

Properties of Multipyramidal Elements

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Abstract. The finite element method is based on the division of the physical domains into a large number of small polytopes with simple geometry. Basically, the most useful finite elements can be divided into two large groups: simplicial elements and hypercubic elements. To keep conformity, triangulating of curved domains with complex geometry requires the usage of various kinds of transitional elements, which are specific for any fixed Euclidean space. The paper deals with a basic problem of the finite element method in the multidimensional spaces - conforming coupling between hypercubic and simplicial meshes. Here we focus on the bipyramidal elements. Some properties of such kind elements are discussed in an arbitrary Euclidean space with a dimension greater than two.

Keywords: Transitional finite elements · Conforming coupling · Pyramidal elements · Bipyramids · Bihexatera

AMS Subject Classifications 65N30 · 52B11

1 Introduction

Most of the papers on the finite element method deal with numerical analysis and error estimate in numerical approximations of the eigenvalue and boundary value problem solutions. But the successful application of the finite element method depends on the quality of the finite element meshes. The quality of meshes is of significant importance especially when the isoparametric multigrid method is applied to solve elliptic problems in domains with complex geometry. The multigrid methods require successive refinements of the initial triangulation. That is why the stability of the sequences of successive triangulations is a crucial point for convergence of the multigrid approximations. The theory of the finite element meshes and advanced discretization methods can be considered as a separate scientific area closely related to the finite element method. We emphasize that the finite element meshes have been applied for computer graphic simulations without solving any boundary or eigenvalue problems. Here, we present some papers that deal with advanced discretization methods.

The efforts of various authors have been devoted to different refinement strategies. Hannukainen et al. [10], Bedregal and Rivara [3], and Perdomo and

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Plaza [20] focus their research on longest-edge bisection methods. The bisection methods have been generalized to arbitrary section methods by Korotov et al. [12]. Other refinement strategies have been demonstrated by Luo et al. [13], Petrov and Todorov [17], and Verstraaten and J. Kosinka [22]. Hermosillo-Arteaga et al. [11] present recent advances in the theory of the finite element meshes. Subdivision techniques in multidimensional Euclidean space have been studied by Brandts et al. [6], Pascucci [19], and Petrov and Todorov [16].

The pyramidal elements are relatively new finite elements. In the three-dimensional case, they have been created by Bedrosian [2] in the early nineties. The pyramids play the role of transitional elements between structured and unstructured finite element meshes. The pyramidal elements have made possible the development of the hex-dominant meshes considered by several authors [14,21,24,25]. The application of the pyramidal elements for constructing hybrid meshes has been described in [4,7–9]. Mathematical models based on composite hybrid meshes have been developed in [14,15]. In the three-dimensional domains, the conforming coupling between hexahedral and tetrahedral meshes does not require complicated transitional elements. The pyramidal elements assure conformity in a natural way, see Fig. 1. This construction has been applied by various authors, see for instance the paper by Yamakawa and Shimada [23]. The situation is completely different in the higher-dimensional spaces. A tesseract pyramid cannot be coupled directly with an arbitrary hexateron since all facets of the hexateron are pentatopes but the pyramid has not got any simplicial four-dimensional facet. Therefore, the hybrid meshes construction in higher-dimensional spaces needs a different approach. New transitional finite elements should be developed.

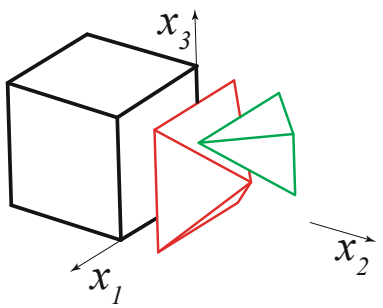

Fig. 1. A conforming coupling between hexahedron and tetrahedron finite elements.

This paper is devoted to the geometrical foundations of the finite element method and in particular the theory of meshes. One of the fundamental problems of the finite element method in an arbitrary n-dimensional Euclidean space is solved in this paper. New k-pyramidal elements are obtained and their properties are discussed. Specifically, in the five-dimensional space new cubic bipyramids

and bihexateron elements are developed. A conforming coupling between penteracts and hexatera is demonstrated. Bipyramidal elements in arbitrary higher-dimensional space are uniformly refined. A plenty of new notions like k-pyramidal elements, conforming chains etc. are introduced.

Further, the paper is organized as follows. New k-pyramidal elements are defined and investigated in Sect. 2. A conforming ensemble of penteract and hexateron meshes are described in Sect. 3. Concluding remarks are presented in Sect. 4.

2 Subdivision Properties of the Bipyramidal Elements

Throughout the whole paper, the upper index stands for the dimension of the polytope.

