A Parameterization of the Klein Bottle by Isometric Transformations in $\mathbb{R}^4$ with Mathematica

Ricardo Velezmoro-León[1], Nestor Javier Farias-Morcillo[1], Robert Ipanaqué-Chero[1(✉)], José Manuel Estela-Vilela[1], and Judith Keren Jiménez-Vilcherrez[2]

[1] Universidad Nacional de Piura, Urb. Miraflores s/n, Castilla, Piura, Peru
{rvelezmorol,nfariasm,ripanaquec}@unp.edu.pe
[2] Universidad Tecnológica del Perú, Av. Vice Cdra 1-Costado Real Plaza, Piura, Peru
C19863@utp.edu.pe

Abstract. The Klein bottle plays a crucial role in the main modern sciences. This surface was first described in 1882 by the German mathematician Felix Klein. In this paper we describe a technique to obtain the parameterization of the Klein bottle. This technique uses isometric transformations (translations and rotations) and the moving frame associated with the unit circumference lying on the xy-plane. The process we follow is to start with the parametrization of the Euclidean cylinder, then continue with the parameterization of the Möbius strip, after that with the parameterization of the torus of revolution and finally, in a natural way, we describe the aforementioned technique. With the parameterization of the Klien bottle obtained, it is easy to show that it can be obtained by gluing two Möbius strips. Additionally, the parameterizations of the n-twisted and n-turns Klein bottles are obtained. All geometric calculations and geometric interpretations are performed with the Mathematica symbolic calculus system.

Keywords: Parameterization · Klein Bottle · Isometric transformations in $\mathbb{R}^4$ · Moving frame

1 Introduction

The Klein bottle plays a crucial role in the main modern sciences [3,4,12,13,15]. This surface was first described in 1882 by the German mathematician Felix Klein. It may have been originally named the Kleinsche Fläche ("Klein surface") and then misinterpreted as Kleinsche Flasche ("Klein bottle"), which ultimately may have led to the adoption of this term in the German language as well [2]. There are parameterizations in R4 of this surface, the two most popular are: non-intersecting 4D parametrization (modeled from the flat torus) and the so-called 4D Möbius tube [17,18].

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In this paper we describe a technique to obtain the parameterization of the Klein bottle. This technique uses isometric transformations such as translations and rotations [11]. In addition, it makes use of the moving frame associated with the unit circumference that lies in the xy-plane [7,11]. We have chosen to follow a process that begins with the description of a technique to parameterize the Euclidean cylinder in $\mathbb{R}^3$, continues with the description of the same technique to parameterize the Möbius strip in $\mathbb{R}^3$, continues with the description of another technique to parameterize the torus of revolution in $\mathbb{R}^4$ and ends, inductively, with the description of the technique to parameterize the Klein bottle in $\mathbb{R}^4$. After obtaining the parameterization of the Klein bottle, it became easy to separate it into two parts, both parts being a separate Möbius strip. In addition, the technique was useful to find the parameterization of the n-twisted and n-turns Klein bottles are obtained [5,9]. All geometric calculations and geometric interpretations are performed with the Mathematica v.11.0 symbolic calculus system [19].

The structure of this paper is as follows: Sect. 2 introduces the basic concepts related to surfaces and quotient spaces, as well as the parametrization of the immersion of the Klein bottle in three-dimensional space. Then, Sect. 3 introduces the technique to parameterize the Klein bottle by means of isometric transformations. Here also the parameterizations of the n-twisted and n-fold Klein bottles are obtained. Finally, Sect. 4 closes with the main conclusions of this paper.

