

A Computational Analysis of the Hopmoc Method Applied to the 2-D Advection-Diffusion and Burgers Equations

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Abstract. This paper shows the accuracy of the Hopmoc method when applied to a partial differential equation that combines both nonlinear propagation and diffusive effects. Specifically, this paper shows the numerical results yielded by the Hopmoc algorithm when applied to the 2-D advection-diffusion and Burgers equations. The results delivered by the Hopmoc method compare favorably with the Crank-Nicolson method and an alternating direction implicit scheme when applied to the advection-diffusion equation. The experiments with the 2-D Burgers equation also show that the Hopmoc algorithm provides results in agreement with several existing methods.

Keywords: Hopscotch method · Modified method of characteristics · Semi-lagrangian approach · Two-dimensional advection-diffusion equation · Two-dimensional Burgers equation

1 Introduction

Practitioners apply various approaches in the approximate solution of differential equations. Some of these approaches are based on a decomposition of operators, such as the alternating direction method [1–3], Implicit-Explicit Method (IMEX) [4,5], and the Odd-Even Hopscotch method [6–8].

The Hopmoc algorithm (see [9] and references therein) joins concepts of the Odd-Even Hopscotch method and the modified method of characteristics (MMOC for short) [10]. The Hopmoc algorithm is similar to the Odd-Even Hopscotch method because both approaches decompose the grid points into two

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O. Gervasi et al. (Eds.): ICCSA 2021, LNCS 12949, pp. 111–120, 2021.
https://doi.org/10.1007/978-3-030-86653-2_8

subsets. The two subsets have their unknowns separately updated within a one-time semi-step. Moreover, each subset performs one explicit update and one implicit update of its unknowns. The Hopmoc algorithm evaluates time semi-steps along characteristic lines in a semi-Lagrangian approach as the MMOC operates. Recently, we showed numerical simulations of the Hopmoc method applied to the 1-D modified Burgers equation [11,12].

This paper shows the computational results yielded by the Hopmoc method when applied to the 2-D advection-diffusion equation. This paper also compares the results provided by the Hopmoc algorithm with six recent approaches when applied to the 2-D Burgers equation.

The remainder of this paper is structured as follows. Section 2 describes the Hopmoc method. Section 3 shows the numerical results. Finally, Sect. 4 addresses the conclusions.

2 Hopmoc Method

The Hopmoc method operates similarly to the Odd-Even Hopscotch method and uses an MMOC-like strategy. One can write a differential operator for the Hopmoc algorithm as

$$L_h \overline{\overline{u}}^n_{i,j} = d \left[\frac{\overline{\overline{u}}^n_{i-1,j} - 2\overline{\overline{u}}^n_{i,j} + \overline{\overline{u}}^n_{i+1,j}}{(\Delta x)^2} + \frac{\overline{\overline{u}}^n_{i,j-1} - 2\overline{\overline{u}}^n_{i,j} + \overline{\overline{u}}^n_{i,j+1}}{(\Delta y)^2} \right]. \tag{1}$$

Using operator (1), the Hopmoc method performs two semi-steps

$$\overline{u}^{n+\frac{1}{2}}_{i,j} = \overline{\overline{u}}^n_{i,j} + \delta t \left(\theta^n_{i,j} L_h \overline{\overline{u}}^n_{i,j} + \theta^{n+\frac{1}{2}}_{i,j} L_h \overline{u}^{n+\frac{1}{2}}_{i,j} \right) \text{ and}$$
$$u^{n+1}_{i,j} = \overline{u}^{n+\frac{1}{2}}_{i,j} + \delta t \left(\theta^{n+\frac{1}{2}}_{i,j} L_h \overline{u}^{n+\frac{1}{2}}_{i,j} + \theta^{n+1}_{i,j} L_h u^{n+1}_{i,j} \right)$$

where

$$\theta^n_{i,j} = \begin{cases} 1, \text{ if } (n+i+j) \text{ is odd, and} \\ 0, \text{ if } (n+i+j) \text{ is even} \end{cases}$$

with $\left(\overline{x}^{n+\frac{1}{2}}_i, \overline{y}^{n+\frac{1}{2}}_i \right) = (x_i - v_1 \delta t, y_j - v_2 \delta t)$ and $\left(\overline{\overline{x}}^n_i, \overline{\overline{y}}^n_i \right) = (x_i - 2v_1 \delta t, y_j - 2v_2 \delta t)$.