Definition 1. *Two polytopes W_1 and W_2 are called elements from the same congruence class if one of them is obtained from the other by a rigid motion.*

In the early forties, Freudenthal has divided the n-dimensional hypercube [5] into $n!$ simplicial elements all of them from the same class. We call these simplices Freudenthal elements.

Definition 2. *We define the class $[\hat{E}^n]$ [16] of all simplices that are congruent to the quadruple Freudenthal element*

$$\hat{E}^n = [\hat{e}_1(2,0,\ldots,0),\ \hat{e}_2(2,2,0\ldots,0,0),$$

$$\hat{e}_3(1,1,1\ldots,0,0),\ldots,\hat{e}_{n-1}(1,1,\ldots,1,0),$$

$$\hat{e}_n(1,1,\ldots,1,1),\ \hat{e}_{n+1}(0,0,\ldots,0)].$$

Example 1. *In the five-dimensional case the class $[\hat{E}^5]$ is defined by the hexateron*

$$\hat{E}^5 = [\hat{e}_1(2,0,0,0,0), \hat{e}_2(2,2,0,0,0), \hat{e}_3(1,1,1,0,0),$$

$$\hat{e}_4(1,1,1,1,0), \hat{e}_5(1,1,1,1,1), \hat{e}_6(0,0,0,0,0)].$$

The elements of the class $[\hat{E}^n]$ can be obtained by coupling four Freudenthal elements. We denote the gravity center and the set of all vertices of the polytope W by $G(W)$ and $V(W)$ correspondingly.

Definition 3. *The polytope W_{-i} is obtained from $W[w_1, w_2, \ldots, w_{n+1}]$, $n \in \mathbf{N}$, by removing the node with the number i.*

Definition 4. *Let C^{n-1} be a $(n-1)$-dimensional facet of the hypercube C^n. The $(n-1)$-dimensional hypercubic pyramid $P^n[V(C^{n-1}), G(C^n)]$ is called canonical.*

The sequence $\{\nu_n\}$ is defined by $\nu_n = 2^{n-2} + 2$.

Definition 5. *The bipyramid B^n is a n-dimensional polytope with ν_n, $n \geq 3$ vertices b_i that satisfy:*

- *b_i $i = 1, 2, \ldots, n-2$ are vertices of $(n-2)$-dimensional hypercube C^{n-2};*
- *the line segment $b = [b_{\nu_n - 1}, b_{\nu_n}]$ is not collinear to the $(n-2)$-dimensional subspace of $\mathbf{R}^n$ determined by the hypercube C^{n-2};*
- *$b \cap C^{n-2} = \emptyset$.*

We consider some cases that clarify the role of the bipyramids in the lower-dimensional Euclidean spaces.

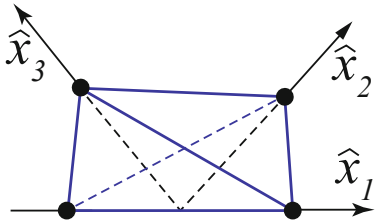

Fig. 2. The reference three-dimensional bipyramid.

The two-dimensional pyramid is the well-known triangle but a two-dimensional bipyramid does not make sense. Therefore, we begin with the three-dimensional case. The reference three-dimensional bipyramid is presented in Fig. 2. From Definition 5 with $n = 3$, we see that the three-dimensional bipyramid is actually a simplex but this is valid only in the three-dimensional case. The square pyramid is a transitional element in the interface subdomain between structured and unstructured meshes. Multigrid methods require stable refinement of all elements in the coarse triangulation. The optimal refinement strategy, see Fig. 3 for the square pyramidal elements have been described by Ainsworth and Fu [1]. All elements

$$[b_6, b_{10}, b_{11}, b_{12}],\ [b_7, b_{10}, b_{12}, b_{13}],\ [b_8, b_{10}, b_{13}, b_{14}],\ [b_9, b_{10}, b_{11}, b_{14}]$$

are three-dimensional bipyramidal elements (tetrahedra). The chain

$$\text{CUBE} \rightleftarrows \text{PYRAMID} \rightleftarrows \text{TETRAHEDRON}$$

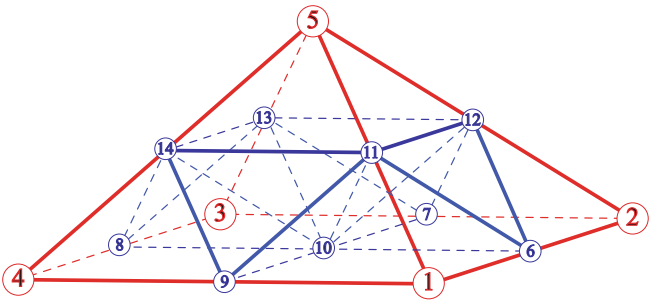

Fig. 3. The optimal partition of the square pyramid.

assure conforming coupling between structured and unstructured three-dimensional meshes, see Fig. 1.