2 Basic Concepts

2.1 Surfaces and Quotient Spaces

Definition 1. *A surface or 2-dimensional manifold is a topological space with the same local properties as the familiar plane of Euclidean geometry [10].*

Definition 2. *Let X be a topological space and ρ be an equivalence relation on X. If $p : X \to X/\rho$, $x \mapsto [x]$ is the natural surjective map, then the collection Ω of all subsets $U \subset X/\rho$ such that $p^{-1}(U)$ is an open set of X, forms the largest topology on X/ρ such that p is continuous. The set X/ρ is called a quotient space of X, with the quotient topology Ω and p is called an identification map [1].*

Example 1. Let $I \times I$ be the unit square.

a. The Euclidean cylinder is the quotient space obtained from unit square $I \times I$ by identifying $(0, t)$ with $(1, t)$, for all $t \in I$ (see Fig. 1).
b. The torus of revolution is the quotient space obtained from unit square $I \times I$ by identifying $(t, 0)$ with $(t, 1)$ and also $(0, t)$ with $(1, t)$, for all $t \in I$ (see Fig. 1).
c. The Möbius strip is the quotient space obtained from unit square $I \times I$ by identifying $(0, t)$ with $(1, 1 - t)$, for all $t \in I$ (see Fig. 1).

d. The Klein bottle is the quotient space obtained from unit square $I \times I$ by identifying $(0, t)$ with $(1, t)$ and also $(t, 0)$ with $(1, 1 - t)$, for all $t \in I$. Since this identification can only be properly represented in $\mathbb{R}^4$, a representation of it is usually made in $\mathbb{R}^3$, which is called immersion of the Klein bottle in $\mathbb{R}^3$ (see Fig. 2).

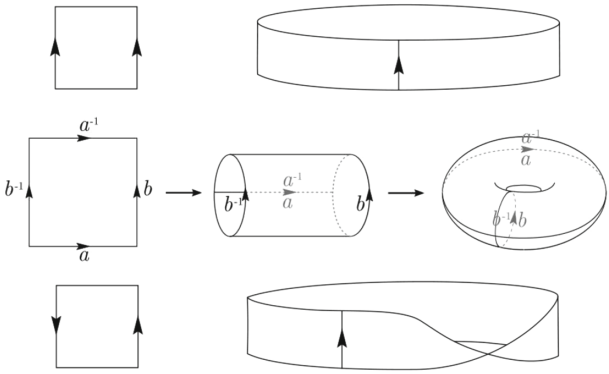

Fig. 1. Cylinder (top), Torus (middle) and Möbius strip (bottom) as the quotient space of unit square.

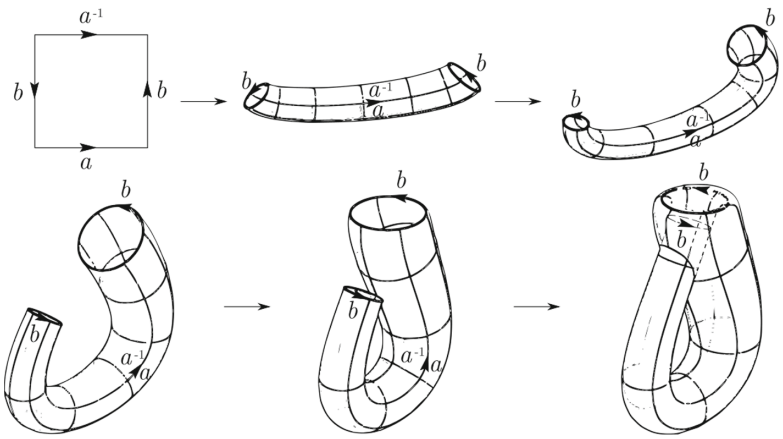

Fig. 2. Klein bottle, immersed in $\mathbb{R}^3$, as the quotient space of unit square.

2.2 Parameterizations of the Immersion of the Klein Bottle in $\mathbb{R}^3$ as a Tubular Surface

Let $\alpha(t) = (\alpha_1(t), \alpha_2(t))$, $t \in [a, b]$, be a curve lying on the xy-plane satisfying $\|\alpha'(t)\| \neq 0$. Let $\mathbf{k} = (0, 0, 1)$ be the z-axis unit vector and $\mathbf{T} = \frac{\alpha'}{\|\alpha'\|}$ be the unit tangent vector field of $\alpha(t)$. Let $\mathbf{N} = \mathbf{k} \times \mathbf{T}$. Then the couple of unit vectors $(\mathbf{N}, \mathbf{k})$ is a moving frame orthogonal to $\alpha'(t)$ and can be used to construct a tube around $\alpha(t)$ as follows:

$$tube(u, v) = \alpha(u) + r(u)\,(\cos v\mathbf{N} + \sin v\mathbf{k})\ , \quad (u, v) \in [a, b] \times [0, 2\pi]\,,$$

where the scalar continuous function $\mathbf{r}(t)$ gives the radius of the tube [8].

