

Calculation of the Differential Geometry of the Intersection of Implicit Hypersurfaces in $\mathbb{R}^4$ with Mathematica

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Abstract. In geometry, knowing the differential properties of a curve at a point is very important, as it would help us understand its behavior around that point; even more so when said curve is generated by the intersection of hypersurfaces in $\mathbb{R}^4$. This paper describes a new Mathematica package, `Frenet4D`, that allows to visualize and calculate the Frenet frame of the curve given by the transversal intersection of three implicit hypersurfaces in $\mathbb{R}^4$ thus like the visualization of the respective hypersurfaces. The output obtained is consistent with Mathematica's notation and results. To show the performance of the package, several illustrative and interesting examples are described.

Keywords: Implicit hypersurface · Implicit–implicit–implicit intersection · Transversal intersection

1 Introduction

The implicit-implicit-implicit hypersurface intersection problem in $\mathbb{R}^4$, to calculate the differential properties of the intersection curve, has been fully developed in [6,7] based on [8] as well as in [11], although using different techniques. The results of both investigations provide explicit formulas to perform the above calculations, but all these calculations are done manually and this is a cumbersome process. Furthermore, none of these investigations shows geometric interpretations in their results.

The use of symbolic calculation systems to speed up calculations and perform geometric interpretations on research results related to various areas of mathematics is increasingly widespread [1,3–5,12]. However, we have not found any work that complements the results of the investigations mentioned in the previous paragraph. For this reason, we decided to carry out this investigation.

This paper describes a new Mathematica package `Frenet4D` that allows us to calculate the Frenet frame of the transverse intersection of three implicit hypersurfaces in $\mathbb{R}^4$, as well as the three curvatures, it also allows us to visualize the

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Frenet frame, at a given point on the curve and the hypersurfaces, demonstrate to the user a very useful tool for teaching and visualization. The results obtained are consistent with Mathematica's notation and results. The performance of the package is discussed throughout the paper using several illustrative and interesting examples.

The paper is organized as follows: Sect. 2 reviews the formulas necessary to calculate the properties of the differential geometry of a curve obtained by the intersection of three hypersurfaces implicit in $\mathbb{R}^4$. Then, Sect. 3 introduces the new Mathematica package, `Frenet4D`, and describes the commands implemented within. The performance of the package is also discussed in this section by using some illustrative examples. Finally, the conclusions are in Sect. 4.

2 Mathematical Preliminaries

Let us first introduce some notations and definitions (Many of the definitions and much of the terminology in this section may be found in [2,3,6,7,11,13]). The dot product of two vectors $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$ in $\mathbb{R}^4$ is defined by

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

the norm of the vector u is

$$\|u\| = \sqrt{u \cdot u}$$

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$, the cross product of three vectors u, v, w is defined as

$$u \times v \times w = -\begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}.$$

Some properties of cross product:

i. $u \times v \times w = v \times w \times u = w \times u \times v = -w \times v \times u = -v \times u \times w = -u \times w \times v$
ii. $u \cdot u \times v \times w = 0$
iii. If $u \times v \times w = 0$ with $u \neq 0, v \neq 0$ y $w \neq 0$ then it may be the case that $u \parallel v$, $u \parallel w$, $v \parallel w$ or $u \parallel v \parallel w$
iv. $\lambda (u \times v \times w) = (\lambda u) \times v \times w = u \times (\lambda v) \times w = u \times v \times (\lambda w)$

We will denote the differentiation of a curve with respect to the arc length s as $\alpha'(s) = \frac{d\alpha}{ds}$, $\alpha''(s) = \frac{d^2\alpha}{ds^2}$, $\alpha'''(s) = \frac{d^3\alpha}{ds^3}$ and $\alpha^{(iv)}(s) = \frac{d^4\alpha}{ds^4}$.

2.1 Regular Curve in $\mathbb{R}^4$

Let $\alpha : I \to \mathbb{R}^4$ be a regular curve parametrized by the arc length s, then from elementary differential geometry, we have

$$\begin{pmatrix} \alpha'(s) \\ \alpha''(s) \\ \alpha'''(s) \\ \alpha^{(iv)}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \kappa_1 & 0 & 0 \\ -\kappa_1^2 & \kappa'_1 & \kappa_1\kappa_2 & 0 \\ -3\kappa_1\kappa'_1 & -\kappa_1^3 + \kappa''_1 - \kappa_1\kappa_2^2 & 2\kappa'_1\kappa_2 + \kappa_1\kappa'_2 & \kappa_1\kappa_2\kappa_3 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b_1(s) \\ b_2(s) \end{pmatrix}$$

where $t(s)$ is the unit tangent vector field, $n(s)$ is the unit principal normal vector field, $b_1(s)$ is the first unit binormal vector field, $b_2(s)$ is the second unit binormal vector field of α and $k_i\,(i=1,2,3)$ are the ith curvature functions of the curve α. The Frenet frame along α is the orthonormal frame $\{t,n,b_1,b_2\}$ is given by

