounding errors quantitatively, we process the computation with quadruple precision arithmetic, and 50 decimal digits computations. Errors between double precision arithmetic and 50 decimal digits

$$\max_j \frac{\left|\vec{x}^{(\text{double})}(k\Delta t, s_j) - \vec{x}^{(50\ \text{digits})}(k\Delta t, s_j)\right|}{\left|\vec{x}^{(50\ \text{digits})}(k\Delta t, s_j)\right|} \tag{4}$$

and those between quadruple precision arithmetic and 50 decimal digits

$$\max_j \frac{\left|\vec{x}^{(\text{quadruple})}(k\Delta t, s_j) - \vec{x}^{(50\ \text{digits})}(k\Delta t, s_j)\right|}{\left|\vec{x}^{(50\ \text{digits})}(k\Delta t, s_j)\right|} \tag{5}$$

are shown in Table 3. In the results errors in $t \le 0.342$ are small as rounding errors, and oscillations in Fig. 2 (b) commonly appears in those with quadruple



(a) $\Delta t = 4 \times 10^{-5}$, double precision ($t = 60\Delta t$)

(b) $\Delta t = 4 \times 10^{-5}$, 50 digits ($t = 60\Delta t$)

(c) $\Delta t = 2 \times 10^{-5}$, double precision ($t = 120\Delta t$)

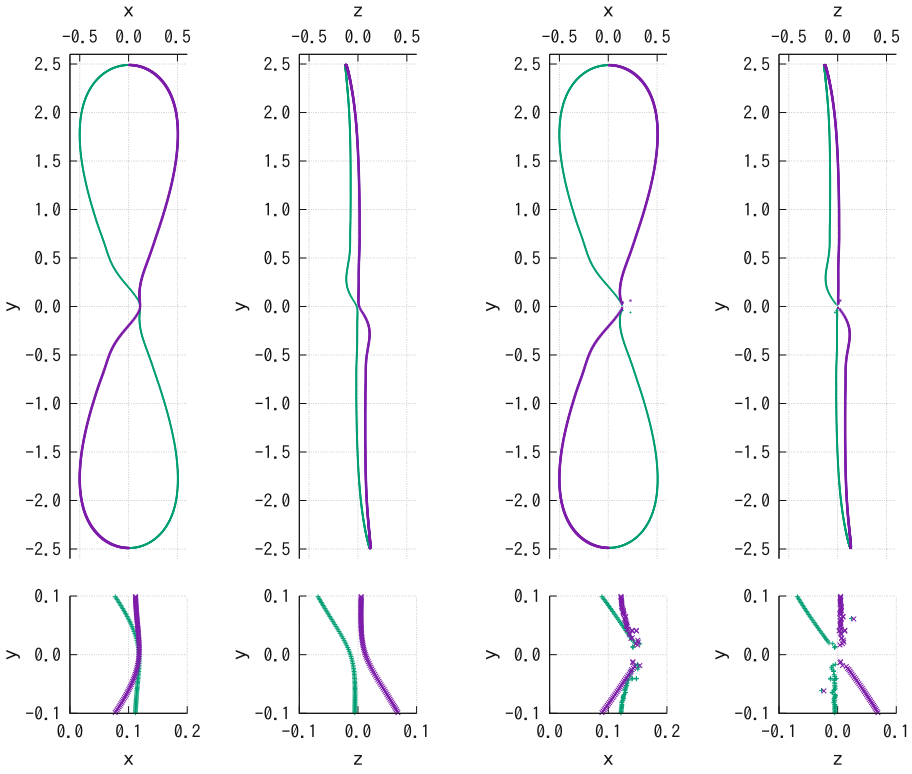
Fig. 1. Numerical results for figure-of-eight vortex filament, $t = 0.0024$, $N = 8192$. The results (a) and (b) has oscillations, while the results (c) is smooth.

precision arithmetic and 50 digits computation. This implies that rounding errors do not give any influence to the oscillation in $t \leq 0.342$, including Fig. 2 (b). Therefore numerical results are reliable in terms of rounding errors.