M. Trott's [14] proposal is to define:

$$\alpha(t) = \left(\frac{1}{t^4 + 1}, \frac{t^2 + t + 1}{t4 + 1}\right), \quad t \in (-\infty, +\infty)\,,$$

$$r(t) = \frac{84t^4 + 56t^3 + 21t^2 + 21t + 24}{672(1 + t^4)}$$

and the resulting image is shown in Fig. 3.

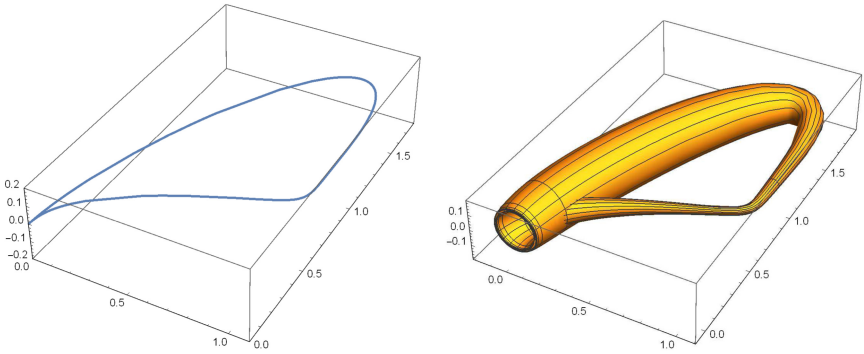

Fig. 3. Central curve α (left) and Klein bottle according to M. Trott's definition (right).

Whereas, G. Franzoni [8] proposes the following definition:

$$\alpha(t) = (a(1 - \cos t), b \sin t(1 - \cos t))\,, \quad t \in [0, 2\pi]\,,$$

$$r(t) = c - d(t - \pi)\sqrt{t(2\pi - t)}\,.$$

and the resulting image, with $(a, b, c, d) = \left(20, 8, \frac{11}{2}, \frac{2}{5}\right)$, is shown in Fig. 4.

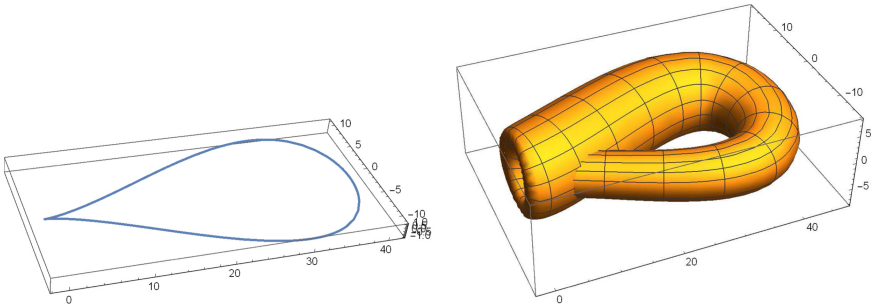

Fig. 4. Central curve α (left) and Klein bottle according to G. Franzoni's definition (right).

3 Parameterizations of Some Surfaces by Isometric Transformations

In this section, two isometric transformations are used: translations and rotations, as well as a moving frame to parameterize the Euclidean cylinder, the Möbius strip, the torus of revolution, and the Klein bottle. The first two in $\mathbb{R}^3$ and the last two in $\mathbb{R}^4$. Although the first three parameterizations are widely known, they are mentioned as a preamble to the fourth parameterization that we propose in this paper. To trace the surfaces construction process in $\mathbb{R}^4$ a trimetric-trimetric model according to [16] is used. This is,

$$O = \{0,0,0\}, \quad \hat{\mathcal{B}} = \left\{ \tfrac{3}{5\sqrt{3}}\,(-1,-1,-1)\,, (1,0,0), (0,1,0), (0,0,1) \right\} \quad \text{and}$$

$$\varphi(\mathbf{p}) = \left(p_2 - \frac{3p_1}{5\sqrt{3}},\ p_3 - \frac{3p_1}{5\sqrt{3}},\ p_4 - \frac{3p_1}{5\sqrt{3}} \right).$$