Each time step with size Δt in the Hopmoc method encompasses three stages as follows [9].

1. At time step t_n, the Hopmoc method obtains $\overline{\overline{u}}$ for all stencil points $\left(\overline{\overline{x}}_i, \overline{\overline{y}}_j \right)$, for $i = 1, ..., N$ and $j = 1, ..., N$ using an interpolation method.
2. At time semi-step $t_{n+\frac{1}{2}}$, the method computes $\overline{u}^{n+\frac{1}{2}}_{i,j}$ using the explicit (implicit) operator for stencil points for which $n + 1 + i + j$ is odd (even).

3. At time semi-step t_{n+1}, the method computes $u_{i,j}^{n+1}$ using the implicit (explicit) operator for stencil points for which $n+1+i+j$ is odd (even).

The method solves neighbor stencil points at the previous time semi-step. Thus, the implicit approach does not demand to solve a linear system in the same way as the Odd-Even method performs. Figure 1 shows two time semi-steps of the Hopmoc algorithm.

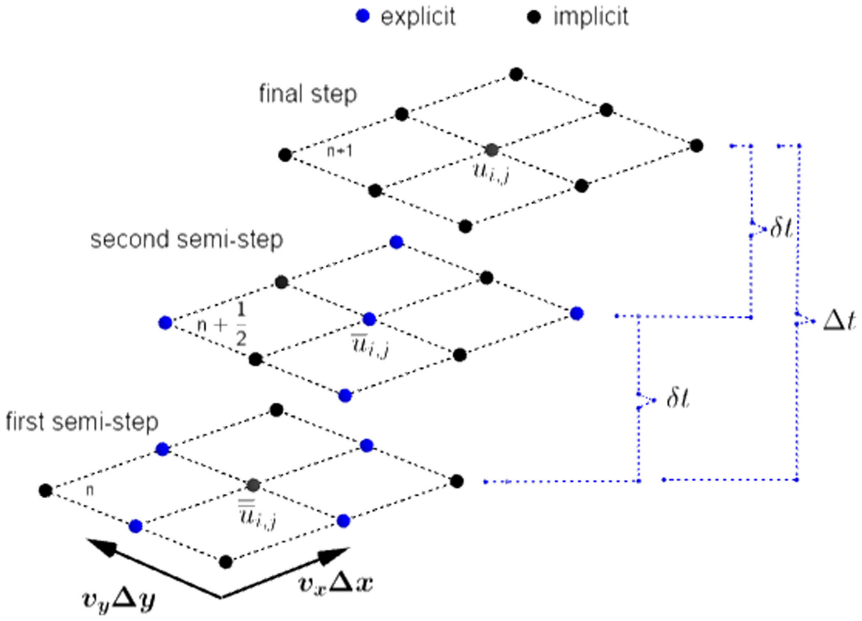

Fig. 1. Two time semi-steps of the Hopmoc method.

3 Computational Analysis

This section shows the results yielded by the Hopmoc method along with a bilinear interpolation [13]. The MacBook Air workstation used in the executions of the simulations featured an Intel® Core™ i5 1.6 GHz processor (Intel; Santa Clara, CA, United States). We implemented the method using the C programming language.

Section 3.1 shows the results yielded by the Hopmoc method when applied to the 2-D advection-diffusion equation and compares the results returned by the Hopmoc method with the Crank-Nicolson and an alternating direction implicit scheme when applied to the same equation. Section 3.2 compares the results delivered by the Hopmoc algorithm with six existing methods when applied to the 2-D Burgers equation.

