Construction of Polyhedra Whose Vertices are Points on Curve Which Lying on Lemniscatic Torus with Mathematica

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Abstract. Polyhedra are widely used in art, science and technology. Faced with this situation, the following research question is formulated: Can new polyhedral structures be generated from another mathematical object such as a lemniscatic torus? For this question, the results we obtained are two particular cases whose vertices are points that belong to curves that lie on a lemniscatic torus: the first, a new polyhedron that has regular trapezoids in the equatorial zone, and the second, one that has triangles equal to each other. For both polyhedra, there exists an antipodal symmetry in the Arctic and Antarctic zones. Emphasis is placed on the construction of two convex polyhedra above mentioned: a one with 18-faces and other with 36-faces, using the scientific software *Mathematica v.11.2*. We also determine their total areas which respectively approximate 9.51 R^2 and 10.44 R^2. Likewise, the volume of each one is approximately 2.41 R^3 and 4.19 R^3, respectively. Moreover, they being inscribed in a sphere of radius R, and their opposite faces are not parallel.

Keywords: Lemniscatic torus · Polyhedral · Wolfram mathematica

1 Introduction

In general, a torus of revolution is defined considering a circle of radius $b > 0$ in the plane $y = 0$, whose center lie at the point $(a, 0, 0)$ with $a > b$. If a is on the circle, then we rotate it around the z-axis. Next we generate a surface called Torus of Revolution [1]. But according to [2,5,6], lemniscatic tori are generated taking into account the parametric definition of a lemniscate, which lies on a plane parallel to the xz-plane and rotating it in the z-axis.

It is interesting to study polyhedra due to their connection in the field of science, these being the basic structures of microorganisms, forms of tissues and crystals, and architecture [4].

Unlike [8], where a dodecahedron construction was made starting from a regular pentagon, in this paper to construct the 18-faces polyhedron we start from a particular curve that lies on a lemniscate torus which has its equatorial zone in

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the xy-plane. And taking values for the argument t of the curve we obtain two hexangles, which lie in the positive and negative part of the z-axis. From this a sphere is constructed where the 18-faces polyhedron is inscribed. By means of an analogous process the 36-faces polyhedron is also constructed. In this way, for first one, we obtain a new polyhedra that has regular trapeziums in the equatorial zone, And for second one, we no longer get trapeziums mentioned above, but triangles equal to each other. For both polyhedra, there exists antipodal symmetry in the arctic and antarctic zones.

The lateral area, total area and volume are calculated to these polyhedra, making use of geometric results of areas and volumes of 3D objects already known to the student. The generation of these polyhedra will allow students to acquire spatial and programming skills, as well as a high capacity for deductive reasoning. It will also stimulate interest in seeking the connections of these new structures with the field of science [7].

The structure of this paper is as follows: Sect. 2 introduces the mathematical definitions of the lemniscate, a lemniscatic torus, and two curves that lie on such a lemniscatic torus. In addition, two different partitions are defined on the domain of the curve to produce control points that generate the polyhedra. Then, Sect. 3 introduces the construction of a new 18-faces irregular polyhedron from 12 points which are vertices of two hexangles, and to these points two suitable points are added as poles for the construction of the 18-faces polyhedron inscribed in a R-radius sphere. Here, the lateral areas, total, and the volume of this polyhedron are also calculated. Next, in Sect. 4 an irregular polyhedron with 36 faces is built inscribed in a R-radius sphere. This polyhedron is generated from 20 points of which 18 are vertices of two regular nonagons that are near the equatorial zone, and 2 are the poles. Additionally, lateral and total area, and volume of this polyhedron are calculated. Finally, Sect. 5 closes with the main conclusions of this paper.

2 Mathematical Preliminaries

In general form, we define a lemniscate in the following way:

$$Lemniscate(v, A, B) = \left(B, \frac{A\cos v}{1+\sin^2 v}, \frac{A\cos v \sin v}{1+\sin^2 v}\right), \quad 0 < v < 2\pi,$$

which rests on the plane $x = B$ and also this lemniscate is inscribed in a circle that is on the same plane $x = B$ and has a radius A. Keep in mind rotation matrix

$$rot(u) = \begin{pmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lemniscatic Torus TL [6] is defined by:

$$TL(u, v, A, B) = Rot(u) \cdot (Lemniscate(v, A, B))^t; 0 < u < 2\pi, \quad -\frac{\pi}{2} < v < \frac{\pi}{2}.$$

If we perform the matrix product, we have the following:

$$TL((u,v),(A,B)) = \left(B\cos u - \frac{A\cos v \sin u}{1+\sin^2 v}, \right.$$
$$\left. B\sin u + \frac{A\cos v\cos u}{1+\sin^2 v}, \frac{A\cos v \sin v}{1+\sin^2 v} \right).$$