3.1 Parameterization of the Euclidian Cylinder in $\mathbb{R}^3$

Let $\beta(t) = (\cos t, \sin t, 0)$, $t \in [0, 2\pi]$, be the unit circumference lying on the xy-plane. Let $\mathbf{k} = (0,0,1)$ be the z-axis unit vector and $\mathbf{T} = \beta'(t) = (-\sin t, \cos t, 0)$ be the unit tangent vector field of $\beta(t)$. Let $\mathbf{N} = \mathbf{k} \times \mathbf{T}$. Let $\delta(s) = \left(0, \frac{s}{2}\right)$, $s \in [-1,1]$, be a segment lying on the normal plane associated with β; that is, the plane generated by $\mathbf{N}$ and $\mathbf{k}$. Then the couple of unit vectors $(\mathbf{N}, \mathbf{k})$ is a moving frame orthogonal to $\beta'(t)$ and can be used to construct a Euclidean cylinder around $\beta(t)$ as follows:

$$cyl(u,v) = \beta(u) + 0 \cdot \mathbf{N}(u) + \frac{v}{2}\mathbf{k}\,, \quad (u,v) \in [0,2\pi] \times [-1,1]\,.$$

This is

$$cyl(u,v) = \left(\cos(u), \sin(u), \frac{v}{2}\right), \quad (u,v) \in [0,2\pi] \times [-1,1] \tag{1}$$

and some resulting images are shown in Fig. 5.

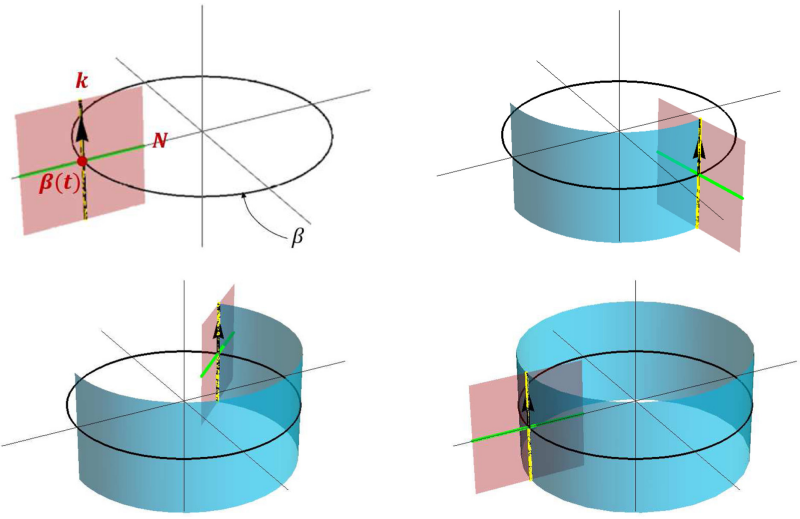

Fig. 5. Circumference β and the moving frame $(\mathbf{N}, \mathbf{k})$ (top-left) and Euclidean cylinder construction process according to Eq. 1 definition (top-right,bottom-left,bottom-right).

3.2 Parameterization of the Möbius strip in $\mathbb{R}^3$

Let $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be the rotation matrix around a point in plane. Let $\beta(t) = (\cos t, \sin t, 0)$, $t \in [0, 2\pi]$, be the unit circumference lying on the xy-plane. Let $\mathbf{k} = (0, 0, 1)$ be the z-axis unit vector and $\mathbf{T} = \beta'(t) = (-\sin t, \cos t, 0)$ be the unit tangent vector field of $\beta(t)$. Let $\mathbf{N} = \mathbf{k} \times \mathbf{T}$. Let $\delta(s) = \left(\frac{s}{2}, 0\right)$, $s \in [-1, 1]$, be a segment lying on the normal plane associated with β; that is, the plane generated by $\mathbf{N}$ and $\mathbf{k}$. Then the couple of unit vectors $(\mathbf{N}, \mathbf{k})$ is a moving frame orthogonal to $\beta'(t)$ and can be used to construct a Möbius strip around $\beta(t)$ as follows:

