

The Mirror Reflection Principle and Probabilistic Invariants in the Theory of Multiple Photon Scattering

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Abstract. In the framework of the statistical interpretation of multiple photon scattering as a stationary Markov process, which was given in well-known works by V.V. Sobolev and S. Ueno, the author has formulated the principle of mirror spatial-angular symmetry for total probabilities of photons exiting a homogeneous slab of finite optical thickness $\tau_0 < \infty$. Based on this principle, linear second kind Fredholm integral equations have been obtained, as well as linear singular integral equations, for new objects of the classical radiative transfer theory, namely probabilistic invariants in the case of arbitrary distribution and power of primary energy sources in the medium. Besides, a unified probabilistic function for photons exiting a homogeneous slab was constructed. For finding unique regular solutions of the obtained linear singular integral equations, by analogy with the classical theory, additional integral relations are given, depending on the presence or absence of roots or pseudo-roots in respective characteristic equations. Given the analytic connection between probabilistic and photometric invariants, it has been shown that their calculations by standard computational methods, such as discretization technique or method of successive approximations, lead to substantial savings of computer resources and provide a more convenient form for representing theoretical data and results of numerical modelling of radiation fields of an atmosphere in the visible spectrum range of 0.6–0.8 μm.

Keywords: Mirror reflection principle · Total probability of photon exiting · Probabilistic invariants · Basic boundary-value problem · Phase function · Single scattering albedo · Optical thickness · Mirror symmetrical optical levels · Mirror symmetry vision lines · Unified probabilistic function

1 Basic Probabilistic Functions and Their Main Properties

Let's consider a homogeneous slab of finite optical thickness $\tau_0 < \infty$ (without a reflecting bottom), in which there occur processes of multiple anisotropic photon scattering and absorption with the single-scattering albedo Λ and $(1-\Lambda)$ respectively, and the arbitrary phase function $P(\cos\gamma)$, where γ is a scattering angle. We ignore the effects of radiation polarization and refraction. The distribution and power of primary external

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and internal energy sources $g(\tau, \eta, \varphi)$ in such a slab are deemed arbitrary and isotropic. Here we focus on the spatial angular structure, nonlocal properties of symmetry and invariance of scalar radiation exiting the medium, as well as on probabilistic interpretation of multiple anisotropic photon scattering in the considered homogeneous slab. We mostly pay attention to one of the problems of the classical radiative transfer theory, in which the medium is illuminated by parallel solar rays falling on its upper $\tau = 0$ or lower $\tau = \tau_0$ border and creating luminosity of a square perpendicular to them, equal to πS. In this case, the basic boundary-value problem of the classical theory of scalar radiative transfer [1, 2] is of a probabilistic nature, meaning that the phase function $P(\cos\gamma)$ which determines the probability of photon scattering at an angle γ (in interaction with the medium's elementary volume dV, and the particles albedo Λ equal to the probability of their "survival" in the initial scattering event), including the basic phenomenological equation of radiative transfer, from the statistical point of view allows to consider the multiple scattering process in terms of a stochastic and uniform Markov chain process with given initial probabilities of distributions of photon absorption and scattering volume coefficients in each current point along their motion path [3–5].

Thus, in the framework of the statistical interpretation of the classical radiative transfer theory, its basic task consists in finding, at each fixed yet arbitrary optical level τ, a spatial-angular distribution of probabilities of photons exiting the medium across the upper ($\tau = 0$) or lower ($\tau = \tau_0$) border at given probabilities that characterize basic optical parameters and initial spatial-angular distribution of probabilities of initial (single) photon scattering and absorption by the medium elementary volume dV. Given the above, to solve the set task, we will use Sobolev' probabilistic method in its original basic statement [6], including generalization to the case of non-isotropic multiple photon scattering [7].

Let the function $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)d\varpi$ determine the probability of such an event where a photon moving in the direction $\theta' = \arccos\eta'$ to the outward normal of the given homogeneous slab at the azimuth φ', and then absorbed at the optical depth τ, exits the medium after multiple scatterings through its upper border $\tau = 0$ at the angle $\theta = \arccos\eta$ to the outward normal at the azimuth φ inside the solid angle $d\omega = \sin\theta\, d\theta\, d\varphi$ (Fig. 1).

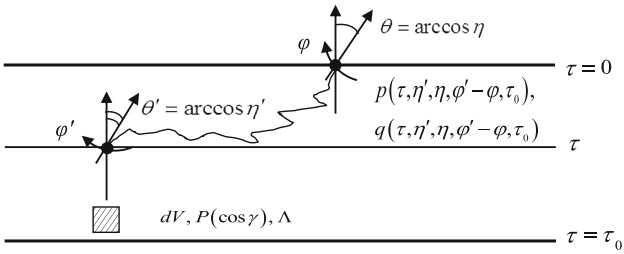


Fig. 1. Probabilities $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ of photons exiting the upper border $\tau = 0$ of the homogeneous slab $[0, \tau_0]$ at their absorption and subsequent radiation at an arbitrary optical thickness τ.

Along with this probabilistic function, we consider the probability that the photon emitted at the optical depth τ in the direction $\theta' = \arccos\eta'$ relative to the outward normal and having the azimuth φ' will exit the medium after multiple scatterings through its upper border $\tau = 0$ at the angle $\theta = \arccos\eta$ to the outward normal at azimuth φ inside the solid angle$d\omega$. Note that according to [1, 6, 7], finding probabilistic functions $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ allows to fully solve the basic boundary-value problem of the scalar radiative transfer theory in the visible range of spectra 0.6–0.8 μm in the case of a homogeneous slab of an arbitrary optical thickness $\tau_0 \leq \infty$ without a reflecting bottom.