In above equation of the TL we have two constants A and B in $\mathbb{R}$ and two parameters u and v. The parameters will be replaced by differentiable ordinary functions. And so we define the curves in the following way:

$$CTL((u,v),(A,B),(\alpha_1,\alpha_2),t) = TL[(u,v),(A,B)] \;/.\; \{u \to \alpha_1, v \to \alpha_2\} \quad (1)$$

Here we have infinite possibilities of obtaining curves that will be resting on the TL. To obtain these possibilities we define the Lemniscatic Torus in Mathematica, using (1):

$$TL[\{u_,v_\},\{A_,B_\}] := \left\{ BCos[u] - \frac{(ACos[v]Sin[u])}{(1+Sin[v]^2)}, \right.$$
$$\left. BSin[u] + \frac{(ACos[u]Cos[v])}{(1+Sin[v]^2)}, \frac{(ACos[v]Sin[v])}{(1+Sin[v]^2)} \right\}$$

To visualize the lemniscathic torus, we will use:

$$g1 = ParametricPlot3D[TL[\{u,v\},\{2,1\}],\{u,0,2\pi\},\{v,-Pi/2,Pi/2\},$$
$$PlotStyle \to \{Yellow, Opacity[0.75]\}]$$

As a particular case, we have the given definition of the curves in the TL in Mathematica [8]:

$$CTL[t_,\{A_,B_\},\alpha\colon\{_,_\}] := TL[\{u,v\},\{A,B\}] \;/.\; \{u \to \alpha[[1]], v \to \alpha[[2]]\}$$

Let $A = 2$ and $B = 1$, we get

$$\left\{ Cos[2t] - \frac{2Cos[3t]Sin[2t]}{1+Sin[3t]^2}, Sin[2t] + \frac{2Cos[2t]Cos[3t]}{1+Sin[3t]^2}, \frac{2Cos[3t]Sin[3t]}{1+Sin[3t]^2} \right\}$$

However, we give the following definition to improve the manipulation of the points that are immersed in the generation of polyhedra.

$$cur[t_] := CTL[t,\{2,1\},\{2t,3t\}]$$

We can observe a sketch by

$$g2 = ParametricPlot3D[Cur[t],\{t,0,2Pi\},PlotStyle \to Hue[0.75]]$$

and by $Show[g1,g2]$ we observe the representation of the curve together with the lemniscatic torus (see Fig. 1).

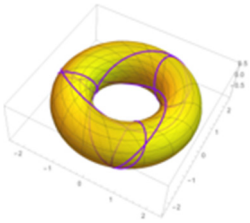


Fig. 1. Lemniscatic torus with $A = 2$, $B = 1$, and the curve $cur[t]$.

3 18-Faces Polyhedron

For domain $t \in [0, 2\pi]$ of the previous curve we will take the following values for the parameter t in the following order $t = \frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{23\pi}{12}$, and then we evaluate the curve at these points in this way:

$$\{p1, p2, p3, p4, p5, p6, p7, p8, p9, p10, p11, p12\} = Cur/@\{Pi/4, Pi/12, 5Pi/12, 5Pi/4, 19Pi/12, 17Pi/12, 7Pi/4, 7Pi/12, 11Pi/12, 3Pi/4, 13Pi/12, 23Pi/12\}$$

The following points are obtained:

$$\left\{\left\{\frac{2\sqrt{2}}{3}, 1, -\frac{2}{3}\right\}, \left\{-\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{2}, \frac{1}{2} + \sqrt{\frac{2}{3}}, \frac{2}{3}\right\}, \left\{\frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2} + \sqrt{\frac{2}{3}}, \frac{2}{3}\right\},\right.$$
$$\left\{-\frac{2\sqrt{2}}{3}, 1, -\frac{2}{3}\right\}, \left\{-\frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{2}, -\frac{1}{2} + \sqrt{\frac{2}{3}}, -\frac{2}{3}\right\}, \left\{-\frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2} - \sqrt{\frac{2}{3}}, \frac{2}{3}\right\},$$
$$\left\{-\frac{2\sqrt{2}}{3}, -1, \frac{2}{3}\right\}, \left\{\frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{2}, -\frac{1}{2} - \sqrt{\frac{2}{3}}, -\frac{2}{3}\right\}, \left\{-\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{2}, -\frac{1}{2} - \sqrt{\frac{2}{3}}, -\frac{2}{3}\right\},$$
$$\left.\left\{\frac{2\sqrt{2}}{3}, -1, \frac{2}{3}\right\}, \left\{\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{2}, \frac{1}{2} - \sqrt{\frac{2}{3}}, \frac{2}{3}\right\}, \left\{\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{2}, -\frac{1}{2} + \sqrt{\frac{2}{3}}, -\frac{2}{3}\right\}\right\}$$