$$\begin{aligned} t(s) &= \alpha'(s) \\ n(s) &= \frac{\alpha''(s)}{\|\alpha''(s)\|} \\ b_2(s) &= \frac{\alpha'(s)\times\alpha''(s)\times\alpha'''(s)}{\|\alpha'(s)\times\alpha''(s)\times\alpha'''(s)\|} \\ b_1(s) &= b_2(s)\times t(s)\times n(s) \end{aligned} \tag{1}$$

the Frenet-Serret formulas along α is given by

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'_1(s) \\ b'_2(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b_1(s) \\ b_2(s) \end{pmatrix} \tag{2}$$

Using 1 and 2 it is possible to see that the curvature functions of α are given by

$$\kappa_1 = \sqrt{\alpha''\cdot\alpha''}, \kappa_2 = \frac{\alpha'''\cdot b_1}{\kappa_1} \text{ and } \kappa_3 = \frac{\alpha^{(iv)}b_2}{\kappa_1\kappa_2}.$$

2.2 Regular Implicit Hypersurface

Let $f(x,y,z,w)=0$ a regular implicit hypersurface i.e. $\nabla f \neq 0$, where $\nabla f = (f_1,f_2,f_3,f_4)$ is the gradient vector of f, $f_p = \frac{\partial f}{\partial x_p}(p=1,2,3,4)$. Then the unit normal vector field of the hypersurface f is given by $N^f = \frac{\nabla f}{\|\nabla f\|}$.

Let $\alpha(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ be a curve on the hypersurface f, then we have

$$\alpha'(s) = (x'_1, x'_2, x'_3, x'_4), \alpha''(s) = (x''_1, x''_2, x''_3, x''_4), \ldots,$$
$$\alpha^{(v)}(s) = \left(x_1^{(v)}, x_2^{(v)}, x_3^{(v)}, x_4^{(v)}\right).$$

2.3 Curves in Hypersurface

Implicit Representation of Space Curve. The implicit representation for a curve in $\mathbb{R}^4$ can be expressed as an intersection curve between three implicit surfaces

$$f(x,y,z,w)=0 \bigcap g(x,y,z,w)=0 \bigcap h(x,y,z,w)=0$$

Curves in Surface Parametrized by Arc-Length. Consider an implicit surface represented by $f : \mathbb{R}^4 \to \mathbb{R}$. A curve $\alpha(s) = (x(s), y(s), z(s), w(s))$ in which it is fulfilled $f(\alpha(s)) = 0$ defines a curve on an implicit hypersurface $f(x, y, z, w) = 0$.

3 The Package Frenet4D: Some Illustrative Examples

In this section, we show the `Frenet4D` package that will allow us to quickly and efficiently calculate the Frenet apparatus for the curve generated by the intersection of 3 implicit-implicit-implicit hypersurfaces at a point of the same; being very significant the fact that it can provide us with a graphical view in $\mathbb{R}^4$ of the hypersurfaces and the Frenet apparatus at the point given that it would be very difficult to obtain manually. In our opinion, the best way to show the power and functionalities of our package is through examples, these examples have already been discussed in [6,11] and whose answers are completely compatible with those provided by the package, so this is the approach that we will follow in this section. Firstly, we load the package:

```
<<Frenet4D.m
```

The commands incorporated in this package are:

```
Graphics4D, ImplicitPlot4D, Frenet4DForImplicit
```

The `Graphics4D` command allows you to obtain the graphics of primitives in 4D, i.e., points, lines, arrows, and polygons. The `ImplicitPlot4D` command plots the graphs of implicitly defined 4D solids. And, with the `Frenet4DForImplicit` command, the differential properties of curves given as the intersection of three implicit hypersurfaces are obtained.

Example 1. Consider the intersection of the three implicit hypersurfaces given by $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 2 = 0$, $h(x, y, z, w) = x^2 + y^2 + z^2 - w^2 = 0$ and $g(x, y, z, w) = x^2 + y^2 - x = 0$ and we calculate the Frenet frame $\{t, n, b_1, b_2\}$ along of the intersection curve α, as well as $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are the first, second and third curvetures of $\alpha(s)$ at the point P0 $= \left\{\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, 1\right\}$.