$$mob(u, v) = \beta(u) + \phi_1(u, v)\mathbf{N}(u) + \phi_2(u, v)\mathbf{k}\,, \quad (u, v) \in [0, 2\pi] \times [-1, 1]\,,$$

where $\phi(u, v) = R\left(\frac{u}{2}\right) \bullet (\delta(v))^\top$. This is

$$mob(u, v) = \left(\cos(u) - \frac{1}{2}v\cos\left(\frac{u}{2}\right)\cos(u), \sin(u) - \frac{1}{2}v\cos\left(\frac{u}{2}\right)\sin(u), \right.$$
$$\left. \frac{1}{2}v\sin\left(\frac{u}{2}\right)\right)\,, \quad (u, v) \in [0, 2\pi] \times [-1, 1] \quad (2)$$

and some resulting images are shown in Fig. 6.

Also, if we do:

$$\phi(u, v) = R\left(\frac{nu}{2}\right) \bullet (\delta(v))^\top\,, \quad n = 1, 2, \ldots$$

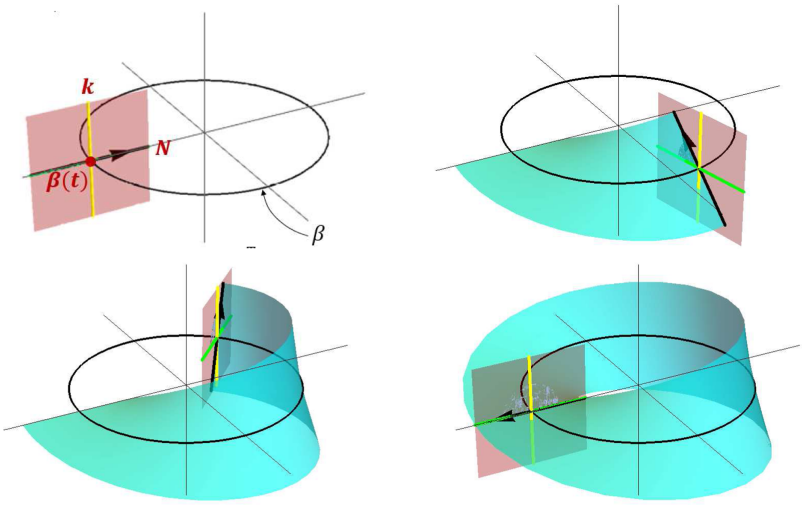

Fig. 6. Circumference β and the moving frame $(\mathbf{N}, \mathbf{k})$ (top-left) and Möbius strip construction process according to Eq. 2 definition (top-right,bottom-left,bottom-right).

and we consider $(u, v) \in [0, 2\pi] \times [0, 2\pi]$, we will obtain n-twisted Möbius strips (see Fig. 7).

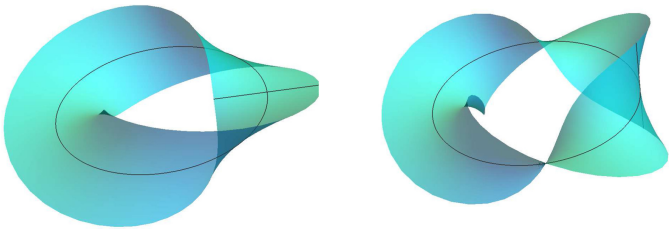

Fig. 7. 2-twisted Möbius strip (left) and 3-twisted Möbius strip (right).

Similarly, if we do:

$$\phi(u, v) = R\left(\frac{u}{2n}\right) \bullet (\delta(v))^{\top}, \quad n = 1, 2, \ldots$$

and we consider $(u, v) \in [0, 2\pi] \times [0, 2n\pi]$, we will obtain n-fold Möbius strips (see Fig. 8).