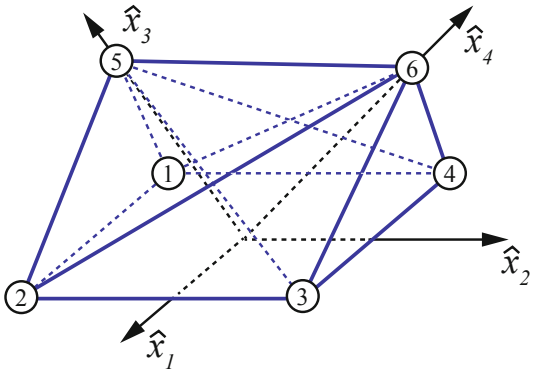

Fig. 4. The reference bipyramid.

The cubic pyramidal elements cannot guarantee conforming coupling between tesseract and pentatope meshes in the four-dimensional case. For this purpose, Petrov et al. [18] have created an additional bipyramidal finite element, Fig. 4. The four-dimensional bipyramid $B[b_i, i = 1, 2, \ldots, 6]$ can be divided into two pentatopes

$$B = S_1[b_1, b_2, b_3, b_5, b_6] \cup S_2[b_1, b_3, b_4, b_5, b_6],$$

that is why it could also be called bipentatope. The transitional four-dimensional bipyramid is related to tesseract partitions in the interface subdomains. Let us consider a tesseract element C from the interface subdomain. The transitional bipyramid is constructed as follows: the sixth and the fifth vertices are the tesseract center and the center of a three-dimensional facet C^3 of the element C; the other four nodes form a two-dimensional facet of C^3. The cubic bipyramid has also been applied to refine an arbitrary canonical cubic pyramid. Petrov et al.

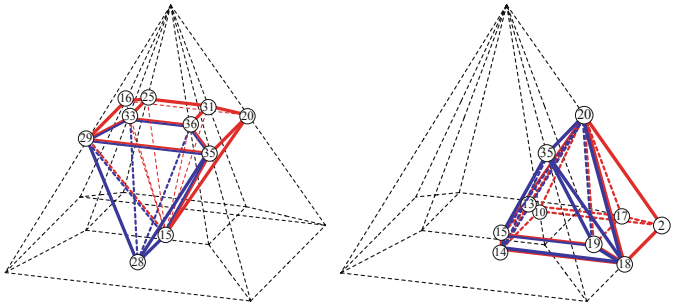

Fig. 5. Couplings between a cubic pyramid and a bipentatope.

have defined a partition operator that divides a canonical cubic pyramid into cubic pyramids and cubic bipyramids, Fig. 5. They have proved that all bipyramids from this partition are invariant and from the same congruence class. Each nondegenerated four-dimensional bipyramid B^4 (see Fig. 4) has two square pyramid facets B^4_{-5} and B^4_{-6}, and four tetrahedron facets

$$[b_1, b_2, b_5, b_6],\ [b_1, b_4, b_5, b_6],\ [b_2, b_3, b_5, b_6],\ [b_3, b_4, b_5, b_6].$$

The conformity in the four-dimensional space is assured by the chain

$$\text{TESSERACT} \rightleftarrows \text{CUBIC PYRAMID} \rightleftarrows \text{SQUARE BIPYRAMID} \rightleftarrows \text{PENTATOPE}\ .$$

The bipyramids has a significant advantages since the reference bipyramid can be easily divided into a set of congruent elements. To this end let us consider the five-dimensional reference element

$$\hat{B}^5 = [\hat{b}_1(-1,-1,-1,0,0), \hat{b}_2(1,-1,-1,0,0), \hat{b}_3(1,1,-1,0,0),$$

$$\hat{b}_4(-1,1,-1,0,0), \hat{b}_5(-1,-1,1,0,0), \hat{b}_6(1,-1,1,0,0),$$

$$\hat{b}_7(1,1,1,0,0), \hat{b}_8(-1,1,1,0,0), \hat{b}_9(0,0,0,1,0), \hat{b}_{10}(0,0,0,0,1)],$$

see Fig. 6. The base of $\hat{B}^5$ can be divided into six Freudenthal elements $\hat{F}^3_i$. Then

$$\hat{B}^5 = [\hat{H}_i,\ i = 1, 2, \ldots 6,\ b_9, b_{10}],$$

$$\hat{H}_i = [V(\hat{F}^3_i),\ b_9, b_{10}].$$

On the other hand, the pyramidal elements cannot be divided only into pyramidal elements. Dividing a pyramidal element so that each edge to be partitioned into two segments we obtain gaps that are bipyramids. For a three-dimensional example we refer the reader to the paper written by Ainsworth and Fu [1], see also Fig. 3, and for the four-dimensional one to the paper by Petrov et al. [18]. Each penteract can be divided into 80 bipyramids.