Table 3. Maximum relative Euclidean distance in figure-of-eight vortex filament, the estimated reconnection time is $t \approx 0.3385$

t	Error in double (4)	Error in quadmath (5)
0.320	7.63×10^{-13}	4.28×10^{-31}
0.338	1.14×10^{-12}	3.65×10^{-31}
0.342	1.34×10^{-8}	9.41×10^{-27}
0.344	1.60×10^{2}	1.01×10^{-2}

In this case, most of computational time are consumed in finding tangent vectors by FFT and summation as discretization of the Biot-Savart integral. Table 4 shows them with $\Delta t = 2 \times 10^{-5}$ and $N = 8192$ for $0 \leq t \leq 0.4$ ($= 20000\Delta t$). We use FFTW3 with the `FFTW_MEASURE` flag for double and quadruple precision arithmetic [1], and the iterative algorithm [4,8] for 50 decimal digits implemented by the authors. Quadruple precision arithmetic and 50 decimal digits computation consume extraordinary longer time than those of double precision arithmetic, since their arithmetic are implemented as software libraries. However,



(a) $t = 0.320$, before reconnection (b) $t = 0.342$, after reconnection

Fig. 2. Numerical results for before and after of the reconnection in figure-of-eight vortex filament with $\Delta t = 2 \times 10^{-5}$ and double precision arithmetic. After reconnection ($t \approx 0.3385$), disturbance appears illustrated in (b).

it is effective to verify the computational results, and it means that the proposed methods are effective and practical for evaluation of accumulation of rounding errors.

4.2 Tent-Shaped Vortex Filament

We consider another example started from an initial shape

$$\begin{aligned} x(p) &= \pm c \cos\theta \cosh p, \\ y(p) &= \frac{c}{m} \sinh p, \\ z(p) &= -c \sin\theta \cosh p, \end{aligned} \tag{6}$$

with $c = 0.1$, $m = 0.35$, $\theta = \pi/4$ with $p \in \mathbb{R}$. Note that this consists of two distinct unbounded curves illustrated in Fig. 3. We allocate nodes $\{\vec{x}(p_i)\,;\, -N \leq i \leq N\}$ defined by

Table 4. Computational time for figure-of-eight vortex filament (unit : d = days, h = hours, m = minutes, s = seconds)

	Tangent	Summation	Total
double precision	47 s	24 m 11 s	25 m 09 s
quadruple precision	12 m 30 s	1 d 18 h 28 m	1 d 18 h 41 m
exflib, 50 digits	19 m 45 s	2 d 15 h 21 m	2 d 15 h 42 m

$$p_i = \sinh\left(\frac{\pi}{2}\sinh(ih)\right), \quad h = \frac{\pi}{2N}$$

with $N = 800$. Thus there are $2N+1 = 1601$ nodes on one of two curves (totally 3202 points).

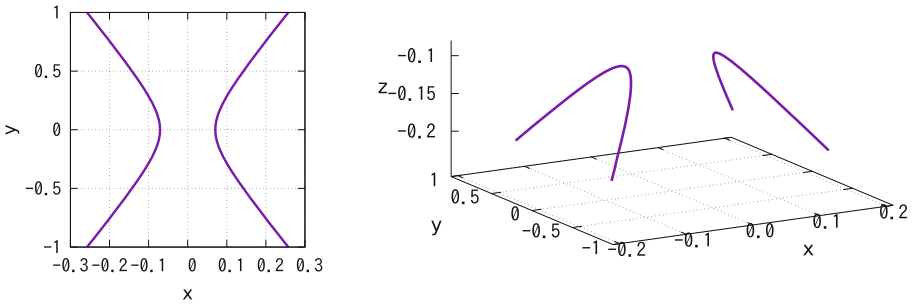


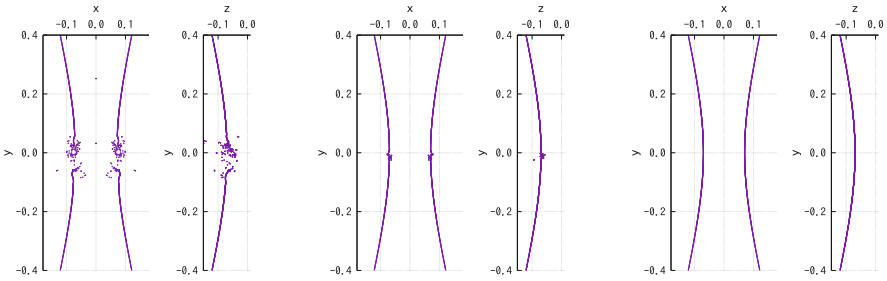
Fig. 3. Initial condition of tent-shaped vortex filament (6)

By starting from the initial nodes with $\Delta t = 4 \times 10^{-5}$, the numerical results at $t = 0.008$ $(= 200\Delta t)$ have disturbance depicted in Fig. 4 (a) and (b), where double precision arithmetic and 50 decimal digits computations are employed respectively. The results indicate that oscillation is amplification of discretization errors. On the other hand, disturbance does not appear in Fig. 4 (c) if smaller $\Delta t = 2 \times 10^{-5}$ is adopted.

