

Asymptotic Quadrature Based Numerical Integration of Stochastic Damped Oscillators

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Abstract. This paper is focused on the numerical solution of stochastic harmonic damped oscillators, characterized by a nonlinear high-oscillating time-varying forcing term, coupled with a random force driven by an additive Wiener noise. The scheme is based on asymptotic quadratures together with the application of the variation of constants formula. Theoretical issues are provided and the numerical evidence on selected popular related physical models is also given.

Keywords: Stochastic differential equations · Stochastic oscillators · Periodic time varying force · Asymptotic quadrature

1 Introduction

Mathematical modelling of oscillating phenomena plays a crucial role in the scientific literature, both for deterministic and stochastic scenarios. Usually, stochastic oscillators are obtained by introducing noisy components into existing deterministic oscillatory models as described, for instance, in [1–5,17–20,27,33] and references therein.

In this work, we are interested in oscillators described a second order differential equation of the form:

$$\ddot{x} = -\omega^2 x + \eta\dot{x} + g_\sigma(t) + \varepsilon\xi(t), \tag{1}$$

where $\xi(t)$ is a white noise process satisfying $\mathbb{E}|\xi(t)\xi(t')| = \delta(t-t')$, being $\delta(\cdot)$ the Dirac delta function. The parameters ω and η are real constants, representing the frequency and the damping of the oscillations. Equations of the form (1) typically model stochastic harmonic damped oscillators driven by both a deterministic time-dependent force and a random Gaussian forcing term. In particular, the high oscillations in the model are determined by the deterministic forcing term

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$g_\sigma(t)$ via the parameter σ and, in general, not by the high frequencies, as in [25] for the deterministic setting.

Stochastic oscillators (1) are very popular in literature as a relevant physical model (see, for instance, [19,20,27] and references therein), therefore a robust numerics to approximate their solutions is a crucial topic to be properly addressed.

The existing numerical literature offers a focus on several aspects regarding the discretizations of stochastic oscillators; we refer, for instance, to [3,4,6–8,11–13,15,16,28,29,31,32] and references therein. Here we present a numerical strategy suited for problem (1), combining the variation of constants formula and a quadrature rule based on the asympotic expansion [23,24]. The idea consists in proposing an appropriate extension to the approach of [26] and [25], in a stochastic setting, for a problem of the form (1). In particular, in [26], the author introduces methods for the efficient numerical approximation of linear and nonlinear systems of highly oscillatory ordinary differential equations and shows how an appropriate choice of quadrature rule improves the accuracy of approximation as the frequencies of the oscillations grow. The variation of constants formula, as a starting point to design numerical methods suited for oscillatory problems, is also very employed in the deterministic setting, see [21,22] and reference therein.

The aforementioned discretization of problem (1) is described in Sect. 2, where we also recall the essential aspects of quadrature formulas based on asymptotic expansions. In Sect. 3, we provide the numerical evidence, confirming the superiority of the proposed method in comparison to a standard solver.

2 Description of the Numerical Scheme

Equation (1) is equivalent to the following first order system of two equations in the variables X_t (the position of the oscillating particle) and V_t (its velocity):

$$\begin{cases} dX_t = V_t dt, \\ dV_t = -\omega^2 X_t dt + \eta V_t dt + g_\sigma(t) dt + \varepsilon dW_t. \end{cases} \tag{2}$$

We are particularly interested in the case

$$g_\sigma(t) = a(t) e^{i\sigma h(t)},$$

where $g'(t) \neq 0$, for $t \geq 0$ (see [25]). By setting the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & \eta \end{bmatrix},$$

the variation of constants formula applied to the system (2) is given by

$$\begin{pmatrix} X_t \\ V_t \end{pmatrix} = e^{tA} \begin{pmatrix} X_0 \\ V_0 \end{pmatrix} + \int_0^t e^{sA} \mathbf{e} \;\; g_\sigma(s) ds + \varepsilon \int_0^t e^{sA} \mathbf{e} \; dW_s, \tag{3}$$

where $\mathbf{e} = [0 \;\; 1]^T$.

Moreover, setting $\mathbf{y}_{n+1} = \begin{pmatrix} X_{n+1} \\ V_{n+1} \end{pmatrix}$, a discretization in the stochastic part reads as

$$\mathbf{y}_{n+1} = e^{hA}\mathbf{y}_n + e^{hA}\int_0^h e^{-sA}\mathbf{e}\ g_\sigma(s)ds + \varepsilon e^{hA}\ \mathbf{e}\ \Delta W_n, \tag{4}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$ denotes the Wiener increment. Clearly, in order to obtain a full discretization to (2), we must approximate the integral involving the deterministic force $g_\sigma(t)$.