First we define the hypersurfaces f, g and h

```
f[x_, y_, z_, w_] := x^2 + y^2 + z^2 + w^2 - 2
h[x_, y_, z_, w_] := x^2 + y^2 + z^2 - w^2
g[x_, y_, z_, w_] := x^2 + y^2 - x
P0 = {1/2, 1/2, Sqrt[2]/2, 1};
```

Frenet4DForImplicit returns the Frenet frame $\{t, n, b_1, b_2\}$ along α, as well as $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are the first, second and third curvetures of $\alpha(s)$ at the point P0 $= \left\{\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, 1\right\}$.

```
{TT, NN, BB1, BB2, k1, k2, k3} = Frenet4DForImplicit[{
  f[x,y,z,w] == 0, g[x,y,z,w] == 0, h[x,y,z,w] == 0 }, {x,y,z,w},
  Point -> P0]
```

$$\left\{\left\{-\sqrt{\frac{2}{3}},0,\frac{1}{\sqrt{3}},0\right\},\left\{\frac{1}{\sqrt{39}},2\sqrt{\frac{3}{13}},\sqrt{\frac{2}{39}},0\right\},\left\{\frac{2}{\sqrt{13}},-\frac{1}{\sqrt{13}},2\sqrt{\frac{2}{13}},0\right\},\right.$$
$$\left.\{0,0,0,-1\},\frac{2\sqrt{\frac{13}{3}}}{3},-\frac{6\sqrt{2}}{13},0\right\}$$

The frenet4D variable is defined as follows:

```
frenet4D =
  Graphics4D[{  AbsolutePointSize[20], Point[P0],
   Red, Arrow[{P0, P0 + .5 TT}], Green, Arrow[{P0, P0 + .5 NN}],
   Cyan, Arrow[{P0, P0 + .5 BB2}], Blue, Arrow[{P0, P0 + .5 BB1}],
   ViewPoint -> {-0.8677623819257761, 3.0335880200886844,
   1.2224288007442063},
   ViewVertical -> {-0.12496031476425706, 0.2964760771820533,
   0.9468298978130967}  }];
```

The following sentences allow us to obtain the graph of the hypersurface $f(x,y,z,w) = x^2+y^2+z^2+w^2-2=0$.

```
S1 = ImplicitPlot4D[
   f[x, y, z, w] == 0, {x, 0.35, 0.75}, {y, 0.35, 0.75}, {z, 0.35,
    0.75}, {w, 0.75, 1.25}, Red];
Show[S1, frenet4D]
```

See Fig. 1 (top-left).

The following sentences allow us to obtain the graph of the hypersurface $g(x,y,z,w) = x^2+y^2-x=0$.

```
S2 = ImplicitPlot4D[
  g[x, y, z, w] == 0, {x, 0.35, 0.75}, {y, 0.35, 0.75}, {z, 0.35,
   0.75}, {w, 0.75, 1.25}, Green]; Show[S2, frenet4D]
```

See Fig. 1 (top-right).

The following sentences allow us to obtain the graph of the hypersurface $h(x,y,z,w) = x^2+y^2+z^2-w^2=0$.

```
S3 = ImplicitPlot4D[
  h[x, y, z, w] == 0, {x, 0.35, 0.75}, {y, 0.35, 0.75}, {z, 0.35,
   0.75}, {w, 0.75, 1.25}, Cyan]; Show[S3, frenet4D]
```

See Fig. 1 (bottom-left).

The following sentences allow us to obtain the graphs in the orthonormal frame $\{t,n,b_1,b_2\}$ at the point P0 $= \left\{\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2},1\right\}$, the red vector represents

the unit tangent vector, the green vector represents the unit normal vector, the blue vector represents the unit vector b_1 and the celestial vector represents the unit vector b_2 of the intersection curve of the hypersurfaces f , g and h .

```
Show[S1, S2, S3, frenet4D]
```

See Fig. 1 *(bottom-right).*

Example 2. Consider the intersection of the three implicit hypersurfaces given by $f(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w})=(\mathrm{y}-\mathrm{w})^3+z=0$, $h\,(x,y,z,w)=(y-w)^2-x=0$, $g\,(x,y,z,w)=y+w-1=0$. and we calculate the Frenet frame $\{t,n,b_1,b_2\}$ along of the intersection curve α, as well as $\kappa_1\,(s)$, $\kappa_2\,(s)$ and $\kappa_3\,(s)$ are the first, second and third curvetures of $\alpha\,(s)$ at the point $\mathrm{P0}=\{1,0,1,1\}$.

First we define the hypersurfaces f , g and h

```
f[x_, y_, z_, w_] := (y - w)^3 + z
h[x_, y_, z_, w_] := (y - w)^2 - x
g[x_, y_, z_, w_] := y + w - 1
P0 = {1, 0, 1, 1};
```