Taking the length of all these points, using: $Norm/@\{p1, p2, p3, p4, p5, p6, p7, p8, p9, p10, p11, p12\}$, we obtain that the length is constant, i.e., the points lie in the lemniscatic torus and the curve, and on the sphere of radius $\sqrt{\frac{7}{3}}$. We will select the points $\{p2, p3, p6, p7, p10, p11\}$ on the northern hemisphere that form a polygon whose sides measure $-\frac{2\sqrt{2}}{3} + \sqrt{3}$ and $\frac{4\sqrt{2}}{3}$:

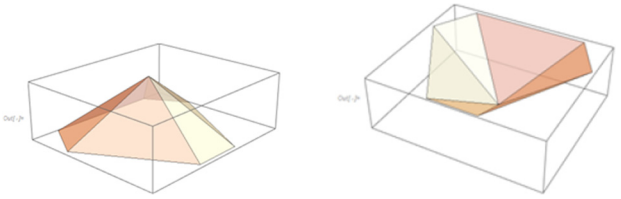
$$\left\{\left\{-\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{2}, \frac{1}{2} + \sqrt{\frac{2}{3}}, \frac{2}{3}\right\}, \left\{\frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2} + \sqrt{\frac{2}{3}}, \frac{2}{3}\right\},\right.$$
$$\left\{-\frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2} - \sqrt{\frac{2}{3}}, \frac{2}{3}\right\}, \left\{-\frac{2\sqrt{2}}{3}, -1, \frac{2}{3}\right\}, \left\{\frac{2\sqrt{2}}{3}, -1, \frac{2}{3}\right\},$$
$$\left.\left\{\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{2}, \frac{1}{2} - \sqrt{\frac{2}{3}}, \frac{2}{3}\right\}\right\}$$

and from now on we will call it hexangle. It should be understood that a hexangle is a 3-equal-interspersed-sides 6-sides-polygon.

In the same way, for the points $\{p4, p5, p8, p9, p12, p1\}$ on the southern hemisphere, namely:

$$\left\{\left\{-\frac{2\sqrt{2}}{3}, 1, -\frac{2}{3}\right\}, \left\{-\frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{2}, -\frac{1}{2} + \sqrt{\frac{2}{3}}, -\frac{2}{3}\right\},\right.$$
$$\left\{\frac{\sqrt{2}}{3} - \frac{\sqrt{3}}{2}, -\frac{1}{2} - \sqrt{\frac{2}{3}}, -\frac{2}{3}\right\}, \left\{-\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{2}, -\frac{1}{2} - \sqrt{\frac{2}{3}}, -\frac{2}{3}\right\}$$
$$\left.\left\{\frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{2}, -\frac{1}{2} + \sqrt{\frac{2}{3}}, -\frac{2}{3}\right\}, \left\{\frac{2\sqrt{2}}{3}, 1, -\frac{2}{3}\right\}\right\}$$

it follows that they form the same hexangle as the one on the northern hemisphere. The two hexangles define a zone which we will call: the equatorial zone, which consists of 12 points. We can add the north pole point $pn = \left\{0, 0, \sqrt{\frac{7}{3}}\right\}$ and the south pole point $ps = \left\{0, 0, -\sqrt{\frac{7}{3}}\right\}$ of the sphere centered at the origin. Using Mathematica, we can visualize the northern and southern hemispheres with their respective poles (see Fig. 2a and Fig. 2b).

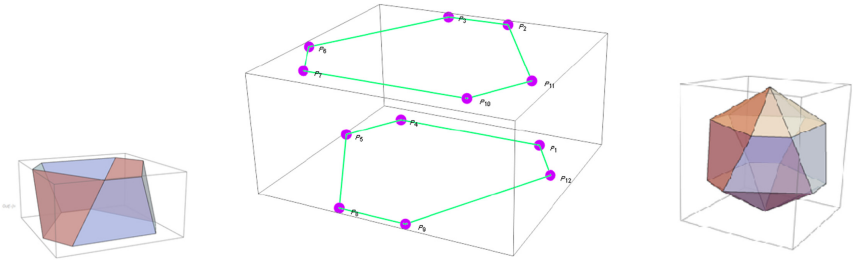


(a) Northern hemisphere, it is a pyramid

(b) Southern hemisphere, it is a pyramid

Fig. 2. Northern and southern Hemisphere

In the equatorial zone we have isosceles trapeziums, see Fig. 3a, it is also verified that the outline of the roof and basis of the figure are hexangles Fig. 3b. When we paste Fig. 2a, Fig. 2b and Fig. 3a we obtain polyhedron 1, see Fig. 3c.



(a) They are six isosceles trapezoids.

(b) From Fig.3a the roof and basis are hexangle.

(c) Sketch of the obtained polyhedron.

Fig. 3. Quadrilaterals of equatorial zone and 18-faces polyhedron.