3.1 Advection-Diffusion Equation

Consider the numerical solution of the 2-D advection-diffusion equation

$$u_t + v_1 u_x + v_2 u_y - \frac{1}{Re}(u_{xx} + u_{yy}) = 0 \tag{2}$$

where $Re = \frac{\rho \cdot \nu \cdot L}{\mu}$ is the Reynolds number, ρ is the density, ν is the characteristic velocity, L is a characteristic linear dimension, and μ is the dynamic viscosity. Eq. (2) is defined in the domain $\Omega = \{(x, y) : 0 \leq x \leq 1; 0 \leq y \leq 1\}$ with analytical solution $u(x, y, t) = cos(\pi(x - v_x t))cos(\pi(y - v_y t))e^{-2Re \cdot \pi^2 \cdot t}$ and boundary conditions $u(x, y, t) = cos(\pi(x - v_x t))cos(\pi(y - v_y t))e^{-2Re \cdot \pi^2 \cdot t}$. The accuracy of the approach is measured in terms of errors norms

$$\|\epsilon\|_\infty = ||u_{exact} - u_{computed}||_\infty = \max_{i,j} \left| u_{i,j}^{exact} - u_{i,j}^{computed} \right|$$

and

$$\|\epsilon\|_2 = ||u_{exact} - u_{computed}||_2 = \left(\sum_{i=0}^{N} \sum_{j=0}^{N} \left| u_{i,j}^{exact} - u_{i,j}^{computed} \right|^2 \right)^{1/2}$$

where u_{exact} and $u_{computed}$ represent the exact and computed solutions, respectively.

Table 1 shows error norms $\|\epsilon\|_\infty$ and $\|\epsilon\|_2$ when applying the Hopmoc method to Eq. (2). The first part of Table 1 shows the results yielded by the Hopmoc method when varying δt, δx, and δy for the same $\frac{\delta t}{\Delta x \cdot \delta y}$. The second part of Table 1 reveals that the method yields small numerical error when using different Reynolds number regimes. In particular, for a very small Reynolds number regime, the numerical errors are large when compared with the other simulations. This phenomenon indicates that the method presents characteristics to be employed to turbulent flows [14]. Table 2 shows numerical errors when applying the Hopmoc method to Eq. (2) using $t_f = 10^{-4}$, $Re = 5.0$, and $v_x = v_y = 1.0$.

Table 1. Results yielded by the Hopmoc method when applied to Eq. (2) using $\delta t = 10^{-5}$, $t_f = 0.8$, $v_x = v_y = 1.0$, and several Reynolds number regimes (t denotes time and s denotes seconds).

Mesh	Re	$\|\epsilon\|_\infty$	$\|\epsilon\|_2$	t(s)
10×10	1.00	2.2104e−02	1.2495e−02	0.4
20×20		2.2540e−02	1.2472e−02	0.4
30×30		2.2653e−02	1.2453e−02	0.4
40×40		2.2703e−02	1.2443e−02	0.6
50×50	100.00	1.8454e−02	4.0593e−03	0.7
	10.00	1.8454e−02	4.5660e−03	0.7
	1.00	2.2740e−02	1.2436e−02	0.6
	0.10	3.2200e−02	2.0704e−02	0.6
	0.01	3.5453e−02	2.3400e−02	0.7

Table 3 shows the results yielded by the Hopmoc method in simulations using different values for δt. In simulations using the same Δx and Δy, we conclude that when δt tends to zero, the numerical errors of the method tend to zero. These results show that when respecting conditions of consistency, the Hopmoc is unconditionally stable.

Table 2. Results yielded by the Hopmoc method when applied to Eq. (2) using $Re = 5.0$, $t_f = 10^{-4}$, $v_x = v_y = 1.0$.