Frenet4DForImplicit returns the Frenet frame $\{t,n,b_1,b_2\}$ along α, as well as $\kappa_1\,(s)$, $\kappa_2\,(s)$ and $\kappa_3\,(s)$ are the first, second and third curvetures of $\alpha\,(s)$ at the point $P0=\{1,0,1,1\}$.

```
{TT, NN, BB1, BB2, k1, k2, k3} = Frenet4DForImplicit[{
  f[x,y,z,w] == 0, g[x,y,z,w] == 0, h[x,y,z,w] == 0 }, {x,y,z,w},
  Point -> P0]
```

$$\left\{\left\{\frac{2\sqrt{\frac{2}{3}}}{3},-\frac{1}{3\sqrt{6}},\sqrt{\frac{2}{3}},\frac{1}{3\sqrt{6}}\right\},\left\{-\frac{17}{6\sqrt{21}},\frac{11}{6\sqrt{21}},\frac{5}{2\sqrt{21}},-\frac{11}{6\sqrt{21}}\right\},\right.$$
$$\left.\left\{\frac{3}{2\sqrt{7}},\frac{3}{2\sqrt{7}},-\frac{1}{2\sqrt{7}},-\frac{3}{2\sqrt{7}}\right\},\left\{0,-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right\},\frac{8\sqrt{\frac{7}{3}}}{81},-\frac{6}{28\sqrt{2}},0\right\}$$

The frenet4D variable is defined as follows:

```
frenet4D =
  Graphics4D[{  AbsolutePointSize[20], Point[P0],
   Red, Arrow[{P0, P0 + .5 TT}], Green, Arrow[{P0, P0 + .5 NN}],
   Cyan, Arrow[{P0, P0 + .5 BB2}], Blue, Arrow[{P0, P0 + .5 BB1}],
   ViewPoint -> {-0.8677623819257761, 3.0335880200886844,
   1.2224288007442063},
   ViewVertical -> {-0.12496031476425706, 0.2964760771820533,
   0.9468298978130967}  }];
```

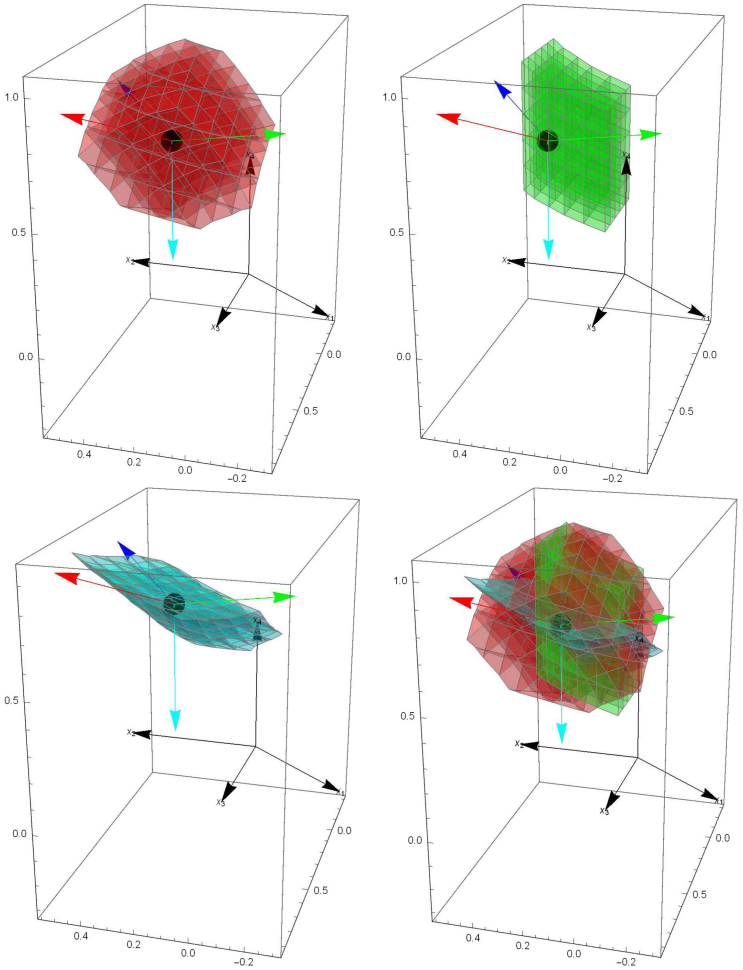

Fig. 1. Hypersurfaces $f(x, y, z, w) = 0$ (top-left), $g(x, y, z, w) = 0$ (top-right), $h(x, y, z, w) = 0$ (bottom-left) and graphical visualization of $f(x, y, z, w) = 0 \bigcap g(x, y, z, w) = 0 \bigcap h(x, y, z, w) = 0$ at the point $P0 = \left\{\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, 1\right\}$(bottom-right)

The following sentences allow us to obtain the graph of the hypersurface $f(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}) = (\mathrm{y} - \mathrm{w})^3 + z = 0$

```
S1 = ImplicitPlot4D[
   f[x, y, z, w] == 0, {x, 0.75, 1.25}, {y, -0.25, 0.35}, {z, 0.75,
    1.25}, {w, 0.75, 1.25}, Red];
Show[S1, frenet4D]
```

See Fig. 2 (top-left).