The points of the lemniscatic torus were normalized to generalize them to polyhedra inscribed in a sphere of radius R.

$$Cur[t_] := \left\{ Cos[2t] - \frac{2Cos[3t]Sin[2t]}{1+Sin[3t]^2}, Sin[2t] + \frac{2Cos[2t]Cos[3t]}{1+Sen[3t]^2}, \frac{2Cos[3t]Sin[3t]}{1+Sin[3t]^2} \right\};$$

$NewPolyhedron[R_] := Module[\{nn = 4, n, polo1, polo2, p1, p2, p3, p4, p5, p6, p7, p8, p9, p10, p11, p12, lisnNew1, lisNew2, lisnNew3, lisNew4, lisnNew5, lisNew6, lis1, lis2, lis3\}, \{p1, p2, p3, p4, p5, p6, p7, p8, p9, p10, p11, p12\} = R$
$Normalize/@(Cur/@\{Pi/4, Pi/12, 5Pi/12, 5Pi/4, 19Pi/12, 17Pi/12, 7Pi/4, 7Pi/12, 11Pi/12, 3Pi/4, 13Pi/12, 23Pi/12\}); lis1 = \{p1, p2, p3, p4, p5, p6, p7, p8, p9, p10, p11, p12\}; lis2 = \{p2, p3, p6, p7, p10, p11\}; lis3 = \{p4, p5, p8, p9, p12, p1\}; listNew1 = Partition[Append[Append[lis1, lis1[[1]]], lis1[[2]]], nn, 2];$
$lisNew2 = Polygon/@Table[Append[lisNew1[[i]], lisNew1[[i]][[1]]], \{i, Length[lisNew1]\}]; n = length[lis2]; polo1 = \{0, 0, R\}; Polo2 = \{0, 0, -R\};$

$lisNew3 = Partition[Append[Append[lis2, lis2[[1]]], lis2[[2]]], 2, 1]; lisNew4 = Polynog/@Table[Append[Append[lisNew3[[i]], polo1], lisNew3[[i]][[1]]], \{i, n\}];$
$lisNew5 = Partition[Append[Append[lis3, lis3[[1]]], lis3[[2]]], 2, 1]; lisNew6 = Polygon/@Table[Append[Append[lisNew5[[i]], polo2], lisNew5[[i]][[1]]], \{i, n\}];$
$Join[lisNew4, lisNew2, LisNew6]]$

Let's visualize Fig. 4 when $R = 1$ and $R = 2$ using above code.

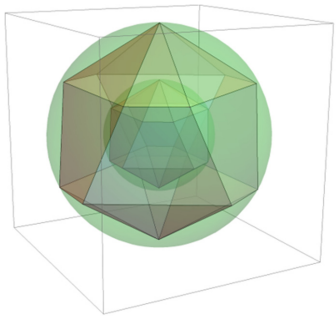


Fig. 4. There are two polyhedra, one is inscribed in a sphere with radius 1 and the other in a sphere of radius 2.

To achieve a generic result in the sense that it is a polygon inscribed in a circle of radius R, we would have to normalize the points and multiply them by R, that is, the distance from each point to the origin is equal to $R > 0$.

$$lis = \{q1, q2, q3, q4, q5, q6, q6, q7, q8, q8, q9, q10, q11, q12\} = \\ R * Normalize/@\{p1, p2, p3, p4, p5, p6, p7, p8, p9, p10, p11, p12\}$$

So we have: $Norm/@lis//Simplify = \{R, R, R, R, R, R, R, R, R, R, R, R\}$

It is clear that the north pole is $pn = \{0, 0, R\}$ and the south pole is $ps = \{0, 0, -R\}$. Now, regarding the total area, we have the equatorial zone where there are six isosceles trapeziums. We also have two pyramids, one in the northern hemisphere and the other in the southern hemisphere, both are equal to each other, except for one isometry. This can be seen in Fig. 5.

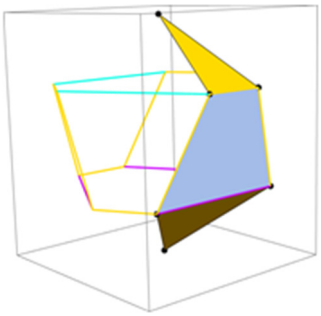


Fig. 5. Types of areas to be calculated in each region.