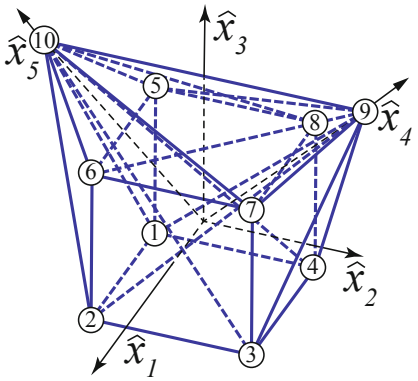

Fig. 6. The five-dimensional reference bipyramid.

Let us continue with the n-dimensional case. We introduce the ν_n-node reference element

$$\hat{B}_k^n \ : \ \begin{cases} -(1-\hat{x}_{n-1}-\hat{x}_n) \le \hat{x}_i \le 1-\hat{x}_{n-1}-\hat{x}_n, \quad i=1,2,\ldots,n-2 \\ 0 \le \hat{x}_{n-1} \le 1-\hat{x}_n, \\ 0 \le \hat{x}_n \le 1 \end{cases} .$$

The largest index in the local numbering of each pyramidal element is reserved for the apex. Analogously, both apexes of the bipyramid are locally numbered by the largest indices.

Definition 6. *The set of n-dimensional hypercube, n-dimensional simplex, and all transitional elements that assure a conforming coupling between them is said to be a conforming chain.*

The whole spectre of the reference transitional elements in the n-dimensional Euclidean space with a integer $n \ge 3$ looks as follows

$$\hat{B}_k^n \ : \ \begin{cases} -\left(1-\sum_{j=k+1}^n \hat{x}_j\right) \le \hat{x}_i \le 1-\sum_{j=k+1}^n \hat{x}_j, \ \ i=1,2,\ldots,k \\ 0 \le \hat{x}_i \le 1-\sum_{j=i+1}^n \hat{x}_j, \ \ i=k+1,k+2,\ldots,n-1 \\ 0 \le \hat{x}_n \le 1 \end{cases} .$$

Definition 7. *All affine equivalent elements to the element $\hat{B}_k^n$ are called transitional elements.*

The particular cases when $k=1,\ n-2,\ n-1$ are of significant importance. The element B_1^n is a simplex, B_{n-2}^n is a bipyramid, and B_{n-1}^n is a hypercubic pyramid. Following these denotations we can say that the polytope B_{n-k}^n is a n-dimensional k-pyramid.

Further, we concentrate on the properties of the bipyramids. Describing the properties of the bipyramids we omit the index $n-2$ and write B^n instead of B^n_{n-2} for notational simplicity. We consider two canonical pyramids P_1^n and P_2^n that share a common $(n-2)$-hypercubic pyramid $P_{1,2}^{n-1}$. (Note that each five-dimensional canonical pyramid is a tesseract pyramid.) We suppose that the pyramid P_1^n shares a common facet with a hypercubic element $\overline{C}^n$. The latter means that P_1^n should not be refined. Additionally, the transitional elements are only located in the interface subdomain, which is canonical. Moreover, the facet $P_{1,2}^{n-1}$ should not be divided. Therefore, the transitional bipyramid B^n, in this case, should be defined by

$$B^n = [V(P_{1,2}^{n-1}), G(P^n_{2,-(2^{n-1}+1)})]. \tag{1}$$

Let

$$\mu_n = \begin{cases} 1, & \text{if } n = 3; \\ 2^{n-4}(n-2)!, & n \geq 4. \end{cases}$$

Theorem 1. *The n-dimensional transitional bipyramid B^n $n \geq 3$ can be partitioned uniformly into μ_n quadruple Freudenthal elements.*

Proof. Let B^n be a transitional bipyramid defined by (1). Both pyramids P_1^n and P_2^n determine a hypercube C^n with a center the common apex of both pyramids. The bases of P_2^n and $P_{1,2}^{n-1}$ are denoted by C^{n-1} and C^{n-2}.

(i) The three-dimensional case is trivial since the bipyramid B^3 is a single indecomposable quadruple Freudenthal element.

(ii) We consider the four-dimensional case as an exception of the general case when $n \geq 5$. The transitional bipyramid B^4 has six nodes. We define a pentatope $S[s_1, s_2, \ldots, s_5]$ by:

- the vertex s_5 is chosen to be the center of the tesseract determined by the cubic pyramids P_1^4 and P_2^4;
- the vertex s_4 is the center of $P^4_{2,-9}$;
- the vertices s_i $i = 1, 2, 3$ form a square corner of $P^3_{1,2,-5}$.

Obviously, we can create only two pentatopes following this construction. Both of them belong to $[\hat{E}]$.