2 Probabilistic Invariants in the Theory of Multiple Photon Scattering

Let us turn now to studying the sought-for nonlocal spatial-angular properties of mirror symmetry for the probabilities $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ of photons exiting the homogeneous slab of finite optical thickness $\tau_0 < \infty$, in which the geometric and optical symmetry axes coincide with its middle $\tau_0/2$. In the traditional approach to considering those properties [1, 2] for upgoing $\eta' < 0$ and downgoing $\eta' > 0$ vision lines, usually one arbitrary but fixed optical level τ is selected [6, 7], and then the well-known theorem of optical reciprocity of scattering processes in a given slab [8]. Modification of this standard approach as proposed by the author [9, 10] implies "splitting" of the initially selected arbitrary level τ into two separate optical levels that are mirror symmetrical in relation to the middle of the initial homogeneous slab $\tau_0/2$. In that case, instead of one current optical level τ, the second level $(\tau_0 - \tau)$ is introduced, which is mirror symmetrical to the initial level τ in relation to the level $\tau_0/2$. As demonstrated below, the consideration of two probabilistic functions $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ for two arbitrary upgoing and downgoing mirror symmetrical directions of photon propagation η' and $-\eta'$ leads to the formation, at optical depths τ and $\tau_0 - \tau$ in directions η' and $-\eta'$, of new probabilistic constructions which possess invariance properties at a simultaneous shift of the current optical depths $\tau \Leftrightarrow \tau_0 - \tau$ and change of directions $\eta' \Leftrightarrow -\eta'$ when vision lines are rotating. This property of spatial-angular mirror symmetry extends and supplements the generally accepted traditional principle of optical reciprocity [11], being a foundation for formulating a more general principle of mirror reflection and spatial-angular mirror symmetry of probabilistic functions $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ in an extended homogeneous slab of finite optical thickness $\tau_0 < \infty$ at any power and arbitrary spatial distribution of primary isotropic energy sources $g(\tau)$ within it.

Following the above-described approach, we consider the behavior of the initial photon (I) absorbed at an arbitrary optical depth τ from the direction of its propagation $\theta' = \arccos\eta'$ at azimuth φ'. At the same arbitrary optical level τ let us introduce the probabilistic function $p_+(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0)$, which determines the probability of the photon exiting the medium across its lower border $\tau = \tau_0$ in the downgoing direction $\theta = \arccos\eta$ at azimuth $(\varphi' - \varphi)$. Thus, at an arbitrarily selected optical level τ the sum of probabilities $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0)$ determines

the total probability $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ of the photon (I) exiting the medium at the angle $\theta = \arccos\eta$ at azimuth $(\varphi' - \varphi)$:

$$p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) + p(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0) \tag{2.1}$$

Along with the photon (I) absorbed at the level τ from the direction $\theta' = \arccos\eta'$ at azimuth φ', let us consider a "mirror" photon (I_*) absorbed at the mirror symmetrical level $\tau_* = \tau_0 - \tau$ from mirror symmetrical direction $\theta'_* = \pi - \arccos\eta' = \arccos\eta_*$ at azimuth φ' (Fig. 2). Obviously, the function $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ characterizes the probability of exit from the given homogeneous slab $[0, \tau_0]$ across its upper border $\tau = 0$ from the mirror symmetrical level $\tau_* = \tau_0 - \tau$ in the direction $\theta = \arccos\eta$ at azimuth φ of the "mirror" photon (I_*) moving in the direction $\eta'_* = -\eta'$ at azimuth φ'. The function $p_*[(\tau_0 - \tau)_*, -\eta'_*, \eta, \varphi' - \varphi, \tau_0]$, inversely, determines the probability that the "mirror" photon (I_*) moving in the direction $(-\eta'_*)$ at azimuth φ' will exit the medium across its lower border $\tau = \tau_0$ from the mirror symmetrical level $(\tau_0 - \tau)_* = \tau$.

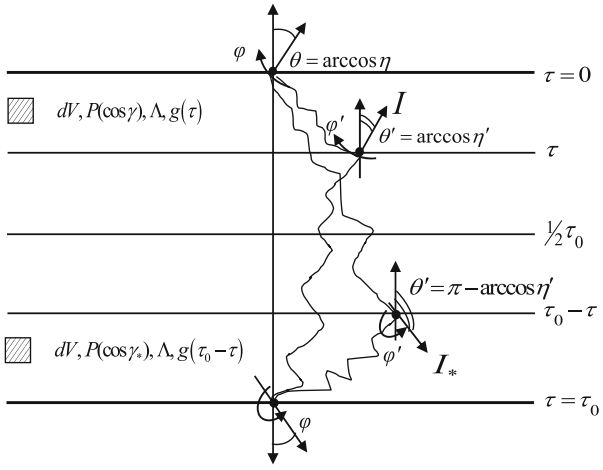


Fig. 2. The geometry of initial (I) and "mirror" (I_*) photons propagation in the homogeneous slab of finite optical thickness $\tau_0 < \infty$.

As follows from Fig. 2, spatial and angular characteristics of the initial (I) and "mirror" (I_*) photons, as well as the propagation directions $(\eta', -\eta'_*)$, and optical levels τ, $(\tau_0 - \tau)$ and $\tau_*(\tau_0 - \tau)_*$ are mirror symmetrical. In this sense, the photons (I) and (I_*) are photons are "mirror" reflections of each other with respect to the natural level of optical symmetry $\tau_0/2$ of the homogeneous slab under consideration. Therefore, the total probability $p_{\pm,*}$ of the "mirror" photon (I_*) exiting the medium across its upper $\tau = 0$ or lower $\tau = \tau_0$ borders from the mirror symmetrical level $(\tau_0 - \tau)$ in the direction (θ, φ) after multiple scatterings is equal to the sum of the following probabilities:

$$\begin{aligned} p_{+,*}(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0) &= p_*[(\tau_0 - \tau)_*, -\eta'_*, \eta, \varphi' - \varphi, \tau_0] + p_*(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0) \\ &= p[\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0] + p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0). \end{aligned} \tag{2.2}$$

The comparison of relations (2.1) and (2.2) leads us to constructing the first probabilistic invariant for the initial photon (I) and its "mirror" reflection (I_*) in the homogeneous slab of finite optical thickness $\tau_0 < \infty$:

$$p_{+,*}(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0) = p_+(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0) = p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0). \tag{2.3}$$

Thus, the total probabilities of exiting the medium for the initial (I) and "mirror" (I_*) photons from the mirror symmetrical levels τ and τ_* are equal in the mirror directions of their propagation η' и η'_*. Note that relation (2.3) determines the probabilistic equivalence between the mirror optical levels τ and $\tau_0 - \tau$ with respect to the optical symmetry axis of the given homogeneous slab $\tau_0/2$ with simultaneous mirror symmetry of propagation directions η' and $-\eta'$ of the initial photon (I) and its "mirror" analogue (I_*) exiting the medium across any of its outside borders $\tau = 0$ and $\tau = \tau_0$. On the other hand, given the optical homogeneousity of the slab with respect to the phase function $P(cos\gamma)$ and parameter Λ, the relation of probabilistic equivalence (2.3) can be interpreted as a relation of spatial-angular invariance for the total probability of the initial photon (I) exiting the medium from the optical level τ or from its mirror level $\tau_* = \tau_0 - \tau$ in the case of their reciprocal translation $\tau \Leftrightarrow \tau_0 - \tau$ and combined replacement of the photon propagation directions $\eta' \Leftrightarrow -\eta'$. This conclusion is of fundamental nature and allows, in the case of optical homogeneousity of a given slab, to view the noted property for the value $p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ as a fundamental property of spatial-angular invariance [9, 10] for the total probability of photon exiting the initial homogeneous slab $[0, \tau_0]$ for its linear semigroup transformations of the type of spatial shift $\tau \Leftrightarrow \tau_0 - \tau$ and simultaneous spatial rotation of the vision line $\eta' \Leftrightarrow -\eta'$ [12, 13].