To find the area we need to automate the calculation of the area of an n-sided convex polygon based on the area of a triangle. For the area of the triangle:

$$AreaTri[P1:\{_,_,_\},P2:\{_,_,_\},P3:\{_,_,_\}\}] :=$$
$$\frac{1}{2}Norm[Cross[P2-P1,P3-P1]]$$

Area of the convex polygon: $AreaPolyConvex[P:\{_,_,_\}..\}] := Module[\{A\}, A = AreaTri/@table[\{P[[1]],P[[i]],P[[i+1]]\},\{i,2,Length[P]-1\}];Suma[A[[i]],\{i,Length[A]\}]\}]$

With respect to the trapeziums observed in Fig. 5, there are 6, and therefore the area would be: $a1 = 6AreaPolyConvex[\{q1,q2,q3,q4\}]//Simplify$

$$= \frac{1}{14}\left(\sqrt{4259-1584\sqrt{6}}+4\sqrt{194-24\sqrt{6}}\right)R^2$$

Regarding to the congruent triangles of the northern and southern hemispheres identified by p_2, pn, p_3 there are six, see Fig. 5:

$$a2 = 6AreaTri[\{q2,\{0,0,R\},q3\}] = \frac{1}{14}\sqrt{3791-1176\sqrt{6}+576\sqrt{14}-560\sqrt{21}}R^2$$

Regarding to the congruent triangles of the southern and northern hemispheres identified by p_4, ps, p_5 there are six, see Fig. 5:

$$a3 = 6AreaTri[\{q4,\{0,0,-R\},q1\}]//Simplify = \frac{8}{7}\sqrt{17-2\sqrt{21}}R^2$$

Therefore, the total area of polyhedron 1 is:

$$a1+a2+a3//Simplify \approx 9.50799R^2$$

To calculate the volume we will carry out the following procedure: We first calculate the volume of the pyramid in the northern hemisphere and multiply it by 2, since the pyramid in the southern hemisphere is congruent with the northern one.

Let's calculate the center of gravity of the hexangle in the northern hemisphere:

$$cg = (q2+q3+q6+q7+q10+q11)/6//Simplify = \left\{0,0,\frac{2R}{\sqrt{21}}\right\}$$

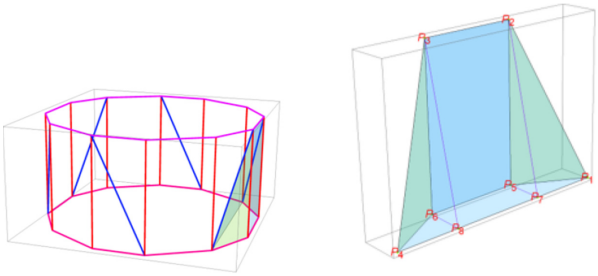
Thus, we have that the height of the pyramid is obtained from:

$$\{0,0,R\}-cg = \left\{0,0,R-\frac{2R}{\sqrt{21}}\right\}$$

Therefore, the volume of the two pyramids is:

$$Vpira = \frac{2}{3}AreaPolyConvex[\{q2,q3,q7,q10,q11\}]\left(R-\frac{2R}{\sqrt{21}}\right)//Simplify$$
$$-\frac{(-21+2\sqrt{21})(19+8\sqrt{70-24\sqrt{6}}+4\sqrt{70+24\sqrt{6}})R^3}{882\sqrt{3}}$$

Second, we calculate the volume of the equatorial zone, which is inscribed within a prism that has dodecangle bases, and the height is 2 times the distance from the origin to the center of gravity of the northern hemisphere hexangle: $h = \frac{4R}{\sqrt{21}}$. See Fig. 6a.



(a) Polyhedron in equatorial zone inscribed in a polyhedron whose bases are dodecangle.

(b) Polyhedral volume to removen.

Fig. 6. Calculating the volume of the Polyhedron in the equatorial zone inscribed in a polyhedron with dodecangle bases.

In Fig. 6a is shown that the part of the equatorial zone of polyhedron 1 is inscribed in a prism whose basis is a dodecangle. We define a dodecangle as a convex polygon with twelve angles and sides, with the characteristic that its internal angles are equal and its non-adjacent sides are equal.

Now, the volume of polyhedron 1 is the volume of the prism whose basis is the dodecangle, whose vertices are shown by *sup*, plus that of the two pyramids minus six times the volume of polyhedron, which is shown in Fig. 6b. The area of the basis is calculated based on the points:

$$sup = Table\left[Append\left[Delete\left[lis[[i]], -1\right], \frac{2}{\sqrt{21}}\right], \{i, 12\}\right] //Simplify$$

$$\left\{\left\{2\sqrt{\frac{2}{21}}R, \sqrt{\frac{3}{7}}R, \frac{2R}{\sqrt{21}}\right\}, \left\{\frac{\left(9-2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(3+2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\}\right.$$
$$\left\{\frac{\left(-9+2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(3+2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\}, \left\{-2\sqrt{\frac{2}{21}}R, \sqrt{\frac{3}{7}}R, \frac{2R}{\sqrt{21}}\right\},$$
$$\left\{-\frac{\left(9+2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(-3+2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\}, \left\{-\frac{\left(9+2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(3-2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\},$$

$$\left\{-2\sqrt{\frac{2}{21}}R, -\sqrt{\frac{3}{7}}R, \frac{2R}{\sqrt{21}}\right\}, \left\{\frac{\left(-9+2\sqrt{6}\right)R}{6\sqrt{7}}, -\frac{\left(3+2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\},$$
$$\left\{\frac{\left(9-2\sqrt{6}\right)R}{6\sqrt{7}}, -\frac{\left(3+2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\}, \left\{2\sqrt{\frac{2}{21}}R, -\sqrt{\frac{3}{7}}R, \frac{2R}{\sqrt{21}}\right\},$$
$$\left\{\frac{\left(9+2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(3-2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\}, \left\{\frac{\left(9+2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(-3+2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\}\Bigg\}$$