2.1 Discretization Based on Asymptotic Expansions

Following [25], we recall that an asymptotic expansion of the integral

$$I = \int_0^t e^{-sA} g_\sigma(s)ds \tag{5}$$

is readily available, in fact,

$$I = \int_0^t e^{-sA}a(s)e^{i\omega h(s)}ds \sim -\sum_{r=0}^{\infty} \frac{1}{(-i\omega)^{r+1}}\left[x_r(t)\frac{e^{ih(t)}}{h'(t)} - x_r(t)\frac{e^{ih(0)}}{h'(0)}\right],$$

where

$$x_0(t) = e^{-tA}a(t) \quad x_{r+1} = \frac{d}{dt}\frac{x_r(t)}{g'(t)}, \quad r \in \mathbb{Z}.$$

Note that the functions x_r are non-oscillatory. By considering [24] (in particular Lemma 2.1), a truncation of the series to the s-th term

$$Q_A^s = -\sum_{r=0}^{s} \frac{1}{(-i\omega)^{r+1}}\left[x_r(t)\frac{e^{ih(t)}}{h'(t)} - x_r(t)\frac{e^{ih(0)}}{h'(0)}\right] \tag{6}$$

provides an efficient approximation to the integral (5) if ω is a large number.

The numerical strategy we propose is based on the semi-discretization (4), where the integral (5) is approximated by employing formula (6) with $s = 2$.

Finally, the following result (see [24]) analyzes the accuracy of the quadrature formula (6), establishing its asymptotic order of convergence.

Theorem 1. *For every smooth h and a it is true that*

$$I - Q_A^s \sim \mathcal{O}(\omega^{-s-1}) \quad \omega \longrightarrow \infty.$$

3 Numerical Experiments

Motivated by [1,19,25], we propose the following system

$$\begin{cases} dX_t = V_t dt, \\ dV_t = -\omega^2 X_t dt + \eta V_t dt + e^{i\sigma h(t)} dt + \varepsilon dW_t, \end{cases} \tag{7}$$

where $\varepsilon = 0.3$. We set $\omega^2 = 50$, $\eta = -10$, and the initial conditions $X_0 = 0.5$, $V_0 = 0.5$. We test the method introduced in Sect. 2, considering the cases $h(t) \equiv 1$ and $h(t) = -t^2 + t$.

In Fig. 1, we show the behaviour of the method when $h(t) = -t^2 + t$ and $\sigma = 80$. Moreover, we observe the behaviour of the mean-square error in the position of the numerical solution, choosing $h(t) \equiv 1$, with $\sigma = 70$ (Fig. 2) and $\sigma = 150$ (Fig. 3). Furthermore, we propose a comparison between our approach and the classical Stochastic Exponential Euler method [30].

We can see in all the provided experiments that the Stochastic Exponential Euler method has problem of convergence, as visible in Tables 1, 2 and 3, beacuse it is not able to accurately catch the high oscillations of the solution, as our approach is, evidently, able to do. Indeed, as visibles from Figs. 1, 2 and 3 and the aforementioned tables, the introduced method is clearly convergent, also for high values of the frequencies.

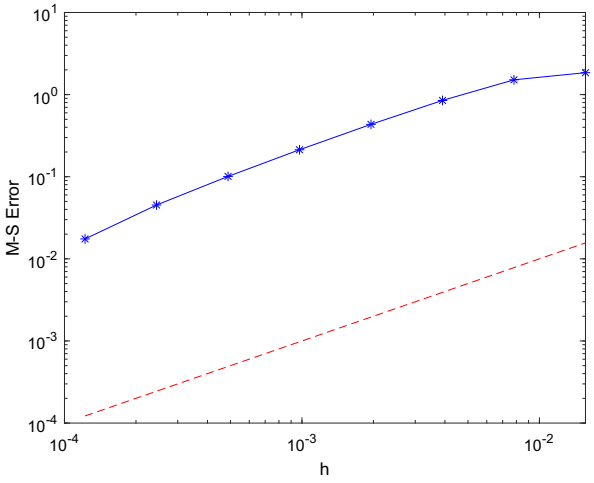

Fig. 1. Mean-square error in the position of the numerical solution to problem (7) with $h(t) = -t^2 + t$ and $\sigma = 80$, computed by the asymptotic quadrature method introduced in Sect. 2. The pattern of the mean-square error is plotted vs the employed stepsize, in log-log scale.