Another probabilistic characteristic of scalar radiation fields that allows for its invariance interpretation and respectively symmetrization at arbitrary mirror optical levels τ and $\tau_0 - \tau$ in mirror directions of photon propagation η' and $-\eta'$, is the value of spatial-angular non-symmetry for the total probability of photon existing across the upper $\tau = 0$ or lower $\tau = \tau_0$ borders of the considered homogeneous slab $[0, \tau_0]$. It is obvious that for the initial photon (I) the value of probabilistic non-symmetry can be determined by the following relation:

$$p_-(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0) = p(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0) - p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0). \tag{2.4}$$

We have a similar relation for the "mirror" photon (I_*):

$$p_{-,*}(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0) = p_*(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0) - p[(\tau_0 - \tau)_*, -\eta'_*, \eta, \varphi' - \varphi, \tau_0] \tag{2.5}$$

From the comparison of (2.4) and (2.5) follows the second probabilistic invariant $p_-(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$, which at the mirror symmetrical optical levels τ и $\tau_0 - \tau$ in mirror symmetrical directions of photon propagation η' and $-\eta'$ (2.3) has an additional property of invariance in the case of reciprocal and simultaneous translation of effective variables $\tau \Leftrightarrow \tau_0 - \tau$ and $\eta' \Leftrightarrow -\eta'$ in relation (2.4):

$$p_-(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = -p_-(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0) \tag{2.6}$$

Furthermore, knowledge of probabilistic invariants $p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $p_-(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ allows to uniquely determine values of their constituents

according to a simple linear scheme as follows:

$$p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = \frac{p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) - p_-(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)}{2}, \quad (2.7)$$

$$p(\tau_0 - \tau, \eta', \eta, \varphi' - \varphi, \tau_0) = \frac{p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) + p_-(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)}{2}. \quad (2.8)$$

Note that the above consideration for $p_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ is also valid for probabilistic invariants $q_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$. By reproducing it, we can easily obtain the following relations:

$$q_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) + q(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0), \quad (2.9)$$

$$q_-(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = q(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0) - q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0), \quad (2.10)$$

$$q_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = q_+(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0), \quad (2.11)$$

$$q_-(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = -q_-(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0). \quad (2.12)$$

It is obvious that all the above remarks regarding the probabilistic invariants $p_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ to the full extent refer to the probabilistic invariants $q_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$.

Thus, by using relations (2.3) and (2.4) local properties of optical reciprocity of the phase function $P(\eta', \eta, \varphi' - \varphi)$, which were formulated in [8] for elementary volume dV of a scattering and absorbing medium, and similar properties of angular symmetry of brightness coefficients $\rho(\eta', \eta, \varphi' - \varphi, \tau_0)$ and $\sigma(\eta', \eta, \varphi' - \varphi, \tau_0)$ [14, 15] can be generalized to the whole extended homogeneous slab $[0, \tau_0]$ of finite optical thickness $\tau_0 < \infty$.

3 The Principle of Spatial-Angular Mirror Symmetry for Total Probabilities of Photon Exiting a Homogeneous Slab

The property of equivalence of total probabilities of photon exiting the medium $p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ и $q_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$, which are respectively absorbed or radiated at the mirror symmetrical optical levels τ and $\tau_0 - \tau$ in the mirror symmetrical vision lines η' and $-\eta'$, enables us to formulate a general principle of spatial-angular mirror symmetry for their constituents, namely for probabilistic functions $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$.

For that purpose, along with the original problem A, let us consider a similar problem A_* which is mirror symmetrical to the original problem A with respect to the location of optical levels τ and $\tau_0 - \tau$ and vision lines η' and $-\eta'$, including the spatial distribution of primary isotropic energy sources $g(\tau)$ (Fig. 3).

By combining the probabilities of the initial (I) and "mirror" (I_*) photon exit p and p_*, respectively, from mirror level (τ, τ_0) and $\left[(\tau - \tau_0), (\tau - \tau_0)_*\right]$ across the upper τ

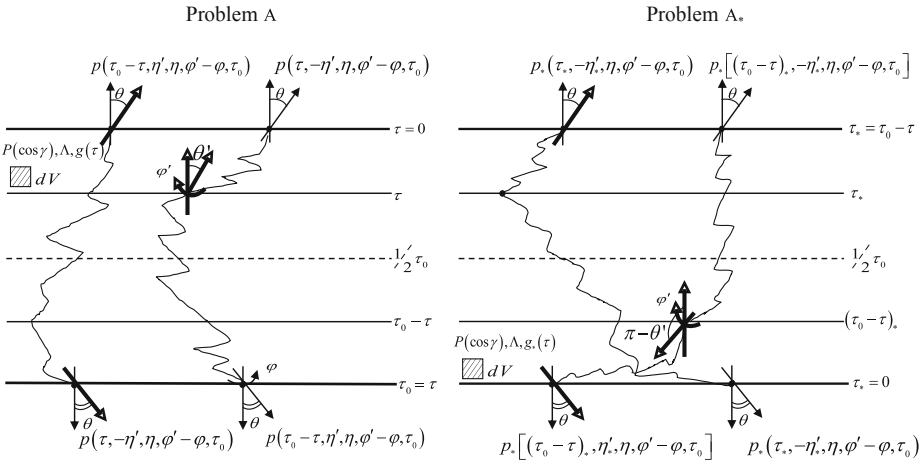


Fig. 3. Total probabilities $p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ of photons exiting the homogeneous slab $[0, \tau_0]$ across its upper $\tau = 0$ and lower $\tau = \tau_0$ borders in the initial (A) and symmetrized (A_*) problems.