$\delta t \times 10^{-5}$	Mesh	$\|\epsilon\|_\infty$	$\|\epsilon\|_2$	t(s)
1.0000	10×10	2.2104e−02	1.2495e−02	0.4
0.2500	20×20	2.2541e−02	1.2473e−02	1.7
0.1111	30×30	2.2655e−02	1.2455e−02	4.3
0.0625	40×40	2.2705e−02	1.2444e−02	10.9
0.0400	50×50	2.2741e−02	1.2437e−02	17.4

Table 3. Results yielded by the Hopmoc method when applied to Eq. (2) using $Re = 1.0$, $t_f = 0.8$, 50×50 grid points, and $v_x = v_y = 1.0$.

$\delta t \times 10^{-5}$	$\|\epsilon\|_\infty$	$\|\epsilon\|_2$	t(s)
100.00	8.2135e+04	1.1126e+04	0.01
10.00	2.2725e−02	1.2419e−02	0.07
1.00	2.2740e−02	1.2436e−02	0.62
0.10	2.2741e−02	1.2437e−02	6.90
0.01	2.2741e−02	1.2437e−02	68.06

3.2 Burgers Equation

The Burgers equation is a simple equation used to comprehend the properties of the Navier-Stokes equations. Furthermore, one obtains the Burgers equation as a result of linking nonlinear wave motion with linear diffusion. As previously mentioned, it is the simplest model for analyzing the combined effect of nonlinear advection and diffusion. As in the Navier-Stokes equations, one can determine a Reynolds number, which indicates the ratio between the advective and the viscous contribution in a flow. The simulation of the flow evolution requires the use of accurate and robust numerical methods [15]. Thus, the Burgers equation arises as a mathematical model from many physical events. Examples of these situations are environmental protection, the flow of a shock wave traveling in a viscous fluid, fluid mechanics, gas dynamic, hydrology, nonlinear acoustics,

sedimentation of two kinds of particles in fluid suspensions under the effect of gravity. Other examples are shallow-water waves, acoustic transmission, traffic flow, turbulence, supersonic flows, and wave propagation in a nonlinear thermoelastic media (see [16] and references therein).

This section shows two experiments. The first compares the results yielded by the Hopmoc method with the Crank-Nicolson method and an alternating direction implicit scheme. The second compares the results produced by the Hopmoc algorithm with six other existing methods.

Comparison with Crank-Nicolson Method and an Alternating Direction Implicit Scheme. This section compares the results yielded by the Hopmoc method with Crank-Nicolson (CN) [17] and the alternating direction implicit (ADI) scheme presented by Saqib et al. [18] when applied to the equation $u_t + uu_x + uu_y - \frac{1}{Re}(u_{xx} + u_{yy}) = 0$ where $(x, y, t) \in \Omega \times (0, T]$ with initial conditions $u(x, y, 0) = u_0(x, y)$, $(x, y) \in \Omega$ given by the exact solution $u(x, y, t) = \frac{1}{1+\frac{(x+y+t)Re}{2}}$ defined in $\Omega = [(x, y) : 0 \leq x, y \leq 1]$. Table 4 (5) shows the results yielded by the Hopmoc method, Crank-Nicolson method, and the alternating direction implicit scheme presented by Saqib et al. [18] in typical grid points when applied to Burgers equation using Reynolds number $Re = 200$, $t_f = 3.0$ ($t_f = 1.0$), $\delta t = 0.0001$, and 20×20 (30×30) grid points.

Table 4. Results yielded by the Hopmoc method, Crank-Nicolson method, and an alternating direction implicit scheme [18] when applied to Burgers equation using Reynolds number $Re = 200$, $t_f = 3.0$, $\Delta t = 0.0001$, and 20×20 grid points.

(x, y)	Exact	Hopmoc	ADI	CN
(0.20,0.20)	1.0000	1.0000	1.0000	1.0000
(0.05,0.05)	1.0000	1.0000	1.0000	1.0000
(0.20,0.05)	1.0000	1.0000	1.0000	1.0000
(0.15,0.15)	1.0000	1.0000	0.9999	0.9998
(0.30,0.30)	1.0000	1.0000	0.9999	0.9998
(0.30,0.20)	1.0000	1.0000	0.9997	0.9999
(0.70,0.70)	1.0000	1.0000	1.0000	1.0000
(0.70,0.15)	1.0000	1.0000	0.9999	0.9998
(0.80,0.30)	1.0000	1.0000	0.9889	0.9998
(0.90,0.90)	1.0000	1.0000	1.0000	1.0000
(0.25,0.75)	1.0000	0.9988	0.9888	0.9887
(0.75,0.25)	1.0000	1.0000	0.9888	0.9887