The choice of $\Delta t = 2 \times 10^{-5}$ and $N = 800$ enable us to reproduce reconnection. Figure 5 (a) and (b) depict numerical results around reconnection time, which is approximately $t = 0.5058$ in the tent-shaped model.

We process the same scheme with quadruple precision arithmetic and 50 decimal digits computations. The errors between double precision arithmetic and 50 decimal digits computation (4), and those between quadruple precision arithmetic and 50 decimal digits computation (5) are shown in Table 5. The results are almost same as the figure-of-eight vortex filament example, and major ingredients of differences for $t \leq 0.5074$ are not rounding errors.

Table 6 shows the computational time for $0 \leq t \leq 0.52$ $(= 26000\Delta t)$. In this case quadruple precision arithmetic and 50 decimal digits computation spent substantially longer computational time than double precision arithmetic.



(a) $\Delta t = 4 \times 10^{-5}$, double precision ($t = 200\Delta t$)

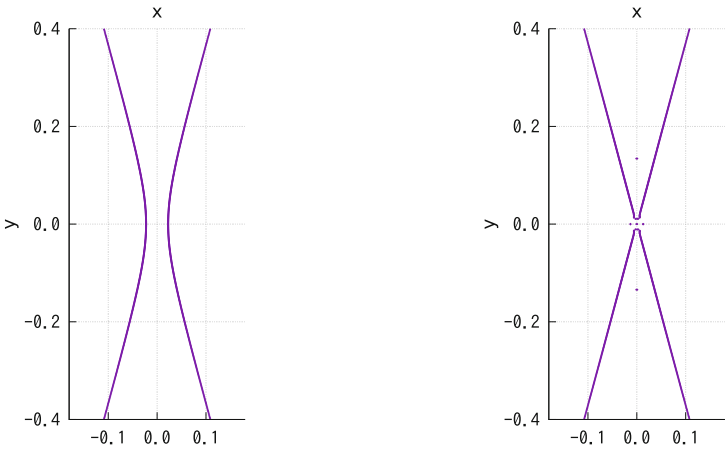
(b) $\Delta t = 4 \times 10^{-5}$, 50 digits ($t = 200\Delta t$)

(c) $\Delta t = 2 \times 10^{-5}$, double precision ($t = 400\Delta t$)

Fig. 4. Numerical results for tent-shaped vortex filament, $t = 0.008$, $N = 800$

Table 5. Maximum relative Euclidean distance in tent-shaped vortex filament, the estimated reconnection time is $t \approx 0.5058$

t	Error in double (4)	Error in quadmath (5)
0.4500	2.20×10^{-12}	7.07×10^{-31}
0.5040	2.47×10^{-12}	7.47×10^{-31}
0.5074	9.48×10^{-8}	1.33×10^{-26}
0.5100	1.21×10^{20}	5.37×10^{3}



(a) $t = 0.4500$, before reconnection

(b) $t = 0.5074$, after reconnection

Fig. 5. Numerical results for reconnection in tent-shaped vortex filament, $\Delta t = 2 \times 10^{-5}$, double precision

Table 6. Computational time for tent-shaped vortex filament (unit : h = hours, m = minutes, s = seconds)

Environment	Time
double	2 m 36 s
quadruple precision	4 h 54 m 26 s
exflib, 50 digits	7 h 58 m 19 s

5 Concluding Remarks

This paper presents an estimate strategy of rounding errors by high-accurate arithmetic. Its feasibility is exhibited in numerical computations of vortex filaments, and reliability of numerical reconnection is concluded from numerical results although the reconnection of vortex filaments is involved with occurrence of singularity of smooth solutions and is sensitive to computational errors. The proposed method gives a quantitative and concrete estimate in spite of consuming substantial amount of computational time.

For high-accurate numerical computation, we have to pay attention to discretization errors in addition to rounding errors. Our numerical experiments in Fig. 1 and Fig. 4 suggest appearance of disturbance much earlier than the reconnection, and it is reduced by taking smaller time steps. As analogous to the numerical analysis of partial differential equations such as the wave equation and the heat equation, it indicates conditions for numerical stability of the scheme. In other words, the balance of temporal and spatial discretization parameter choice is crucial for reliable numerical computations, which will be discussed in a forthcoming paper.

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