$= 0$ and lower $\tau = \tau_0$ borders of the given homogeneous slab $[0, \tau_0]$ in the mirror symmetrical vision lines η' и η'_* relative to the origins τ and η' used in the initial non-symmetrized problem A, we can formulate their probabilistic equivalence in the following basic proposition: in a spatial-angular symmetrized problem, the total probabilities $p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ of multiply scattered photons exiting the medium across the upper $\tau = 0$ and lower $\tau = \tau_0$ border of a homogeneous slab from its symmetrical levels τ and $\tau_0 - \tau$ are equal in the mirror symmetrical vision lines η' and $-\eta'$ in the case of symmetrized location of primary (isotropic) energy sources with respect to level $\tau_0/2$:

$$p(\tau, -\eta', \eta, \varphi' - \varphi, \tau_0)|_A + p_*(\tau_*, -\eta'_*, \eta, \varphi' - \varphi, \tau_0)|_{A_*} = p(\tau_0 - \tau, \eta', \eta, \varphi' - \varphi, \tau_0)|_A + p_*[(\tau_0 - \tau)_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0]|_{A_*} \quad (3.1)$$

$$p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)|_A + p_*(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0)|_{A_*} = p(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0)|_A + p_*[(\tau_0 - \tau)_*, -\eta'_*, \eta, \varphi' - \varphi, \tau_0]|_{A_*} \quad (3.2)$$

From relations (3.1) and (3.2) it follows that the substitution of the initial problem A by the "mirror" problem A_*, and vice versa, with simultaneous and superposed translation of optical levels $\tau \Leftrightarrow \tau_0 - \tau$, $\tau_* \Leftrightarrow (\tau_0 - \tau)_*$ and vision lines $\eta' \Leftrightarrow -\eta'$, $\eta'_* \Leftrightarrow -\eta_*$ in the case of "mirror" substitution of primary (isotropic) energy sources $g(\tau) \Leftrightarrow g_*(\tau)$ with respect to the middle $\tau_0/2$ of the given homogeneous slab $[0, \tau_0]$, leads to their full probabilistic equivalence. In other words, the total probabilities $p_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ of photons exiting a homogeneous slab across its upper $\tau = 0$ and lower $\tau = \tau_0$ borders from symmetrical optical levels $(\tau, \tau_0 - \tau)$ in the mirror directions of photon propagation η' and $-\eta'$ in a symmetrized problem, are equal (Fig. 4).

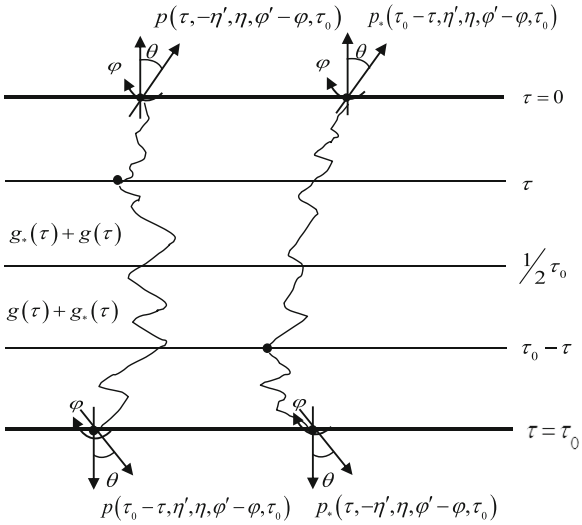


Fig. 4. Combined initial (A) and mirror (A_*) problems to determine the total probability of photon exiting the homogeneous slab [0, τ_0] from its mirror symmetrical optical levels τ и τ_0–τ in the symmetrical directions η' and $-\eta'$ of photon propagation

The same conclusion is also true for probabilistic invariants $q_+(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$. In that case, the following obvious relations:

$$\tau_* = \tau_0 - \tau, \quad (\tau_0 - \tau)_* = \tau, \quad \eta'_* = -\eta', \quad g_*(\tau) = g(\tau_0 - \tau), \tag{3.3}$$

allow to directly and uniquely express probabilistic functions $p_*(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0)$ and $q_*(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0)$ in the "mirror" problem A_* through similar probabilistic functions $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ in the initial problem A:

$$p_*(\tau_*, \eta'_*, \eta, \varphi' - \varphi, \tau_0) = p(\tau_0 - \tau, -\eta', \eta, \varphi' - \varphi, \tau_0), \tag{3.4}$$

$$p_*[(\tau_0 - \tau)_*, -\eta'_*, \eta, \varphi' - \varphi, \tau_0] = p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0). \tag{3.5}$$

Thus, the equivalence of photons fully exit from the symmetrical optical levels τ and $\tau - \tau_0$ in the mirror symmetrical directions of their propagation η' and $-\eta'$ in the initial (A) and mirror (A_*) problems helps us to strictly formulate this fundamental property as the following basic proposition: as a result of multiple scattering and absorption of photons in a homogeneous slab of finite optical thickness $\tau_0 < \infty$ the total probabilities of photons exiting across the slab's upper $\tau = 0$ and lower $\tau = \tau_0$ borders from the symmetrical levels τ and τ–τ_0 in the mirror symmetrical propagation directions η' and $-\eta'$ are equal.

Therefore, it can be stated that the existence of probabilistic invariants $p_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ is resulting from the general statistic interpretation of the multiple photon scattering process in the phenomenological theory of scalar radiative transfer [1, 2, 6, 7].

4 Symmetrization of Main Integral Equations for Basic Probabilistic Functions

The introduction of new probabilistic objects into the classical multiple photon scattering theory, namely the probabilistic invariants $p_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$, allows to define them through linear symmetrized integral and integral differential equations which are different from those in [1, 6, 14]. Indeed, by using relations (2.1)–(2.6) and (2.9)–(2.12), after obvious linear transformations of basic integral differential equations considered in [14], we can obtain their symmetrized analogs:

$$\eta' \frac{\partial q_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)}{\partial \tau} = -q_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) + p_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0), \quad \eta' \in [-1, 1] \cap \eta \in [-1, 1] \cap \varphi \in [0, 2\pi] \cap \tau \in [0, \tau_0]. \tag{4.1}$$