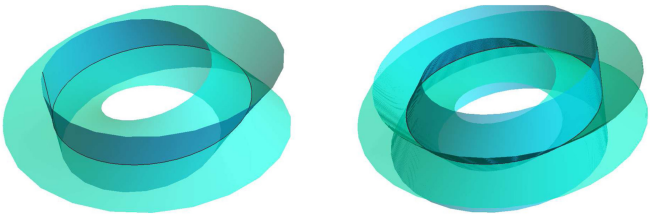

Fig. 8. 2-fold Möbius strip (left) and 3-fold Möbius strip (right).

3.3 Parameterization of the Torus of Revolution in $\mathbb{R}^4$

Let $\beta(t) = (\cos t, \sin t, 0, 0)$, $t \in [0, 2\pi]$, be the unit circumference lying on the xy-plane. Let $\mathbf{k}_1 = (0, 0, 1, 0)$ and $\mathbf{k}_2 = (0, 0, 0, 1)$ be the unit vectors in the z-axis and w-axis directions, respectively, and $\mathbf{T} = \beta'(t) = (-\sin t, \cos t, 0, 0)$ be the unit tangent vector field of $\beta(t)$. Let $\mathbf{N} = \mathbf{k}_1 \times \mathbf{k}_2 \times \mathbf{T}$. Let $\delta(s) = \frac{1}{2}(\cos s, \sin s, 0)$, $s \in [0, 2\pi]$, be a segment lying on the normal plane associated with β; that is, the 3D-space generated by $\mathbf{N}$, $\mathbf{k}_1$ and $\mathbf{k}_2$. Then the triad of unit vectors $(\mathbf{N}, \mathbf{k}_1, \mathbf{k}_2)$ is a moving frame orthogonal to $\beta'(t)$ and can be used to construct a torus of revolution around $\beta(t)$ as follows:

$$tor(u, v) = \beta(u) + \frac{1}{2}\cos v\mathbf{N}(u) + \frac{1}{2}\sin u\mathbf{k}_1 + 0 \cdot \mathbf{k}_2\,, \quad (u, v) \in [0, 2\pi] \times [0, 2\pi]\,.$$

This is

$$tor(u, v) = \left(\frac{1}{2}\cos(u)\cos(v) + \cos(u), \frac{1}{2}\sin(u)\cos(v) + \sin(u), \frac{1}{2}\sin(v), 0\right), \quad (u, v) \in [0, 2\pi] \times [-1, 1] \quad (3)$$

and some resulting image are shown in Fig. 9.

3.4 Parameterization of the Klein Bottle in $\mathbb{R}^4$

Let $R(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$ be the rotation matrix around the y-axis. Let $\beta(t) = (\cos t, \sin t, 0, 0)$, $t \in [0, 2\pi]$, be the unit circumference lying on the xy-plane. Let $\mathbf{k}_1 = (0, 0, 1, 0)$ and $\mathbf{k}_2 = (0, 0, 0, 1)$ be the unit vectors in the z-axis and w-axis directions, respectively, and $\mathbf{T} = \beta'(t) = (-\sin t, \cos t, 0, 0)$ be the unit tangent vector field of $\beta(t)$. Let $\mathbf{N} = \mathbf{k}_1 \times \mathbf{k}_2 \times \mathbf{T}$. Let $\delta(s) = \frac{1}{2}(\cos s, \sin s, 0)$, $s \in [0, 2\pi]$, be a segment lying on the normal plane associated with β; that is, the 3D-space generated by $\mathbf{N}$, $\mathbf{k}_1$ and $\mathbf{k}_2$. Then the triad of unit vectors $(\mathbf{N}, \mathbf{k}_1, \mathbf{k}_2)$ is a moving frame orthogonal to $\beta'(t)$ and can be used to construct a Klein bottle around $\beta(t)$ as follows:

$$kle(u, v) = \beta(u) + \phi_1(u, v)\mathbf{N}(u) + \phi_2(u, v)\mathbf{k}_1 + \phi_3(u, v)\mathbf{k}_2\,,\ (u, v) \in [0, 2\pi] \times [0, 2\pi]\,,$$