Further, we suppose that $n \geq 5$. We define a simplex $S^n[s_1, s_2, \ldots, s_{n+1}]$ as follows. For the last three vertices we have $s_{n+1} = G(C^n)$, $s_n = G(C^{n-1})$ and $s_{n-1} = G(C^{n-2})$. The next $n-5$ vertices are defined in a similar way $s_i = G(C^{i-1})$, $i = 4, 5, \ldots, n-2$. Here, C^k is a k-dimensional facet of the hypercube C^{n-2}. The first three vertices s_i $i = 1, 2, 3$ form a square corner in a two-dimensional facet of C^{n-2}. The element $S^n \in [\hat{E}^n]$ by construction [16] (see the definition of the class $[\hat{P}^n]$ in [16] and [16, Theorem 9]).

The hypercube C^{n-1} can be partitioned into $2^{n-2}(n-1)!$ elements of class $[\hat{E}^n]$ but only μ_n are inside the pyramid $B^n_{-\nu_n}$. Adding the vertex b_{ν_n} of B^n to

each of these $(n-1)$-dimensional simplices we obtain μ_n quadruple Freudenthal elements. Thus the bipyramid B^n is divided by

$$B^n = \bigcup_{i=1}^{\mu_n} S_i^n, \;\; \mathrm{vol}_n(S_i^n \cap S_j^n) = 0, \; i \neq j, \;\; S_i^n \in [S^n].$$

The main idea of the finite element method is to calculate all integrals in the finite element of reference. Theorem 1 guarantees that integrals in an arbitrary transitional element

$$\int_{B^n} f(x)dx = \sum_{i=1}^{\mu_n} \int_{S_i^n} f(x)dx, \;\; S_i^n \in [\hat{E}^n] \;\; \forall i = 1, 2, \ldots, \mu_n$$

can be replaced by integrals in simplicial elements all of them from the same class. This is a significant advantage with respect to the application of the bipyramidal elements. Similar results can be obtained for the other n-dimensional k-pyramids.

Having in mind that the polytopes B_k^n and B_{k+1}^n can share B_k^{n-1} facet the conforming chain in $\mathbf{R}^n$ looks as follows

$$C^n \rightleftarrows B_{n-1}^n \rightleftarrows B_{n-2}^n \rightleftarrows \ldots \rightleftarrows B_2^n \rightleftarrows B_1^n = S^n,$$

where C^n is a n-dimensional hypercube and S^n is a n-dimensional simplex.

3 Transitional Elements in the Five-Dimensional Case

In this section, we restrict ourselves to the five-dimensional space. Let Ω be a bounded simply connected five-dimensional domain with a curved Lipschitz-continuous boundary $\Gamma = \partial\Omega$. To obtain an efficient triangulation of the domain Ω we divide it into several subdomains. We separate as large as possible canonical subdomain $\hat{\Omega} \subset \Omega$. The boundary layer Ω_B is connected to $\hat{\Omega}$ by an interface subdomain Ω_I, which is also a canonical domain. The domain $\hat{\Omega}$ is triangulated by penteract elements and the boundary layer by curved simplicial elements. The transitional elements are located in the interface subdomain Ω_I.

Let $\hat{\Pi} \subset \hat{\Omega}$ and $\Pi_I \subset \Omega_I$ be two adjacent penteract elements that share a common facet

$$T_2[\pi_1, \pi_2, \pi_4, \pi_7, \pi_9, \pi_{10}, \pi_{12}, \pi_{15}, \pi_{17}, \pi_{18}, \pi_{20}, \pi_{23}, \pi_{25}, \pi_{26}, \pi_{28}, \pi_{31}],$$

see Fig. 7. Our goal is to decompose the penteract Π_I into transitional and simplicial elements. First, we divide the element Π_I into ten tesseract pyramids P_i. The tesseract pyramid $P_2[V(T_2), \pi_{33}]$, should not be refined since P_2 and $\hat{\Pi}$ share the common facet $P_{2,-17}$. All neighbors of P_2 are refined by the same way. The opposite pyramid P_8 of the element P_2 is refined only by simplicial elements. That is why we show how one of the neighbors

$$P_1 = [T_1[\pi_i, \; i = 1, 2, \ldots, 16], \pi_{33}]$$

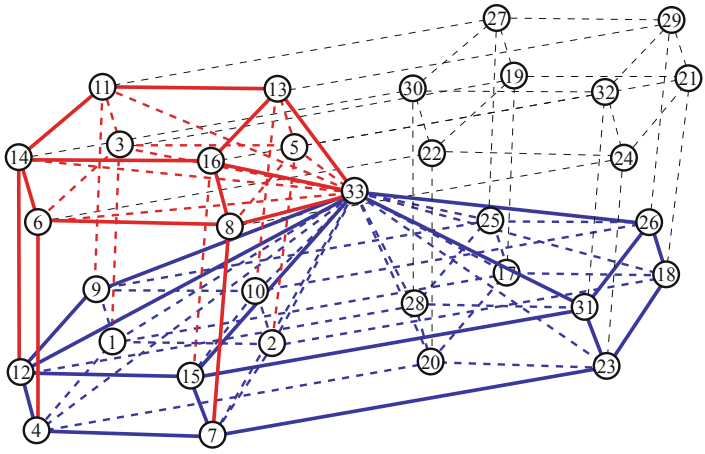

Fig. 7. The penteract Π_I.