Note that Eq. (4.1) is a complete analog of the integral differential equation that in the scalar radiative transfer theory connects the sought-for radiation intensities $I(\tau, \eta', \eta, \varphi' - \varphi_0, \tau_0)$ and source functions $B(\tau, \eta', \eta, \varphi' - \varphi_0, \tau_0)$ at arbitrary optical depths τ [1, 14]. It is obvious that solutions of Eq. (4.1) formally determine probabilistic invariants $q_{\pm}(\tau, \eta', \eta, \varphi' - \varphi_0, \tau_0)$ in the case where the values of $p_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and vice versa. Indeed, taking into account the connection between the probabilistic functions $p(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ [14], definitions (2.1), (2.4), (2.9), (2.10) of probabilistic variants $p_{\pm}(\tau, \eta', \eta, \varphi' - \varphi_0, \tau_0)$ and $q_{\pm}(\tau, \eta', \eta, \varphi' - \varphi_0, \tau_0)$, as well as their invariant properties (2.3), (2.6), (2.11) and (2.12), we can, after simple linear transformation, obtain the following relations:

$$q_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = \int_0^{\tau} p_{\pm}(\tau', \eta', \eta, \varphi' - \varphi, \tau_0) e^{-\frac{\tau - \tau'}{\eta'}} \frac{d\tau'}{\eta'} \pm \delta(\eta' - \eta)\delta(\varphi' - \varphi) e^{-\frac{\tau}{\eta}}, \quad \eta' \in [-1, 1] \cap \eta \in [-1, 1] \cap \varphi \in [0, 2\pi] \cap \tau \in [0, \tau_0]; \tag{4.2}$$

$$p_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = \frac{\Lambda}{4\pi} \int_0^{2\pi} d\varphi'' \int_{-1}^{1} P(\eta'', \eta', \varphi' - \varphi'') q_{\pm}(\tau, \eta', \eta, d(\varphi'' - \varphi), \tau_0) d\eta'', \quad \eta' \in [-1, 1] \cap \eta \in [-1, 1] \cap \varphi \in [0, 2\pi] \cap \tau \in [0, \tau_0]. \tag{4.3}$$

To obtain relations (4.2) and (4.3), we have used the angular symmetry property of phase function $P(-\eta'', -\eta', \varphi - \varphi'') = P(\eta'', \eta', \varphi' - \varphi'')$ [8]. Furthermore, it should be emphasized that the solution of integral differential Eq. (4.1) in view of basic properties (2.1), (2.6), (2.11), and (2.12) for the sought-for probabilistic invariants $p_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ allows to halve the change intervals of effective variable $\tau \in [0, \tau_0]$ and $\eta' \in [-1, 1]$ in comparison to the basic boundary problem of radiative transfer theory [9, 10, 14], and [15], namely: $\tau \in \left[0, \frac{\tau_0}{2}\right]$ and $\eta' \in [-1, 1]$, or alternatively $\tau \in [0, \tau_0]$ and $\eta' \in [0, 1]$.

By substituting then (4.2) into (4.3) and vice versa, we can obtain the following linear second kind Fredholm integral equations for the probabilistic invariants $p_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$:

$$
\begin{aligned}
p_{\pm}(\tau,\eta',\eta,\varphi'-\varphi,\tau_0) &= \frac{\Lambda}{4\pi}\Big[P(\eta,-\eta',\varphi'-\varphi)e^{-\frac{\tau_0-\tau}{\eta}} \pm P(\eta,\eta',\varphi'-\varphi)e^{-\frac{\tau}{\eta}}\Big] \\
&+\frac{\Lambda}{4\pi}\int_0^{2\pi} d\varphi'' \int_0^1 d\eta'' \int_0^{\tau} \big[P(\eta',\eta'',\varphi'-\varphi'')p_{\pm}(\tau',\eta'',\eta,\varphi''-\varphi,\tau_0) \\
&+P(\eta',-\eta'',\varphi'-\varphi'')p_{\pm}(\tau',-\eta'',\eta,\varphi''-\varphi,\tau_0)\big]e^{-\frac{\tau-\tau'}{\eta''}}\frac{d\tau'}{\eta''}, \\
&\eta' \in [-1,1],\ \eta \in [0,1],\ (\varphi'-\varphi) \in [0,2\pi] \cap \tau \in [0,\tau_0];
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
q_{\pm}(\tau,\eta',\eta,\varphi'-\varphi,\tau_0) &= \frac{\Lambda}{4\pi}\int_0^{2\pi} d\varphi'' \int_0^1 d\eta' \int_0^{\tau} \big[P(\eta',\eta'',\varphi'-\varphi'')q_{\pm}(\tau',\eta'',\eta,\varphi''-\varphi,\tau_0) \\
&+P(\eta',-\eta'',\varphi'-\varphi'')q_{\pm}(\tau',-\eta'',\eta,\varphi''-\varphi,\tau_0)\big]e^{-\frac{\tau-\tau'}{\eta''}}\frac{d\tau'}{\eta''} \pm \delta(\eta'-\eta)\delta(\varphi'-\varphi)e^{-\frac{\tau}{\eta}}, \\
&\eta' \in [0,1] \cap \eta \in [0,1] \cap (\varphi'-\varphi) \in [0,2\pi] \cap \tau \in [0,\tau_0].
\end{aligned}
\tag{4.5}
$$

Note that integral Eqs. (4.4) and (4.5) have similar kernels and differ only in their absolute terms. Therefore, numerical solutions of these equations can be obtained by using, for example, one-type convergent iterative schemes with zero approximations which are determined by exact formulas of single photon scattering [9, 10, 14, 15].

Thus, to use the probabilistic interpretation of scalar radiative transfer in a homogeneous slab of finite optical thickness $\tau_0 < \infty$ for building basic symmetrized problems of the theory of multiple non-isotropic photon scattering, it is necessary first to carry out the following combined types of spatial-angular symmetrization:

- symmetrization of optical levels that are mirror positioned relative to the middle $\tau_0/2$ of the given homogeneous slab $[0, \tau_0]$,
- symmetrization of the directions of scattered photons propagation η' and $-\eta'$,
- mirror symmetrization of the locations of primary isotropic energy sources $g(\tau)$ relative to the middle $\tau_0/2$ of the given homogeneous slab $[0, \tau_0]$.

The latter type of symmetrization could be ignored since the probabilities of photons exiting the medium, as noted above, do not depend on the location of primary isotropic energy sources.