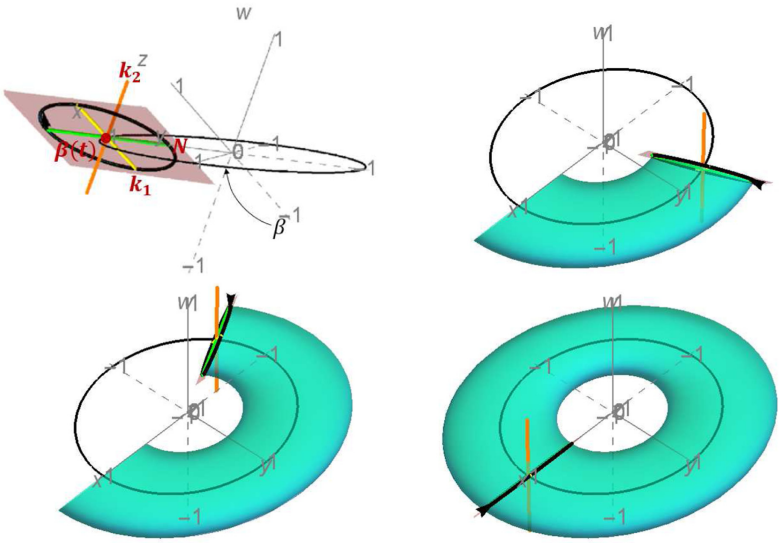

Fig. 9. Circumference β and the moving frame $(\mathbf{N}, \mathbf{k}_1, \mathbf{k}_2)$ (top-left) and torus of revolution construction process according to Eq. 3 definition (top-right, bottom-left, bottom-right). Note that the δ circumference does not leave the plane generated by $\mathbf{N}$ and $\mathbf{k}_1$.

where $\phi(u,v) = R\left(\frac{u}{2}\right) \bullet (\delta(v))^{\top}$. This is

$$kle(u,v) = \left(\frac{1}{2}\cos\left(\frac{u}{2}\right)\cos(u)\cos(v) + \cos(u), \frac{1}{2}\cos\left(\frac{u}{2}\right)\sin(u)\cos(v) + \sin(u), \right.$$
$$\left. \frac{1}{2}\sin(v), -\frac{1}{2}\sin\left(\frac{u}{2}\right)\cos(v)\right), \quad (u,v) \in [0,2\pi] \times [0,2\pi] \qquad (4)$$

and some resulting image are shown in Fig. 10.

It is known that the Klein bottle is the result of gluing two Möbius strips together along their boundary circles [6]. If we plot

$$1 + kle(u,v), \quad (u,v) \in [0,2\pi] \times [0,\pi]$$

and

$$kle(u,v), \quad (u,v) \in [0,2\pi] \times [\pi,2\pi]$$

we will obtain two appropriately separated Möbius strips to visualize this result (see Fig. 11).

Also, if we do:

$$\phi(u,v) = R\left(\frac{nu}{2}\right) \bullet (\delta(v))^{\top}, \quad n = 1,2,\ldots$$

and we consider $(u,v) \in [0,2\pi] \times [0,2\pi]$, we will obtain n-twisted Klein bottles (see Fig. 12).