The following sentences allow us to obtain the graph of the hypersurface $g(x,y,z,w) = y + w - 1 = 0$

```
S2 = ImplicitPlot4D[
  g[x, y, z, w] == 0, {x, 0.75, 1.25}, {y, -0.25, 0.35}, {z, 0.75,
   1.25}, {w, 0.75, 1.25}, Green]; Show[S2, frenet4D]
```

See Fig. 2 (top-right).

The following sentences allow us to obtain the graph of the hypersurface $h(x,y,z,w) = (y-w)^2 - x = 0$

```
S3 = ImplicitPlot4D[
  h[x, y, z, w] == 0, {x, 0.75, 1.25}, {y, -0.25, 0.35}, {z, 0.75,
   1.25}, {w, 0.75, 1.25}, Cyan]; Show[S3, frenet4D]
```

See Fig. 2 (bottom-left).

The following sentences allow us to obtain the graphs in the orthonormal frame $\{t, n, b_1, b_2\}$ at the point $P0 = \{1, 0, 1, 1\}$, the red vector represents the unit tangent vector, the green vector represents the unit normal vector, the blue vector represents the unit vector b_1 and the celestial vector represents the unit vector b_2 of the intersection curve of the hypersurfaces f, g and h.

```
Show[S1, S2, S3, frenet4D]
```

See Fig. 2 (bottom-right).

Example 3. Consider the intersection of the three implicit hypersurfaces given by $f(x,y,z,w) = x^2 + y^2 + z^2 + w^2 - 2 = 0$, $h(x,y,z,w) = (xw + yz)^2 + (xz - yw)^2 - 1 = 0$ and $g(x,y,z,w) = 2xyz - w\left(x^2 - y^2\right) = 0$ and we calculate the Frenet frame $\{t, n, b_1, b_2\}$ along of the intersection curve α, as well as $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are the first, second and third curvetures of $\alpha(s)$ at the point $P0 = \{1, 0, 1, 0\}$.

First we define the hypersurfaces f, g and h

```
f[x_, y_, z_, w_] := x^2 + y^2 + z^2 + w^2 - 2
h[x_, y_, z_, w_] := (x w + y z)^2+(x z - y w)^2 - 1
g[x_, y_, z_, w_] := 2 x y z - w (x^2 - y^2)
P0 = {1, 0, 1, 0};
```

Frenet4DForImplicit returns the Frenet frame $\{t, n, b_1, b_2\}$ along α, as well as $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are the first, second and third curvetures of $\alpha(s)$ at the point $P0 = \{1, 0, 1, 0\}$.

```
{TT, NN, BB1, BB2, k1, k2, k3} = Frenet4DForImplicit[{
  f[x,y,z,w] == 0, g[x,y,z,w] == 0, h[x,y,z,w] == 0 }, {x,y,z,w},
  Point -> P0]
```