5 Linear Singular Integral Equations for Probabilistic Invariants of a Homogeneous Slab of Finite Optical Thickness

For implementing the mirror reflection principle and applying it for effectively modelling the probabilistic functions $p(\tau,\eta',\eta,\varphi'-\varphi,\tau_0)$ and $q(\tau,\eta',\eta,\varphi'-\varphi,\tau_0)$, as well as their probabilistic invariants $p_{\pm}(\tau,\eta',\eta,\varphi'-\varphi,\tau_0)$ and $q_{\pm}(\tau,\eta',\eta,\varphi'-\varphi,\tau_0)$, our main interest is the class of linear singular equations that were originally introduced into the scalar radiative transfer theory by V.V. Sobolev [1, 15] and T.M. Mallikin [16].

Firstly, in order to reduce the total number of independent angle variables, we can use usual Fourier transformations for phase functions $P(\cos\gamma) \equiv P(\eta', \eta'', \varphi' - \varphi'')$:

$$P(\eta', \eta'', \varphi' - \varphi'') = \wp^0(\eta', \eta'') + 2\sum_{m=1}^{M_1} \wp^m(\eta', \eta'') \cos m(\varphi' - \varphi''), \tag{5.1}$$

where azimuthal harmonics $\wp^{\mathrm{m}}(\eta', \eta'')$ are determined as follows:

$$\wp^m(\eta', \eta'') = \frac{1}{2\pi}\int_0^{2\pi} P(\eta', \eta'', \varphi' - \varphi'') \cos m(\varphi' - \varphi'')d\varphi, \quad m = \overline{0, M_1}. \tag{5.2}$$

Then using transformations similar to (5.1) and (5.2) for probabilistic invariants $p_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$, we get

$$p_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = \sum_{m=1}^{M_2} p_\pm^m(\tau, \eta', \eta, \tau_0) \cos m(\varphi' - \varphi), \tag{5.3}$$

$$q_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) = \sum_{m=1}^{M_3} q_\pm^m(\tau, \eta', \eta, \tau_0) \cos m(\varphi' - \varphi), \tag{5.4}$$

where azimuthal harmonics $p_\pm^m(\tau, \eta', \eta, \tau_0)$ and $q_\pm^m(\tau, \eta', \eta, \tau_0)$ can be represented as follows:

$$p_\pm^m(\tau, \eta', \eta, \tau_0) = \frac{1}{2\pi}\int_0^{2\pi} p_\pm(\tau, \eta', \eta, \varphi' - \varphi) \cos m(\varphi' - \varphi)d(\varphi' - \varphi), \tag{5.5}$$

$$q_\pm^m(\tau, \eta', \eta, \tau_0) = \frac{1}{2\pi}\int_0^{2\pi} q_\pm(\tau, \eta', \eta, \varphi' - \varphi) \cos m(\varphi' - \varphi)d(\varphi' - \varphi). \tag{5.6}$$

Henceforth, when using Fourier series (5.1), (5.3), and (5.4), we will select only one common value of the maximum number of azimuthal harmonics $M = \max\{M_1, M_2, M_3\}$, with required adjustment depending on the relation between the values M_1, M_2, and M_3.

Note that the sought-for linear singular integral equations for azimuthal harmonics of the probabilistic invariants $p_\pm^m(\tau, \eta', \eta, \tau_0)$ and $q_\pm^m(\tau, \eta', \eta, \tau_0)$ follow, in view of basic relations obtained on the basis of [14]:

$$\begin{aligned} p_\pm^m(\tau, \eta', \varphi - \varphi, \tau_0) &= \frac{B_\pm^m(\tau, -\eta', \eta, \varphi' - \varphi, \tau_0)}{\pi S}, \\ q_\pm^m(\tau, \eta', \eta, \varphi' - \varphi, \tau_0) &= \frac{I_\pm^m(\tau, -\eta', \eta, \varphi' - \varphi, \tau_0)}{\pi S}, \end{aligned} \tag{5.7}$$

where the photometric values $B_{\pm}^{m}(\tau, \eta', \eta, \tau_0)$ and $I_{\pm}^{m}(\tau, \eta', \eta, \tau_0)$ are, respectively, invariants of azimuthal harmonics of source functions $B^{m}(\tau, \eta', \eta, \tau_0)$ and scalar radiation intensities $I^{m}(\tau, \eta', \eta, \tau_0)$ [10]:

$$\begin{aligned} B_{\pm}^{m}(\tau, \eta', \eta, \tau_0) &= B^{m}(\tau_0 - \tau, -\eta', \eta, \tau_0) \pm B^{m}(\tau, \eta', \eta, \tau_0), \\ B_{\pm}^{m}(\tau, \eta', \eta, \tau_0) &= \pm B_{\pm}^{m}(\tau_0 - \tau, -\eta', \eta, \tau_0) \\ I_{\pm}^{m}(\tau, \eta', \eta, \tau_0) &= I^{m}(\tau_0 - \tau, -\eta', \eta, \tau_0) \pm I^{m}(\tau, \eta', \eta, \tau_0), \\ I_{\pm}^{m}(\tau, \eta', \eta, \tau_0) &= \pm I_{\pm}^{m}(\tau_0 - \tau, -\eta', \eta, \tau_0) \end{aligned} \tag{5.8}$$

from similar equations for azimuthal harmonics invariants of source functions $B_{\pm}^{m}(\tau, \eta', \eta, \tau_0)$ and scalar radiation intensities $I_{\pm}^{m}(\tau, \eta', \eta, \tau_0)$, respectively, [9]:

$$T^{m}(\eta) p_{\pm}^{m}(\tau, \eta', \eta, \tau_0) = \frac{\Lambda}{4\pi}\left[A^{m}(\eta, -\eta') e^{-\frac{\tau_0 - \tau}{\eta}} \pm A^{m}(\eta, \eta') e^{-\frac{\tau}{\eta}}\right]$$

$$\mp \frac{\Lambda}{2}\eta \int_0^1 \frac{p_{\pm}^{m}(\tau, \eta', \eta'', \tau_0) A^{m}(\eta, \eta'')}{\eta - \eta''} d\eta'' \mp \frac{\Lambda}{2}\eta e^{-\frac{\tau_0}{\eta}} \int_0^1 \frac{p_{\pm}^{m}(\tau, \eta', \eta'', \tau_0) A^{m}(\eta, -\eta'')}{\eta + \eta''} d\eta'',$$

$$\eta' \in [-1, 1] \cap \eta \in [0, 1] \cap \tau \in [0, \tau_0], \tag{5.9}$$

$$T^{m}(\eta) q_{\pm}^{m}(\tau, \eta', \eta, \tau_0) = \frac{\Lambda}{4\pi} f_{\pm}^{m}(\tau, \eta', \eta, \tau_0) \mp \frac{\Lambda}{2}\eta \int_0^1 \frac{q_{\pm}^{m}(\tau, \eta', \eta, \tau_0) A^{m}(\eta, \eta'')}{\eta - \eta''} d\eta'' \mp$$

$$\mp \frac{\Lambda}{2}\eta e^{-\frac{\tau_0}{\eta}} \int_0^1 \frac{q_{\pm}^{m}(\tau, \eta', \eta, \tau_0) A^{m}(\eta, -\eta'')}{\eta + \eta''} d\eta'', \ \eta' \in [-1, 1] \cap \eta \in [0, 1] \cap \tau \in [0, \tau_0]. \tag{5.10}$$