So, the volume is:

$$Vdodec = (AreaPolyConvex[sup]Norma[P3 - P6])//Simplify = \frac{16}{7}\sqrt{\frac{6}{7}}R^3$$

We calculate the volume of the polyhedron in Fig. 6b, considering 8 labels for the points that will be needed to calculate the volume, where:

$$\{P1, P2, P3, P4, P5, P6\} =$$
$$\left\{\left\{2\sqrt{\frac{2}{21}}R, \sqrt{\frac{3}{7}}R, -\frac{2R}{\sqrt{21}}\right\}, \left\{\frac{\left(9-2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(3+2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\},\right.$$
$$\left\{\frac{\left(-9+2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(3+2\sqrt{6}\right)R}{2\sqrt{21}}, \frac{2R}{\sqrt{21}}\right\}, \left\{-2\sqrt{\frac{2}{21}}R, \sqrt{\frac{3}{7}}R, -\frac{2R}{\sqrt{21}}\right\},$$
$$\left.\left\{\frac{\left(9-2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(3+2\sqrt{6}\right)R}{2\sqrt{21}}, -\frac{2R}{\sqrt{21}}\right\}, \left\{\frac{\left(-9+2\sqrt{6}\right)R}{6\sqrt{7}}, \frac{\left(3+2\sqrt{6}\right)R}{2\sqrt{21}}, -\frac{2R}{\sqrt{21}}\right\}\right\}.$$

Now, if we consider orthogonal projection of P_5, P_6 on P_4, P_1 we have two points P_8, P_7 which are obtained by $\|P_4 - P_8\| = \|P_7 - P_1\| = x$, $b = \|P_2 - P_3\|$ (minor basis) and $B = \|P_1 - P_4\|$ (major basis) of trapezium $P_1P_5P_6P_4$. $b = Norm[P2 - P3]/Simplify$; $B = Norm[P1 - P4]//Simplify$; $Solve[B == 2x + b, x]$

We take the value of $x = \frac{1}{42}\left(-9\sqrt{7}R + 2\sqrt{42}R + 3\sqrt{77}R\right)$, and we obtain the point $P7 = P1 + x\ (P4 - P1)/Norm[P4 - P1] = \left\{\frac{\left(9+2\sqrt{6}-3\sqrt{11}\right)R}{6\sqrt{7}}, \sqrt{\frac{3}{7}}R, -\frac{2R}{\sqrt{21}}\right\}$ Similarly we obtain the point P_8.

$$P8 = P1 + (x + b)(P4 - P1)/Norm[P4 - P1]//Simplify$$
$$\left\{\frac{(-3+2\sqrt{6}-\sqrt{11})R}{2\sqrt{7}}, \sqrt{\frac{3}{7}}R, -\frac{2R}{\sqrt{21}}\right\}$$

Now we calculate the volume of the polyhedron $P_5, P_6, P_8, P_7, P_2, P_3$ which is half of a prism:

$$Vprisma = \frac{1}{2}\left(AreaPolConvex\left[\{P5, P6, P8, P7\}\right] Norm\left[P3 - P6\right]\right) \approx 0.047R^3$$

Because of tetrahedra volumes $P_4P_8P_6 - P_3$ and $P_1P_5P_7 - P_2$ are congruent, we have:

$$V2 = 2\left(\frac{1}{3}AreaPolyConvex\left[\{P4, P8, P6\}\right]Norm\left[P3 - P6\right]\right) \approx 0.0210578R^3$$

Thus, we have the volume of Fig. 6b:

$$Vprisma + V2//Simplify = -\frac{\left(2 - 8\sqrt{6} - 3\sqrt{11} + 2\sqrt{66}\right)}{63\sqrt{7}}R^3$$

Therefore, the volume of polyhedron 1 is:

$$Vpoliedro1 = Vdodec + Vpira - 6\left(Vprisma + V2\right)//FullSiimplify$$
$$\frac{1}{294}\left(252\sqrt{2} + 7\sqrt{3} + 6\sqrt{7} + 40\sqrt{42} + 4\sqrt{77}\left(-3 + 2\sqrt{6}\right)\right)R^3$$

4 36-Faces Polyhedron

We define the curve from the lemniscatic torus when $A = 8$, $B = 5$ and $\alpha_1(t) = 5t$, $\alpha_2(t) = 7$ with Mathematica.