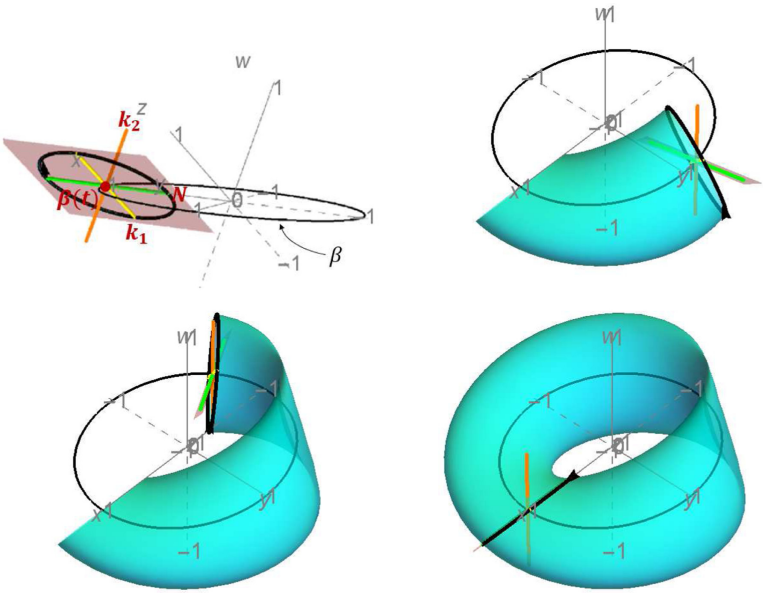

Fig. 10. Circumference β and the moving frame $(\mathbf{N}, \mathbf{k}_1, \mathbf{k}_2)$ (top-left) and torus of revolution construction process according to Eq. 4 definition (top-right,bottom-left,bottom-right). Note that the δ circumference leaves the plane generated by $\mathbf{N}$ and $\mathbf{k}_1$ to effect the rotational motion around the axis generated by $\mathbf{k}_1$.

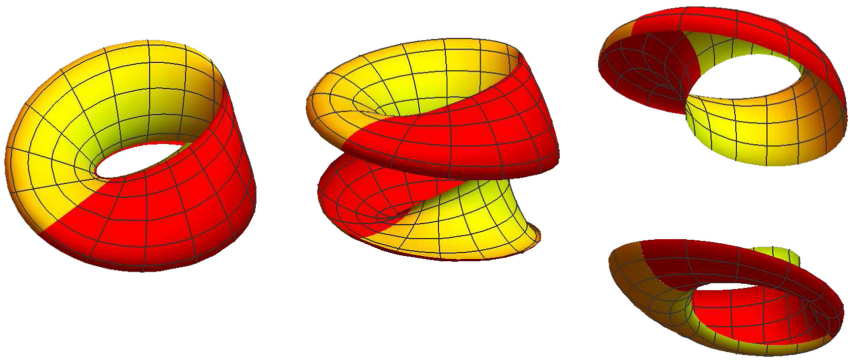

Fig. 11. The Klein bottle as the result of gluing two Möbius strips together along their boundary circles.

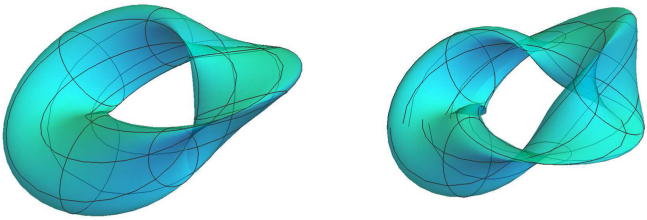

Fig. 12. 2-twisted Klein bottle (left) and 3-twisted Klein bottle (right).

Similarly, if we do:

$$\phi(u,v) = R\left(\frac{u}{2n}\right) \bullet (\delta(v))^{\top}, \quad n = 1, 2, \ldots$$

and we consider $(u, v) \in [0, 2\pi] \times [0, 2n\pi]$, we will obtain n-fold Klein bottles (see Fig. 13).

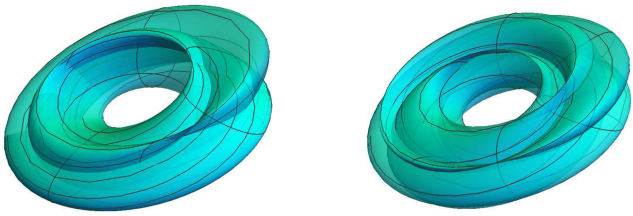

Fig. 13. 2-fold Klein bottle (left) and 3-fold Klein bottle (right).

4 Conclusions

In this paper a technique to obtain the parameterization of the Klein bottle has been described. This technique has used isometric transformations and the moving frame associated with the unit circle lying on the xy-plane. The process followed was to start with the parameterization of the Euclidean cylinder, then continued with the parameterization of the Möbius band, then with the parameterization of the torus of revolution and finally, in a natural way, the aforementioned technique was described. With the parameterization of the Klien bottle obtained, it was easy to show that it can be obtained by gluing two Möbius strips. Additionally, the parameterizations of the n-twisted and n-fold Klein bottles were obtained. All geometric calculations and geometric interpretations were made with the Mathematica symbolic calculation system.

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