Table 1. Observed mean-square orders of convergence of the stochastic exponential Euler method and the asymptotic quadrature method introduced in Sect. 2 and the stochastic exponential Euler method applied to problem (7) with $h(t) = -t^2 + t$ and $\sigma = 80$.

h	Exponential euler	Asymptotic quadrature
0.0001	–	–
0.0002	0.0107	0.0570
0.0005	0.0088	0.1117
0.0010	0.0103	0.2255
0.0020	0.0108	0.4452
0.0039	0.0103	0.8620
0.0078	0.0105	1.5243
0.0156	0.0102	1.8475

Table 2. Observed mean-square orders of convergence of the stochastic exponential Euler method and the asymptotic quadrature method introduced in Sect. 2 and the applied to problem (7) with $f(t) \equiv 1$ and $\sigma = 70$.

h	Exponential euler	Asymptotic quadrature
0.0005	–	–
0.0010	0.0245	0.0199
0.0020	0.0232	0.0387
0.0039	0.0233	0.0757
0.0078	0.0230	0.1493
0.0156	0.0222	0.2879
0.0313	0.0247	0.4933
0.0625	0.0226	0.4559

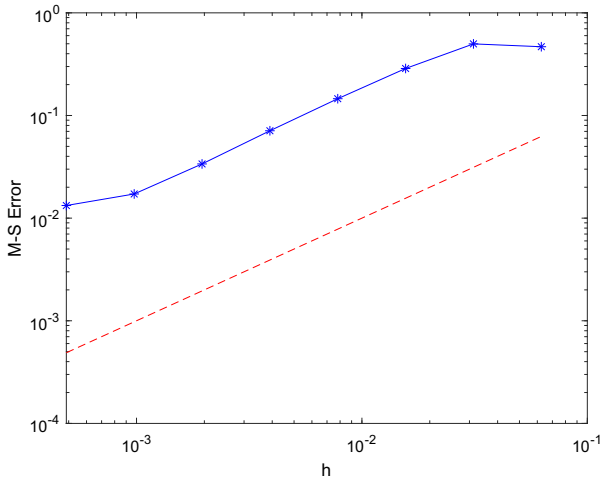

Fig. 2. Mean-square error in the position of the numerical solution to problem (7) with $f(t) \equiv 1$ and $\sigma = 70$, computed by the asymptotic quadrature method introduced in Sect. 2. The pattern of the mean-square error is plotted vs the employed stepsize, in log-log scale.

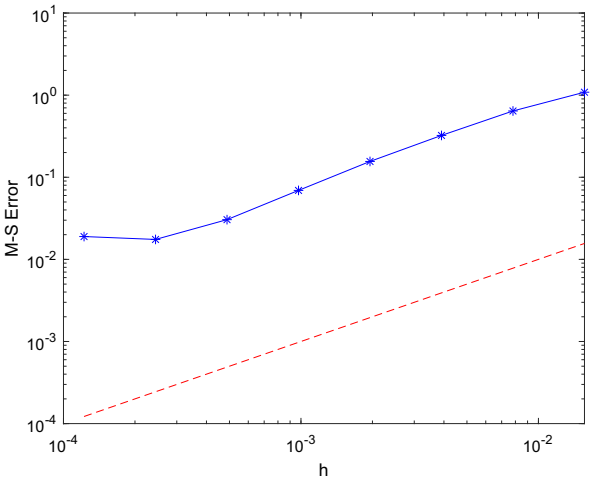

Fig. 3. Mean-square error in the position of the numerical solution to problem (7) with $f(t) \equiv 1$ and $\sigma = 150$, computed by the asymptotic quadrature method introduced in Sect. 2. The pattern of the mean-square error is plotted vs the employed stepsize, in log-log scale.

Table 3. Observed mean-square orders of convergence of the stochastic exponential Euler method and the asymptotic quadrature method introduced in Sect. 2 and the stochastic exponential Euler method applied to problem (7) with $f(t) \equiv 1$ and $\sigma = 150$.

h	Exponential euler	Asymptotic quadrature
0.0001	–	–
0.0002	0.0107	0.0317
0.0005	0.0112	0.0504
0.0010	0.0103	0.0946
0.0020	0.0108	0.0183
0.0039	0.0110	0.3531
0.0078	0.0107	0.6746
0.0156	0.0104	1.1184

4 Conclusions

In this paper, we have combined two typical ingredients of the numerics for deterministic differential problems, i.e., the variation of constants formula and the asymtpotic expansion quadrature, to define a numerical strategy for second order stochastic equation of type (1). The effectiveness of the approach, also briefly described by the provided numerical tests, is also particularly relevant in comparison with existing schemes, such as the classical exponential Euler method for stochastic differential equations. Clearly, such strategy remains tied to the

type of system considered. In this work, we enforce the idea to properly extend, when possibile, efficient numerical strategies for deterministic ODEs to the SDEs setting; such an idea inspired other works in different stochastic settings, as [14] for the development of a theory of stochastic two-step Runge-Kutta methods or [9,10] for the assessment of a theory of nonlinear stability for stochastic numerical schemes.

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