Functions $f_{\pm}^{m}(\tau, \eta', \eta, \tau_0) = f^{m}(\tau_0 - \tau, -\eta', \eta, \tau_0) \pm f^{m}(\tau, \eta', \eta, \tau_0)$, entering into Eq. (5.10), are determined on the basis of the following relations:

$$f^{m}(\tau, \eta', \eta, \tau_0) = \begin{cases} \frac{\Lambda S}{4} A^{m}(\eta, \eta') \frac{\eta}{\eta - \eta'} \left(e^{-\frac{\tau}{\eta}} - e^{-\frac{\tau}{\eta'}}\right), \eta' > 0, \\ \frac{\Lambda S}{4} A^{m}(\eta, \eta') \frac{\eta}{\eta - \eta'} e^{-\frac{\tau}{\eta}} \left[1 - e^{-\left(\frac{1}{\eta} - \frac{1}{-\eta'}\right)(\tau_0 - \tau)}\right], \eta' < 0 \end{cases} \tag{5.11}$$

Azimuthal harmonics of functions $T^{m}(\eta)$ are determined by relations [1, 15]:

$$T^{m}(\eta) = 1 - \Lambda \eta^2 \int_0^1 \frac{A^{m}(\eta', \eta')}{\eta^2 - \eta'^2} d\eta', \quad m = \overline{0, M}. \tag{5.12}$$

Polynomial kernel functions $A^{m}(\eta', \eta')$ в (5.11) are found by the formula [15]:

$$A^{m}(\eta'', \eta') = P_{m}^{m}(\eta') A_{m}^{m}(\eta'', \eta') = \sum_{i=m}^{N} x_i \frac{(i-m)!}{(i+m)!} \mathfrak{R}_{i}^{m}(\eta'') P_{i}^{m}(\eta'), m = \overline{0, M}, \tag{5.13}$$

where $P_i^m(\eta')$ are associated Legendre polynomials, and $\mathfrak{R}_i^m T^m(\eta)$ are Sobolev polynomials calculated on the basis of recurrent relations:

$$(i-m+1)\mathfrak{R}_{i+1,m}^m(\eta'') + (i+m)\mathfrak{R}_{i-1,m}^m(\eta'') = (2i+1-\Lambda x_i)\eta''\mathfrak{R}_{i,m}^m(\eta''),$$

$$\mathfrak{R}_m^m(\eta'') = P_m^m(\eta''), \mathfrak{R}_{m+1}^m(\eta'') = (2m+1-\Lambda x_m)\eta'' P_m^m(\eta''), m = \overline{0, M} \tag{5.14}$$

The coefficients x_i entering into the auxiliary functions $A^m(\eta, \eta')$ are coefficient of expansion of the given phase function $P(\cos\gamma)$ in terms of Legendre polynomials $P_i(\cos\gamma)$:

$$P(\cos\gamma) = \sum_{i=0}^{N} x_i P_i(\cos\gamma), x_i = \frac{2i+1}{2}\int_0^\pi P(\cos\gamma)P_i(\cos\gamma)\sin\gamma d\gamma, x_i < 2i+1. \tag{5.15}$$

We should, however, pay attention to one important situation connected with possible non-uniqueness of solutions for linear singular integral Eqs. (5.9) and (5.10). As shown in [15, 16] if characteristic equations

$$T^m\left(\frac{1}{k_m}\right) = 0, \quad m = \overline{0, M}, \quad T^m\left(\frac{1}{\tilde{k}_m}\right) = 0, \quad m = \overline{0, M} \tag{5.16}$$

have neither roots $k_m \subset [0, 1]$ nor pseudo-roots $\tilde{k}_m \subset]0, 1[$, then the sought for solutions of Eqs. (5.9) and (5.10) are unique. Otherwise, the solutions of those equations are non-unique, there, according to recommendations in [15], we should assume $\eta = \frac{1}{k_m}$ or $\eta = \frac{1}{\tilde{k}_m}$, considering them together with additional integral relations. These relations can be easily obtained on the basis of defining the probabilistic invariants $p_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ taking into account the respective results in [9, 15], and [10]. For example, for the probabilistic invariants $p_\pm(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ in absence of pseudo-roots $\tilde{k}_m$ and existence of only characteristic roots k_m, they appear as follows:

$$\begin{aligned}
&\int_0^1 p_\pm(\tau, \eta', \eta'', \tau_0)A^m\left(\frac{1}{k_m}, \eta''\right)\frac{d\eta''}{1-k_m\eta''} \\
&+ e^{-k_m\tau_0}\int_0^1 p_\pm(\tau, \eta', \eta'', \tau_0)A^m\left(\frac{1}{k_m}, -\eta''\right)\frac{d\eta''}{1+k_m\eta''} \\
&= \frac{1}{2\pi}\left[A^m\left(\frac{1}{k_m}, -\eta'\right)e^{-k_m(\tau_0-\tau)} \pm A^m\left(\frac{1}{k_m}, \eta'\right)e^{k_m\tau}\right], \quad m = \overline{0, M}
\end{aligned} \tag{5.17}$$