Table 5. Results yielded by the Hopmoc method, Crank-Nicolson and an alternating direction implicit scheme [18] when applied to Burgers equation using Reynolds number $Re = 200$, $t_f = 1.0$, $\Delta t = 0.0001$, and 30×30 grid points.

(x, y)	Exact	Hopmoc	ADI	CN
(0.2,0.2)	1.0000	1.0000	1.0000	1.0000
(0.3,0.3)	1.0000	1.0000	0.8515	0.8415
(0.3,0.2)	1.0000	1.0000	0.9000	0.9112
(0.7,0.7)	1.0000	1.0000	0.7515	0.7414
(0.8,0.3)	1.0000	1.0000	0.9876	0.9866
(0.9,0.9)	1.0000	0.9996	0.9996	0.9954

Comparison with Six Other Existing Methods This section illustrates the accuracy of the Hopmoc method when applied to the system of 2-D Burgers equations

$$\begin{cases} u_t + uu_x + vu_y = \frac{1}{Re}(u_{xx} + u_{yy}) \\ v_t + uv_x + vv_y = \frac{1}{Re}(v_{xx} + v_{yy}) \end{cases} \tag{3}$$

with initial conditions $u(x, y, 0) = f(x, y)$, $v(x, y, 0) = g(x, y)$, $(x, y) \in \Omega$, and boundary conditions $u(x, y, t) = \varphi(x, y, t)$, $v(x, y, t) = \psi(x, y, t)$, $x, y \in \partial\Omega$, $t > 0$ where $\Omega = \{(x, y) : a \leq x \leq b, a \leq y \leq b\}$ and $\partial\Omega$ is its boundary, f, g, φ, and ψ are known functions, $u(x, y, t)$ and $v(x, y, t)$ are the velocity componentes to be determined.

We consider the system of 2-D Burgers Eq. (3) with exact solutions $u(x, y, t) = v(x, y, t) = \frac{3}{4} - \frac{1}{4\left(1+e^{(-4x+4y-t)\frac{Re}{32}}\right)}$ using $\Delta x = \Delta y = 0.05$ and $Re = 100$. The initial and boundary conditions are taken from the exact solution. The computational domain is $\Omega = \{(x, y) : 0 \leq x, y \leq 1\}$. Table 6 compares the numerical solutions yielded by the Hopmoc method with exact solution and six existing methods:

- an exponential finite-difference (Expo-FD) method [19];
- an implicit logarithmic finite-difference method (I-LFDM) [20];
- a modified cubic B-spline differential quadrature (MCBDQ) method [21];
- a splitting method (SM) proposed by Shi et al. [16];
- a Crank-Nicolson/Adams-Bashforth (CNAB) scheme [22]
- the alternating direction implicit (ADI) method proposed by Çelikten and Aksan [23].

Table 6 shows the results yielded by seven methods when applied to the system of Burgers Eq. (3) for $k = 0.0001$ and $t_f = 0.5$ at typical grid points. The table shows that the results yielded by the Hopmoc method are compatible with the six other methods.

Table 6. Comparison of numerical and exact solutions of $u(x, y, t)$ and $v(x, y, t)$ for $Re = 100$, $k = 0.0001$, $\Delta x = \Delta y = 0.05$, and $t_f = 0.5$ at some typical grid points.