$$Cur1[t] := CTL[t, \{8, 5\}, \{t, 9t\}]$$
$$Cur1[t]$$
$$\left\{5Cos[t] - \frac{8Cos[9t]Sin[t]}{1 + Sin[9t]^2}, 5Sin[t] + \frac{8Cos[t]Cos[9t]}{1 + Sin[9t]^2}, \frac{8Cos[9t]Sin[9t]}{1 + Sin[9t]^2}\right\}$$

Now, we vary the parameter t in the order given:

$$t = \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \tfrac{17\pi}{36}, \tfrac{35\pi}{36}, \tfrac{25\pi}{36}, \tfrac{43\pi}{36}, \tfrac{11\pi}{12}, \tfrac{17\pi}{12}, \tfrac{41\pi}{36}, \tfrac{59\pi}{36}, \tfrac{49\pi}{36}, \tfrac{67\pi}{36}, \tfrac{19\pi}{12},$$
$$\tfrac{\pi}{12}, \tfrac{65\pi}{36}, \tfrac{11\pi}{36}, \tfrac{\pi}{36}, \tfrac{19\pi}{36}$$

$$m1 = \{r1, r2, r3, r4, r5, r6, r7, r8, r9, r10, r11, r12, r13, r14, r15, r16, r17, r18\} =$$
$$Normalize/@Cur1/@\{\pi/4, 3\pi/4, 17\pi/36, 35\pi/36, 25\pi/36, 43\pi/36, 11\pi/12, 17\pi/12,$$
$$41\pi/36, 59\pi/36, 49\pi/36, 67\pi/36, 19\pi/12, \pi/12, 65\pi/36, 11\pi/36,$$
$$\pi/36, 19\pi/36\}//Simplify;$$

Taking the lists

$$r2, r4, r6, r8, r10, r12, r14, r16, r18 \text{ and } r1, r3, r5, r7, r9, r11, r13, r15, r17$$

two regular nonagons are obtained which will define the equatorial zone together with 18 congruent triangles, as shown in the Fig. 7.

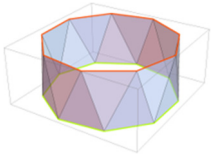


Fig. 7. Regular nonagons in the equatorial zone.

To form the zone of the southern hemisphere we will have to add $\{0, 0, -1\}$:

$$ps = \{\{r2, r4, r6, r8, r10, r12, r14, r16, r18\}, \{0, 0 - 1\}\}$$

To form the zone of the northern hemisphere we will have to add $\{0, 0, 1\}$:

$$pn = \{\{r1, r3, r5, r7, r9, r11, r13, r15, r17\}, \{0, 0, 1\}\}$$

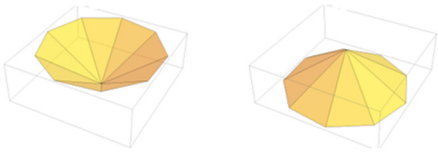


Fig. 8. Constructions in the north pole and south pole area.

Uniting the equatorial zone together with the zone of the southern and northern hemisphere (see Fig. 8) we have the polyhedron and the polyhedron inscribed in a unit sphere with three points $r2, r1, r3$, as shown in Fig. 9. The three points are used to find the area of the equatorial zone.

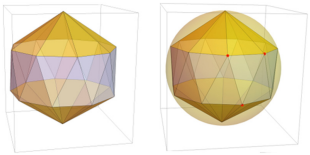


Fig. 9. Polyhedron 2 and polyhedron 2 with 3 points to determine the area located in the equatorial zone.

The one with 18 similar triangles between the northern and southern hemispheres also has 18 congruent triangles in the equatorial zone, but they are different from those in the southern and northern hemispheres, so this polyhedron 2 has 36 faces, 54 edges and 20 vertices of which 18 are on the lemniscatic

torus and 2 in the sphere of unit radius. In addition Euler's formula is satisfaced: $36 - 54 + 20 = 2$. Now, let's calculate the total area of polyhedron 2: First, we calculate the area of a triangle in the southern hemisphere and multiply it by 18. To do this we take $\{r2, r4, \{0, 0, -1\}\}$ and the south pole.

$$azhnzhs = 18 * AreaPolyConvex[\{r2, r4, \{0, 0, -1\}\}]//N \approx 5.99$$

Equatorial zone area: $aze = 18 * AreaPolyConvex[\{r2, r1, r3\}]//N \approx 4.45$

So the total area is: $azhnzhs + aze//N \approx 10.44$

To calculate the volume of the polyhedron 2: First we calculate the volume of the pyramid of the southern hemisphere, whose vertices are:$\{r2, r4, r6, r8, r10, r12, r14, r16, r18\}$, and the height $h1$ is: $h1 = 1 - \frac{h}{2} = 1 - \frac{8}{\sqrt{417}}$. So the volume of the pyramid in the southern hemisphere is:

$$Vpisur = \frac{1}{3} AreaPolyConvex\left[\{r2, r4, r6, r8, r10, r12, r14, r16, r18\}\right]$$
$$h1//Simplify \approx 0.361364$$

The pyramid in the northern hemisphere has the same value as the volume of the pyramid in the southern hemisphere.