6 Unified Probabilistic Function for Photons Exiting a Homogeneous Slab

The methodological importance of probabilistic functions $p^m(\tau, \eta', \eta, \tau_0)$ and $q^m(\tau, \eta', \eta, \tau_0)$, $p_\pm^m(\tau, \eta', \eta, \tau_0)$ and $q_\pm^m(\tau, \eta', \eta, \tau_0)$ considered above lies in the fact

that at mirror symmetrical optical levels τ and $\tau_0 - \tau$ of the given homogeneous slab $[0, \tau_0]$ without a reflecting bottom, in the symmetrical directions of photon propagation η' and $-\eta'$, they form, in a way similar to unified photometric functions for intensities $I_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0)$ and source functions $B_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0)$ [9, 10], new objects of the radiative transfer theory, namely unified probabilistic functions $\mathcal{P}_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0)$ and $\mathcal{Q}_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0)$ for photons exiting the medium [18, 20]:

$$\begin{aligned}\mathcal{P}_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0) &= (\eta' + \eta)p^{m}(\tau_0 - \tau, -\eta', \eta, \tau_0) + (\eta' - \eta)p^{m}(\tau, \eta', \eta, \tau_0) \\ &= \eta' p_{+}^{m}(\tau, \eta', \eta, \tau_0) + \eta p_{-}^{m}(\tau, \eta', \eta, \tau_0), m = \overline{0, M},\end{aligned} \tag{6.1}$$

$$\begin{aligned}\mathcal{Q}_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0) &= (\eta' + \eta)q^{m}(\tau_0 - \tau, -\eta', \eta, \tau_0) + (\eta' + \eta)q^{m}(\tau, \eta', \eta, \tau_0) \\ &= \eta' q_{+}^{m}(\tau, \eta', \eta, \tau_0) + \eta q_{-}^{m}(\tau, \eta', \eta, \tau_0), m = \overline{0, M},\end{aligned} \tag{6.2}$$

Taking into account relations (5.7) and (5.8), instead of relations (6.1) and (6.2) we get:

$$\begin{aligned}\mathcal{P}_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0) &= \frac{1}{\pi S}[(\eta' + \eta)B^{m}(\tau_0 - \tau, -\eta', \eta, \tau_0) + (\eta' + \eta)B^{m}(\tau, \eta', \eta, \tau_0)] \\ &= \frac{1}{\pi S}[\eta' B_{+}^{m}(\tau, \eta', \eta, \tau_0) + \eta B_{-}^{m}(\tau, \eta', \eta, \tau_0)] = \frac{1}{\pi S}B_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0), m = \overline{0, M},\end{aligned} \tag{6.3}$$

$$\begin{aligned}\mathcal{Q}_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0) &= \frac{1}{\pi S}[(\eta' + \eta)I^{m}(\tau_0 - \tau, -\eta', \eta, \tau_0) + (\eta' - \eta)I^{m}(\tau, \eta', \eta, \tau_0)] \\ &= \frac{1}{\pi S}[\eta' I_{+}^{m}(\tau, \eta', \eta, \tau_0) + \eta I_{-}^{m}(\tau, \eta', \eta, \tau_0)] = \frac{1}{\pi S}I_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0), m = \overline{0, M},\end{aligned} \tag{6.4}$$

where photometric invariants $B_{\pm}^{m}(\tau, \eta', \eta, \tau_0)$ and $I_{\pm}^{m}(\tau, \eta', \eta, \tau_0)$ can be determined according to relations (5.8).

Note that knowing non-local properties of spatial-angular symmetry of probabilistic values $p^{m}(\tau, \eta', \eta, \tau_0)$ and $q^{m}(\tau, \eta', \eta, \tau_0)$ in case of inversions of effective variables η' and η, similar to photometric values $B^{m}(\tau, \eta', \eta, \tau_0)$ and $I^{m}(\tau, \eta', \eta, \tau_0)$, enables us to uniquely define them after simple and obvious linear transformations of the unified probabilistic functions $\mathcal{P}_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0)$ and $\mathcal{Q}_{\Sigma}^{m}(\tau, \eta', \eta, \tau_0)$ [9, 10].

Thus, strict relations (6.1)–(6.4) form one and common complex of mutually consistent probabilistic and photometric values. These relations enclose the system of unified probabilistic ($\mathcal{P}_{\Sigma}^{m}$, $\mathcal{Q}_{\Sigma}^{m}$) and photometric (B_{Σ}^{m}, I_{Σ}^{m}) functions, including probabilistic ($p_{\pm}^{m}$, $q_{\pm}^{m}$) and photometric ($B_{\pm}^{m}$, $I_{\pm}^{m}$) invariants for finding numerical and analytical solutions of basic boundary-value problems in the classical scalar radiative transfer theory [14, 15] in the case of homogeneous slabs of finite optical thickness $\tau_0 < \infty$; without a reflecting bottom [18, 20]. In that case, the symmetrization of basic structural functions of this theory, such as fundamental Sobolev functions Φ^{m}, Ambarzumian functions $\varphi^{m}(\tau, \tau_0)$ and $\psi^{m}(\tau, \tau_0)$, Chandrasekhar functions X^{m} and Y^{m}, conducted in [18, 20], in a certain semantic sense, corresponds to the content of the initial probabilistic interpretation [1, 2] of the phenomenological theory of multiple photon scattering and to the probabilistic meaning of the aforementioned structural functions in the visible spectrum range of 0.6–0.8 μm.

7 Conclusion

From computational viewpoint and taking into account strong evidence [17, 18] and the concept of probabilistic invariants, it follows that the application of numerical methods,

for example an angular discretization method, for numerical solutions of linear integral Eqs. (4.4), (4.5), (5.9), (5.10) leads to systems of linear algebraic equations of rank N, i.e. two time lower than for a system of similar integral equations for $p^m(\tau, \eta', \eta, \tau_0)$ and $q^m(\tau, \eta', \eta, \tau_0)$ [14]. Furthermore, from linear second kind Fredholm integral Eqs. (4.4) and (4.5) and linear singular integral Eqs. (5.9) and (5.10) it follows that their regular solutions within the ranges of effective variables $\tau \in [0,\tau_0]$, $\eta' \in [-1,1]$ and $\eta \in [0, 1]$, taking into account basic properties (2.3)–(2.4) and (2.11)–(2.12) of spatial-angular symmetry of probabilistic invariants $p_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$ and $q_{\pm}(\tau, \eta', \eta, \varphi' - \varphi, \tau_0)$, allow to use more narrow ranges of their definition, namely $\tau \in [0, \tau_0/2]$ and $\eta' \in [-1, 1]$, or alternatively $\tau \in [0,\tau_0]$ and $\eta' \in [0,1]$. Understandably, such double decrease in ranges of optical depths τ or angular variables η' will substantially simplify algorithms for numerical solutions of equations like (4.4), (4.5), (5.9), and (5.10), especially in cases of large values $\tau >> 1$ and $\tau_0 >> 1$, and also in the case of strongly elongated phase functions $P(\cos\gamma)$ at $m >> 1$ and $M >> 1$ [19, 20].

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