should be refined, see Fig. 8. The pyramids P_1 and P_2 share a common cubic pyramid

$$P_{1,2}[\pi_1, \pi_2, \pi_4, \pi_7, \pi_9, \pi_{10}, \pi_{12}, \pi_{15}, \pi_{33}].$$

The pyramid $P_{1,2}$, see Fig. 7 and its facets should not be divided. Additionally, we denote the penteract center and the center of the facet T_1 by π_{33} and π_{34}, correspondingly. The point π_{35} is the center of $P_{1,2,-9}$.

We introduce a partition operator $\mathcal{L}$ that refine the bipyramid

$$B_3^5 = [b_i = \pi_i,\ i = 1, 2, \ldots, 10]$$

into twelve hexatera, see Fig. 8 and 9, as follows:

$$\mathcal{L}B = \Big\{[b_1, b_2, b_4, b_9, b_{10}, b_{11}],\ \ [b_1, b_2, b_5, b_9, b_{10}, b_{11}],$$
$$[b_1, b_4, b_5, b_9, b_{10}, b_{11}],\ [b_2, b_3, b_4, b_9, b_{10}, b_{11}],\ [b_2, b_3, b_6, b_9, b_{10}, b_{11}],$$
$$[b_2, b_5, b_6, b_9, b_{10}, b_{11}],\ [b_3, b_4, b_7, b_9, b_{10}, b_{11}],\ [b_3, b_6, b_7, b_9, b_{10}, b_{11}],$$
$$[b_4, b_5, b_8, b_9, b_{10}, b_{11}],\ [b_4, b_7, b_8, b_9, b_{10}, b_{11}],\ [b_5, b_6, b_8, b_9, b_{10}, b_{11}],$$
$$[b_6, b_7, b_8, b_9, b_{10}, b_{11}]\Big\}.$$

Let $H[h_i,\ i = 1, 2, \ldots, 6]$ be an arbitrary hexateron belonging to $\mathcal{L}B$. Then h_6 is the center of the penteract Π_I, h_5 is the center of the tesseract T_1, and h_4 is the center of a three-dimensional facet $P_{1,2,-9}$ of T_1. The other three vertices h_i $i = 1, 2, 3$ form a square corner in a two-dimensional facet of $P_{1,2,-9}$. This construction is valid for all elements of $\mathcal{L}B$ and guarantees that H belongs to $[\hat{E}^5]$.

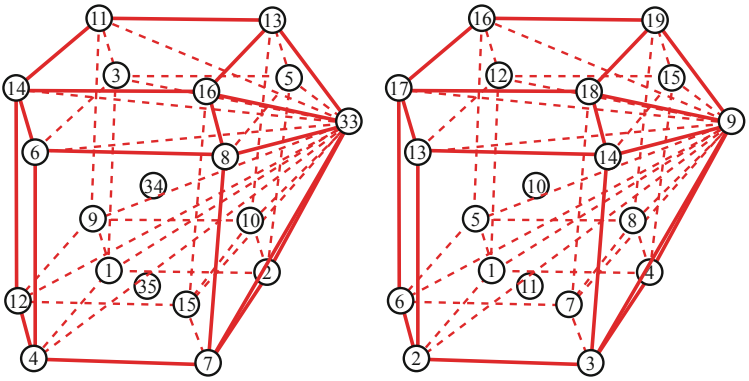

Fig. 8. The global and the local numbering of the tesseract pyramid P_1.

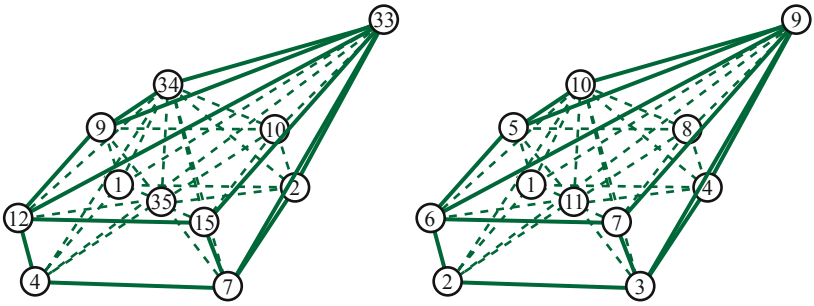

Fig. 9. The global and the local numbering of the cubic bipyramid B^5.