Second, we construct two 18-sided polygons, at the top and bottom of the equatorial zone, with the following lists:

$$polsup = Table\left[Append\left[Delete\left[m1\left[[i]\right], -1\right], \frac{8}{\sqrt{417}}\right], \{i, 1, 18\}\right];$$
$$polinf = Table\left[Append\left[Delete\left[m1\left[[i]\right], -1\right], -\frac{8}{\sqrt{417}}\right], \{i, 1, 18\}\right];$$

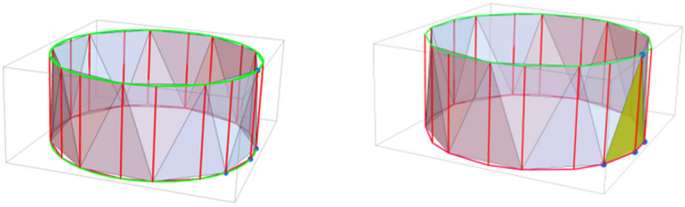


Fig. 10. 18-sided polygons, top to bottom of the equatorial zone, with the *polsup* and *polinf* lists.

In Fig. 10 it is shown that the part of the equatorial zone of polyhedron 2 is inscribed in a prism whose bases are regular nonagon.

Furthermore, we project rays, all parallel to z-axis and we will choose a tetrahedron as shown in the Fig. 10 (right). The four points involved are $polinf[[1]]$,

$polinf[[2]]$, $polinf[[18]]$ for the basis and $polsup[[1]]$ for the top vertex, the height of the tetrahedron is:

$$h = Norm\left[polinf\left[[2]\right] - polsup\left[[2]\right]\right] //Simplify$$

Thus, we have that the volume of the tetrahedron in question is:

$$Vt = \left(\frac{1}{3}\right) AreaPolyConvex\left[\{polinf\left[[1]\right], polinf\left[[2]\right], polinf\left[[18]\right]\}\right]$$
$$h//Simplify \approx 0.00437139$$

Now, we calculate the volume of the prism, whose basis is $polinf$ or $polsup$ and the height is the same as that of the tetrahedron, said volume is:

$$Vpri = AreaPolyConvex[polinf]h//N \approx 2.04$$

Therefore, the volume of the equatorial zone is:

$$Vzecua = Vpri - 18Vt//N \approx 1.96$$

Consequently, we get that the volume of polyhedron 2 is:

$$Vpoliedro2 = 2 * Vpisur + Vzecua - 18Vt//FullSimplify = \frac{4}{3}\pi \approx 4.19$$

5 Conclusions

This article illustrates an approach based on the use of numerical and symbolic programming of the scientific software Mathematica v.11.2 [1]. This approach is used for the construction and analysis of the particular characteristics of the two convex polyhedra, the 18-faces one and the 36-faces one. We started from the equation of the parametrically defined lemniscatic torus. If we substitute the parameters for differentiable ordinary functions, there are infinite possibilities of curves that lie on the lemniscatic torus. Particular cases were taken in such a way that when evaluating the curves, we obtained points that were taken as the vertices of such polyhedra, except for the vertices that coincide with the poles of the sphere that inscribes them.

The construction of the polyhedra under study exhibits interesting characteristics in its construction. In the 18-faces polyhedron, on the faces of the equatorial zone, two roof and basis hexangles were found, whose internal angles are equal and their non-adyacent sides are equal. Likewise, it was observed that the same region is inscribed in a prism whose basis is a dodecangle with equal internal angles and equal non-adjacent sides. Similarly, the 36-faces polyhedron was studied. On the sides of the equatorial zone, a regular nonagon was found as a roof and basis. The interesting thing about this region is that it is inscribed in a prism whose basis is an octadecangle with the characteristic that its internal angles are equal and its non-adjacent sides are equal.

The total area of the polyhedron of 18 and 36 faces was calculated, which is approximately 9.51 R^2 and 10.44 R^2, respectively. Also, the volume is 2.41 R^3 and 4.19 R^3, respectively. In addition, it was found that they satisfy Leonhard Euler's theorem, which indicates the relationship between the number of faces, edges and vertices of a convex polyhedron: $C - R + V = 2$.

The generation of these polyhedra will allow students to acquire spatial and programming skills, as well as a high capacity for deductive reasoning. It will also stimulate interest in seeking the connections of these new structures with the field of science.

Future research includes the extension of the study of the isometries that leave invariant convex irregular polyhedra with 18 and 32 faces. Likewise, the construction of new polyhedra generated from the lemniscatic torus and possible ranges of application in different fields such as morphogeometry can also be carried out.

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