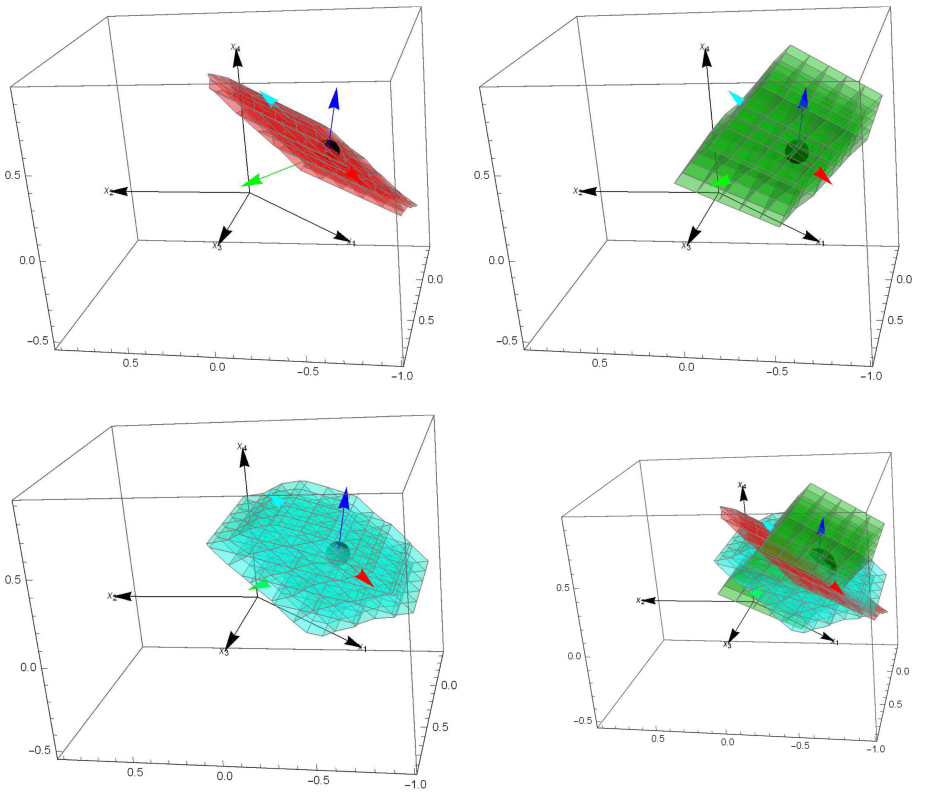

Fig. 2. Hypersurfaces $f(x, y, z, w) = 0$ (top-left), $g(x, y, z, w) = 0$ (top-right), $h(x, y, z, w) = 0$ (bottom-left) and graphical visualization of $f(x, y, z, w) = 0 \bigcap g(x, y, z, w) = 0 \bigcap h(x, y, z, w) = 0$ at the point $P0 = \{1, 0, 1, 1\}$ (bottom-right)

$$\left\{\left\{0, \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right\}, \left\{-\frac{1}{\sqrt{17}}, 0, -\frac{4}{\sqrt{17}}, 0\right\}, \left\{0, -\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right\}, \right.$$
$$\left. \left\{\frac{4}{\sqrt{17}}, 0, -\frac{1}{\sqrt{17}}, 0\right\}, \frac{\sqrt{17}}{5}, -\frac{6}{5\sqrt{17}}, \frac{2}{\sqrt{17}}\right\}$$

The frenet4D variable is defined as follows:

```
frenet4D =
  Graphics4D[{  AbsolutePointSize[20], Point[P0],
   Red, Arrow[{P0, P0 + 2.5 TT}], Green, Arrow[{P0, P0 + 2.5 NN}],
   Cyan, Arrow[{P0, P0 + 2.5 BB2}], Blue, Arrow[{P0, P0 + 2.5 BB1}],
   ViewPoint -> {-0.8677623819257761, 3.0335880200886844,
   1.2224288007442063},
   ViewVertical -> {-0.12496031476425706, 0.2964760771820533,
   0.9468298978130967}  }];
```

The following sentences allow us to obtain the graph of the hypersurface $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 2 = 0$.

```
S1 = ImplicitPlot4D[
  f[x, y, z, w] == 0, {x, 0.30, 1.2}, {y, -0.75, 0.75}, {z, 0.30,
   1.2}, {w, -0.75, 0.75}, Red];
Show[S1, frenet4D]
```

See Fig. 3 (top-left).

The following sentences allow us to obtain the graph of the hypersurface $g(x, y, z, w) = 2xyz - w\left(x^2 - y^2\right) = 0$.

```
S2 = ImplicitPlot4D[
  g[x, y, z, w] == 0, {x, 0.30, 1.2}, {y, -0.75, 0.75}, {z, 0.30,
   1.2}, {w, -0.75, 0.75}, Green]
```

See Fig. 3 (top-right).

The following sentences allow us to obtain the graph of the hypersurface $h(x, y, z, w) = (xw + yz)^2 + (xz - yw)^2 - 1 = 0$.

```
S3 = ImplicitPlot4D[
  h[x, y, z, w] == 0, {x, 0.30, 1.2}, {y, -0.75, 0.75}, {z, 0.30,
   1.2}, {w, -0.75, 0.75}, Cyan]
```

See Fig. 3 (bottom-left).

The following sentences allow us to obtain the graphs in the orthonormal frame $\{t, n, b_1, b_2\}$ at the point P0 $= \{1, 0, 1, 1\}$, the red vector represents the unit tangent vector, the green vector represents the unit normal vector, the blue vector represents the unit vector b_1 and the celestial vector represents the unit vector b_2 of the intersection curve of the hypersurfaces f , g and h.

```
Show[S1, S2, S3, frenet4D]
```

See Fig. 3 (bottom-right).

Example 4. Consider the intersection of the three implicit hypersurfaces given by $f(x, y, z, w) = x^2+y^2+z^2+w^2-1 = 0$,$h(x, y, z, w) = x^2+y^2+z^2-w^2-1 = 0$ and $g(x, y, z, w) = x^2+y^2-x = 0$ and we calculate the Frenet frame $\{t, n, b_1, b_2\}$ along of the intersection curve α, as well as $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are the first, second and third curvetures of $\alpha(s)$ at the point P0 $= \{1, 0, 0, 0\}$.