	(x, y)	Exact	SM	MCBDQ	Expo-FD	I-LFDM	ADI	CNAB	Hopmoc
$u(x, y, t)$	(0,1,0,1)	0.54332	0.54336	0.54412	0.54300	0.54300	0.54299	0.55979	0.54325
	(0,5,0,1)	0.50035	0.50035	0.50037	0.50034	0.50034	0.50034	0.50160	0.50041
	(0,9,0,1)	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50001	0.50000
	(0,3,0,3)	0.54332	0.54338	0.54388	0.54270	0.54269	0.54268	0.54844	0.53957
	(0,7,0,3)	0.50035	0.50034	0.50037	0.50032	0.50032	0.50032	0.50155	0.50051
	(0,1,0,5)	0.74221	0.74228	0.74196	0.74215	0.74215	0.74215	0.74510	0.73468
	(0,5,0,5)	0.54332	0.54333	0.54347	0.54252	0.54251	0.54249	0.54733	0.53397
	(0,9,0,5)	0.50035	0.50034	0.50035	0.50030	0.50030	0.50030	0.50155	0.50056
	(0,3,0,7)	0.74221	0.74236	0.74211	0.74212	0.74211	0.74211	0.74324	0.71934
	(0,7,0,7)	0.54332	0.54332	0.54327	0.54247	0.54246	0.54245	0.54733	0.52921
	(0,1,0,9)	0.74995	0.74995	0.74994	0.74994	0.74994	0.74994	0.74996	0.74981
	(0,5,0,9)	0.74221	0.74237	0.74219	0.74210	0.74210	0.74210	0.74330	0.70472
	(0,9,0,9)	0.54332	0.54331	0.54333	0.54229	0.54228	0.54227	0.54616	0.52726
$v(x, y, t)$	(0,1,0,1)	0.95668	0.956650	0.95589	0.95700	0.95700	0.95701	0.95021	0.95675
	(0,5,0,1)	0.99965	0.99965	0.99963	0.99966	0.99966	0.99966	0.99840	0.99959
	(0,9,0,1)	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	1.00000
	(0,3,0,3)	0.95668	0.95662	0.95612	0.95731	0.95731	0.95732	0.95156	0.96043
	(0,7,0,3)	0.99965	0.99966	0.99964	0.99968	0.99968	0.99968	0.99845	0.99949
	(0,1,0,5)	0.75779	0.75772	0.75804	0.75785	0.75785	0.75785	0.75490	0.76532
	(0,5,0,5)	0.95668	0.95667	0.95654	0.95749	0.95749	0.95751	0.95267	0.96603
	(0,9,0,5)	0.99965	0.99966	0.99965	0.99970	0.99970	0.99970	0.99845	0.99944
	(0,3,0,7)	0.75779	0.75764	0.75789	0.75789	0.75789	0.75789	0.75676	0.78066
	(0,7,0,7)	0.95668	0.95668	0.95673	0.95754	0.95754	0.95755	0.95267	0.97079
	(0,1,0,9)	0.75005	0.75005	0.75006	0.75006	0.75006	0.75006	0.75004	0.75019
	(0,5,0,9)	0.75779	0.75763	0.75781	0.75790	0.75790	0.75790	0.75670	0.79528
	(0,9,0,9)	0.95668	0.95669	0.95667	0.95772	0.95772	0.95773	0.95384	0.97274

4 Conclusions

This paper shows the computational results yielded by the Hopmoc method when applied to the two-dimensional advection-diffusion equation. The Hopmoc algorithm yielded better results than did the Crank-Nicolson and the alternating direction implicit scheme presented by Saqib et al. [18] when applied to the advection-diffusion equation. The results yielded by the Hopmoc method when applied to the 2-D Burgers equation are compatible with six other recent studies presented in the literature.

We plan to integrate the Hopmoc method with backward differential formulas in future works. We also intend to study the Hopmoc algorithm applied to the two-dimensional advection-diffusion equation in conjunction with total variation diminishing techniques [24] and flux-limiting procedures to improve its accuracy.

Researchers solve very large-scale problems in the present day with the use of parallel computations. Massive problems today are of the order of billions of degrees of freedom (as examples, see experiments in parallel using structural [25] and computational fluid dynamics problems [26]). We plan to apply the Hopmoc method in parallel implementations using OpenMP, Galois, and Message Passing Interface systems.

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