Any nondegenerated five-dimensional bipyramid (see Fig. 6) has two facets B_{-10} and B_{-9} that are cubic pyramids and six facets

$$[b_1, b_2, b_3, b_4, b_9, b_{10}],\ [b_1, b_2, b_5, b_6, b_9, b_{10}],\ [b_1, b_4, b_5, b_8, b_9, b_{10}],$$

$$[b_2, b_3, b_6, b_7, b_9, b_{10}],\ [b_3, b_4, b_7, b_8, b_9, b_{10}],\ [b_5, b_6, b_7, b_8, b_9, b_{10}]$$

that are four-dimensional bipyramids (see Fig. 4). The latter means that a five-dimensional bipyramid cannot be coupled conformingly with an arbitrary hexateron. To enable conforming coupling between penteract and simplicial elements in the five-dimensional space, we need another seven-node transitional element B_2^5. The new element is called bihexateron, see Fig. 10, and the reference element is defined as follows

$$\hat{B}_2^5 \ : \ \begin{cases} -\left(1 - \sum_{j=3}^5 \hat{x}_j\right) \le \hat{x}_i \le 1 - \sum_{j=3}^5 \hat{x}_j, \quad i = 1, 2 \\ 0 \le \hat{x}_3 \le 1 - \hat{x}_4 - \hat{x}_5, \\ 0 \le \hat{x}_4 \le 1 - \hat{x}_5, \\ 0 \le \hat{x}_5 \le 1 \end{cases} .$$

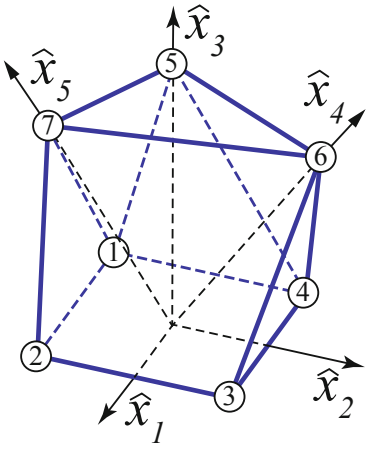

Fig. 10. The reference bihexateron $\hat{B}_2^5$.

An arbitrary nondegenerated bihexateron has seven facets, three of them

$$[\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4, \hat{b}_5, \hat{b}_6],\ [\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4, \hat{b}_5, \hat{b}_7],\ [\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4, \hat{b}_6, \hat{b}_7]$$

are four-dimensional bipyramids and other four are

$$[\hat{b}_1, \hat{b}_2, \hat{b}_5, \hat{b}_6, \hat{b}_7],\ [\hat{b}_1, \hat{b}_4, \hat{b}_5, \hat{b}_6, \hat{b}_7],\ [\hat{b}_2, \hat{b}_3, \hat{b}_5, \hat{b}_6, \hat{b}_7],\ [\hat{b}_3, \hat{b}_4, \hat{b}_5, \hat{b}_6, \hat{b}_7]$$

are pentatopes. Therefore B_3^5 and B_2^5 can share B_2^4. Obviously, a bihexateron and a hexateron can be connected conformingly since the nondegenerated hexateron has six pentatopial facets. Thus, we can assure a conforming coupling between penteracts and hexatera by applying the following chain

$$\text{PENTERACT} \rightleftarrows \text{TESSERACT PYRAMID} \rightleftarrows \text{BIPYRAMID} \rightleftarrows$$
$$\text{BIHEXATERON} \rightleftarrows \text{HEXATERON}\ .$$

4 Conclusion

Two kinds of new transitional finite elements are defined in the five-dimensional Euclidean space. The new seven-node bihexateron element and the new ten-node cubic bipyramid can share a common facet. A conforming coupling between structured and unstructured five-dimensional finite element meshes is obtained based on the chain penteract-tesseract pyramid-bipyramid-bihexateron-hexateron. All kinds of transitional elements are defined in the n-dimensional case, $n \geq 3$. The pyramidal elements are presented as a special kind of k-pyramidal elements. Similar presentation is found for the simplicial elements. The conforming chain is obtained in the n-dimensional case. A uniform refinement of the transitional

bipyramids by quadruple Freudenthal elements is obtained. The uniform refinement essentially reduces the computational complexity of the integral computations in such elements. Additionally, the new reference elements are also triangulated uniformly.