First we define the hypersurfaces f , g and h

```
f[x_, y_, z_, w_] := x^2 + y^2 + z^2 + w^2 - 1
h[x_, y_, z_, w_] := x^2 + y^2 + z^2 - w^2 - 1
g[x_, y_, z_, w_] := x^2 + y^2 - x
P0 = {1, 0, 0, 0};
```

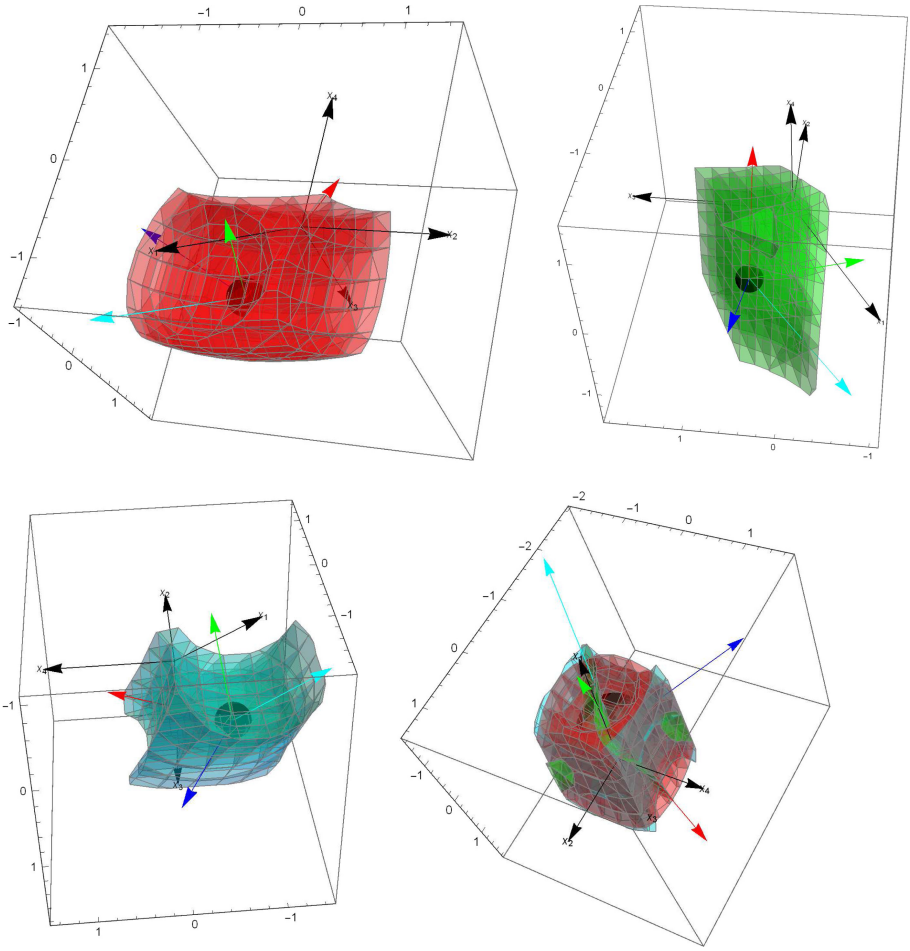

Fig. 3. Hypersurfaces $f(x,y,z,w) = 0$ (top-left), $g(x,y,z,w) = 0$ (top-right), $h(x,y,z,w) = 0$ (bottom-left) and graphical visualization of $f(x,y,z,w) = 0 \bigcap g(x,y,z,w) = 0 \bigcap h(x,y,z,w) = 0$ at the point $\mathrm{P0} = \{1,0,1,1\}$ (bottom-right)

Frenet4DForImplicit returns the Frenet frame $\{t, n, b_1, b_2\}$ along α, as well as $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are the first, second and third curvetures of $\alpha(s)$ at the point $\mathrm{P0} = \{1,0,0,0\}$.

```
{TT, NN, BB1, BB2, k1, k2, k3} = Frenet4DForImplicit[{
  f[x,y,z,w] == 0, g[x,y,z,w] == 0, h[x,y,z,w] == 0 }, {x,y,z,w},
  Point -> P0]
```

$$\left\{\left\{0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right\},\{-1,0,0,0\},\left\{0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right\},\{0,0,0,1\},1,\frac{3}{4},0\right\}$$

The frenet4D variable is defined as follows:

```
frenet4D =
  Graphics4D[{  AbsolutePointSize[20], Point[P0],
   Red, Arrow[{P0, P0 + 1.5 TT}], Green, Arrow[{P0, P0 + 2.5 NN}],
   Cyan, Arrow[{P0, P0 + 1.5 BB2}], Blue, Arrow[{P0, P0 + 1.5 BB1}],
   ViewPoint -> {-0.8677623819257761, 3.0335880200886844,
   1.2224288007442063},
   ViewVertical -> {-0.12496031476425706, 0.2964760771820533,
   0.9468298978130967}  }];
```

The following sentences allow us to obtain the graph of the hypersurface $f\,(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 1 = 0\,.$

```
S1 = ImplicitPlot4D[
  f[x, y, z, w] == 0, {x, 0.75, 1.5}, {y, -0.35, 0.75}, {z, -0.35,
   0.75}, {w, -0.35, 0.75}, Red];
Show[S1, frenet4D]
```

See Fig. 4 (top-left).

The following sentences allow us to obtain the graph of the hypersurface $g\,(x, y, z, w) = x^2 + y^2 + z^2 - w^2 - 1 = 0\,.$

```
S2 = ImplicitPlot4D[
  g[x, y, z, w] == 0, {x, 0.75, 1.5}, {y, -0.35, 0.75}, {z, -0.35,
   0.75}, {w, -0.35, 0.75}, Green]
```

See Fig. 4 (top-right).

The following sentences allow us to obtain the graph of the hypersurface $h\,(x, y, z, w) = x^2 + y^2 - x = 0\,.$

```
S3 = ImplicitPlot4D[
  h[x, y, z, w] == 0, {x, 0.75, 1.5}, {y, -0.35, 0.75}, {z, -0.35,
   0.75}, {w, -0.35, 0.75}, Cyan]
```

See Fig. 4 (bottom-left).

The following sentences allow us to obtain the graphs in the orthonormal frame $\{t, n, b_1, b_2\}$ at the point P0 $= \{1, 0, 0, 1\}$, the red vector represents the unit tangent vector, the green vector represents the unit normal vector, the blue vector represents the unit vector b_1 and the celestial vector represents the unit vector b_2 of the intersection curve of the hypersurfaces f , g and h .

```
Show[S1, S2, S3, frenet4D]
```

See Fig. 4 (bottom-right).

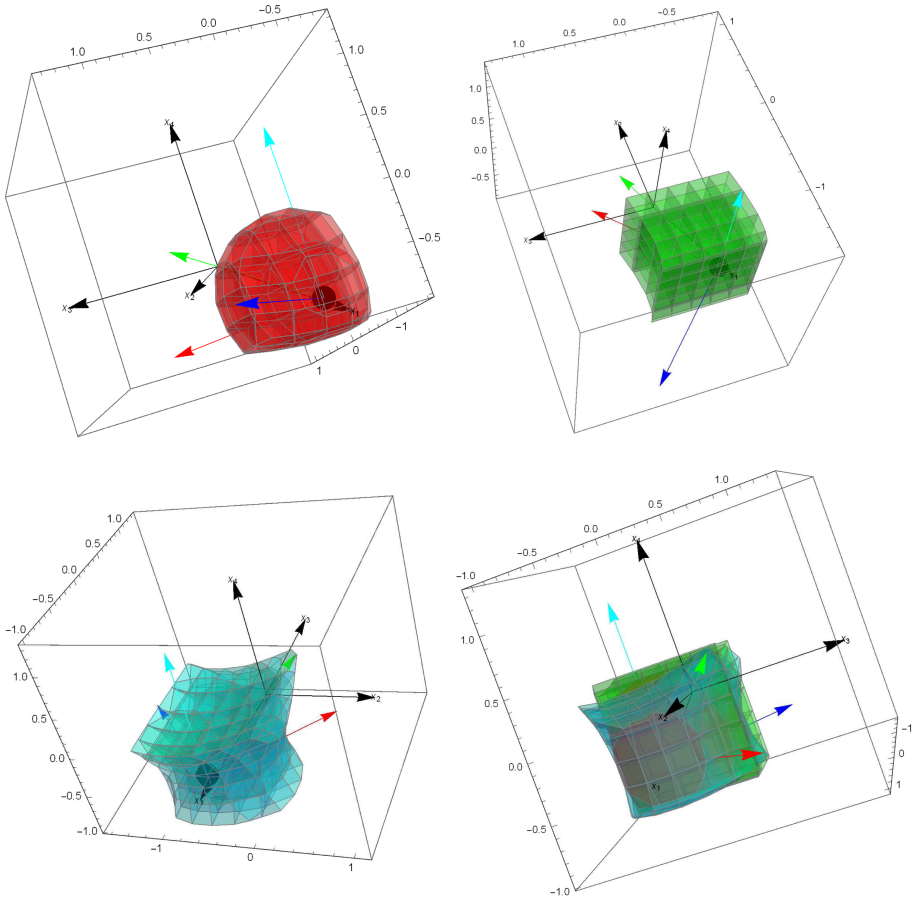

Fig. 4. Hypersurfaces $f(x, y, z, w) = 0$ (top-left), $g(x, y, z, w) = 0$ (top-right), $h(x, y, z, w) = 0$ (bottom-left) and graphical visualization of $f(x, y, z, w) = 0 \bigcap g(x, y, z, w) = 0 \bigcap h(x, y, z, w) = 0$ at the point P0 $= \{1, 0, 1, 1\}$ (bottom-right)

4 Conclusions

This paper proposes a new Mathematica package to calculate the properties of differential geometry of the curves given by the intersection between three implicit hypersurfaces in $\mathbb{R}^4$, according to the results obtained in [2,6,7,9,11,13] and as a continuation of [3–5,14] whose results coincide with those found in this work. The performance of the package is analyzed using two illustrative and interesting examples. All commands have been implemented in Mathematica and are consistent with the Mathematica notation and results [1,10,12]. The program is shorter and more efficient in my experience.

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