References

1. Ainsworth, M., Fu, G.: A lowest-order composite finite element exact sequence on pyramids. Comput. Methods Appl. Mech. Eng. **324**, 110–127 (2017)
2. Bedrosian, G.: Shape functions and integration formulas for three-dimensional finite element analysis. Internat. J. Numer. Methods Engrg. **35**, 95–108 (1992)
3. Bedregal, C.: Maria-Cecilia Rivara, Longest-edge algorithms for size-optimal refinement of triangulations. Comput. Aided Des. **46**, 246–251 (2014)
4. Bergot, M., Cohen, G., Duruflé, M.: Higher-order finite elements for hybrid meshes using new nodal pyramidal elements. J. Sci. Comput. **42**, 345–381 (2010)
5. Bey, J.: Simplicial grid refinement, On Freudenthal's algorithm and the optimal number of congruence classes. Numer. Math. **85**(1), 1–29 (1998)
6. Brandts, J., Korotov, S., Křížek, M.: Simplicial finite elements in higher dimensions. Appl. Math. **52**(3), 251–265 (2007)
7. Ca, C., Qin, Q.-H., Yu, A.: A new hybrid finite element approach for three-dimensional elastic problems. Arch. Mech. **64**(3), 261–292 (2012)
8. Coulomb, J.-L., Zgainski, F.-X., Marechal, Y.: A pyramidal element to link hexahedral, prismatic and tetrahedral edge finite elements. IEEE Trans. Magn. **33**(2), 1362–1365 (1997)
9. Devloo, P., Duran, O., Gomes, S., Ainsworth, M.: High-order composite finite element exact sequences based on tetrahedral-hexahedral-prismatic-pyramidal partitions. hal-02100485 (2019)
10. Hannukainen, A., Korotov, S., Křížek, M.: On numerical regularity of the face-to-face longest-edge bisection algorithm for tetrahedral partitions. Sci. Comput. Programm. **90**(Part A) (2014) Pages 34–41
11. Hermosillo-Arteaga, A., Romo-Organista, M., Magaña del Toro, R., Carrera-Bolaños, J.: Development of a refinement algorithm for tetrahedral finite elements. Rev. Int. métodos numér. cálc. diseño ing. **37**(1), 1–21 (2021)
12. Korotov, S., Plaza, Á., Suárez, J.P.: Longest-edge n-section algorithms: properties and open problems. J. Comput. Appl. Math. **293**, 139–146 (2016)
13. Luo, X., Wang, Z., Wang, M., Shi, X.: A variable-scale refinement triangulation algorithm with faults data. J. Comput. **31**(5), 212–223 (2020)
14. O'Malley, B., Kophazi, J., Eaton, M.D., Badalassi, V., Warner, P., Copestake, A.: Pyramid finite elements for discontinuous and continuous discretizations of the neutron diffusion equation with applications to reactor physics. Prog. Nucl. Energy **105**, 175–184 (2018)
15. Martinelli, L.B., Alves, E.C.: Optimization of geometrically nonlinear truss structures under dynamic loading. REM Int. Eng. J. **73**(3), 293–301 (2020)
16. Petrov, M.S., Todorov, T.D.: Properties of the multidimensional finite elements. Appl. Math. Comput. **391**, 125695 (2021)
17. Petrov, M.S., Todorov, T.D.: Refinement strategies related to cubic tetrahedral meshes. Appl. Numer. Math. **137**, 169–183 (2019)
18. Petrov, M.S., Todorov, T.D., Walters, G., Williams, D.M., Witherden, F.D.: Enabling four-dimensional conformal hybrid meshing with cubic pyramids. Numer. Algorithms (to appear)

19. Pascucci, V.: Slow growing subdivision (SGS) in any dimension: towards removing the curse of dimensionality. EUROGRAPHICS **21**(3), 451–460 (2002)
20. Perdomo, F., Plaza, Á.: Properties of triangulations obtained by the longest-edge bisection. Central Eur. J. Math. **12**(12), 1796–1810 (2014). https://doi.org/10.2478/s11533-014-0448-4
21. Ray, N., Sokolov, D., Reberol, M., Ledoux, F., Lévy, B.: Hex-dominant meshing: mind the gap! Comput. Aided Des. **102**, 94–103 (2018)
22. Verstraaten, T.W., Kosinka, J.: Local and hierarchical refinement for subdivision gradient meshes. Pac. Graph. **37**(7), 373–383 (2018)
23. Yamakawa, S., Gentilini, I., Shimada, K.: Subdivision templates for converting a non-conformal hex-dominant mesh to a conformal hex-dominant mesh without pyramid elements. Eng. Comput. **27**, 51–65 (2011)
24. Yamakawa, S., Shimada, K.: Fully-automated hex-dominant mesh generation with directionality control via packing rectangular solid cells. Int. J. Numer. Methods Eng. **57**, 2099–2129 (2003)
25. Yamakawa, S., Shimada, K.: Increasing the number and volume of hexahedral and prism elements in a hex-dominant mesh by topological transformations. IMR, 403–413 (2003)