

An Algorithm for Polytope Overlapping Detection

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Abstract. The intersection of polytopes is a basic problem of computational geometry with many engineering applications. Intersections of simplices or parallelotopes have been widely used in finite element grid generations. This paper is devoted to an algorithm for detecting overlapping polytopes. We present a new iterative algorithm, which is independent of the dimension of the Euclidean space. The main idea is triangulating the tested polytopes by simplicial finite elements and then investigating couples of potential simplices for an intersection. For that purpose, a method for overlapping detection of arbitrary simplices in $\mathbf{R}^n$ is developed. A detailed description of the pseudocode of the original algorithm is presented. The advantages of the proposed method are demonstrated in the twelve-dimensional case.

Keywords: Overlap detection algorithm · Overlapping domains · Intersection of simplices · Convex polytopes

AMS Subject Classifications: 52B11 · 65N30

1 Introduction

The problem of the intersection of two polytopes in Euclidean space has arisen in many Engineering activities. This includes applications in robotics, computer-aided design, computer graphics, computational mechanics, etc. There are several papers devoted to this problem [1,8,25]. This is one of the significant problems in the area of computational geometry. Most of the authors have used various iterative techniques based on numerical methods in order to establish the overlapping of polytopes [17]. Intersections of polyhedra related to the finite element method have been studied by Descantes et al. [7] and Lee et al. [12]. The main idea in [7] is a common plane to be created that separates both tested polytopes. The algorithm is restricted to the three-dimensional space and convex polyhedra. The paper by Edwards et al. [8] is devoted to the intersection of metrizable infinite-dimensional simplices. Lira et al. have applied finite element meshes in order to detect the intersection of surfaces [15].

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Brandts et al. [5] marked the beginning of a new stage in the development of the finite element method. They have discussed boundary value problems in multidimensional domains as a motivation to discover the finite element method in higher-dimensional Euclidean spaces. The authors have defined areas for further studies as: supercloseness and superconvergence; strengthened Cauchy-Schwarz inequalities; angle conditions for regularity of FEM partitions etc.

The major contribution of the paper is a feasible polytope overlapping detection algorithm. The proposed algorithm is easy for implementing. The validity of the algorithm does not depend on the complexity of the investigated polytopes. The algorithm can be successfully applied to convex polytopes in all n-dimensional Euclidean spaces. A method for identifying an overlapping of arbitrary nondegenerated k-simplex and l-simplex in n-dimensional Euclidean spaces is obtained. Intersections between polytopes of different dimensions are studied as well. The main idea is described in pseudocode. A real comparison between various iterative methods can be established in higher-dimensional spaces. For this purpose, a twelve-dimensional example is used to demonstrate the advantages of the presented algorithm for overlap detection. The new overlap detection algorithm is based on a method for solving systems of linear inequalities. Therefore, a comparison between methods for solving systems of linear inequalities is presented. The numerical tests in the twelve-dimensional Euclidean space indicate that the Motzkin method is superior among all considered methods.

This paper is organized as follows. Preliminary results on methods for solving systems of linear inequalities are presented in Sect. 2. The overlapping of simplices is considered in Sect. 3. The overlap detection algorithm in pseudocode is described in Sect. 4. Examples illustrating the new algorithm are presented in Sect. 5. The obtained results are discussed in Sect. 6.

2 Preliminaries

Our algorithm is based on solving a system of linear inequalities and the dissipation algorithm [20]. This is why we make a brief overview of the methods for solving a system of inequalities. We classify these methods into two major groups: exact methods and methods applying numerical optimizations. The exact methods for solving linear systems of inequalities have a significant advantage over iterative ones because in them the determination of a solution depends only on the round-off errors. On the other hand, any system of linear inequalities can be solved by the Fourier-Motzkin elimination [10]. The Fourier-Motzkin algorithm has been improved by Bastrakov et al. [2], Keßler [11], and Šimeček et al. [24]. But unfortunately, the classic exact methods are provided with an unacceptable computational cost. For instance, the algebraic complexity of the Fourier-Motzkin algorithm grows up to double exponentially concerning the dimensions of the linear system [10]. By reducing the large number of redundant inequalities Khachiyan [14] has obtained a computational cost equal to singly exponential time. In the second group, we mention the applications of the least square method [13], Newton's method [21], the BB-method [23], the conjugate gradient method

for quadratics [28], the relaxation method [26], and the Motzkin method [9] for solving systems of inequalities.

The proposed method requires a system of linear inequalities to be solved. To this end, we choose the Motzkin iterative method. Our choice is based on a comparison between several iterative methods for solving systems of inequalities. The best performance of the Motzkin method in the twelve-dimensional space is demonstrated in Sect. 5. Our algorithm can successfully work with any convergent method for solving a system of linear inequalities.

Throughout the whole paper ε stands for a positive number, which approximates zero and the upper index indicates the dimension of the polytope. The denotations $\underline{x}$ and $\|\cdot\|$ are the radius column vector of the point $x \in \mathbf{R}^n$ and the Euclidean norm in $\mathbf{R}^n$. Let $A\Big(2(n+1)\times n\Big)$ *and* $B\Big(2(n+1)\times 1\Big)$ be matrices with real entries, I_n be the identity matrix of size n, and $x = (x_1, x_2, \dots x_n) \in \mathbf{R}^n$. We say that $\underline{x} < \underline{y}$, $x, y \in \mathbf{R}^n$ if $x_i < y_i \ \forall i = 1, 2, \dots, n$. The object of interest in this section is the linear systems of inequalities

$$A\underline{x} < B \tag{1}$$

presented in a matrix form. We suppose that the block decomposition of the matrix

$$A = \begin{pmatrix} A_{11}(n \times n) \\ A_{21}((n+2) \times n) \end{pmatrix}$$

is provided with an invertible block A_{11}. The analyses of the specific system of inequalities (1) is motivated by the main problem of the paper.

Definition 1. *We say that the system (1) is consistent if (1) has at least one solution.*

We suppose that A_i is the i-th row of A, and

$$\xi = \underset{i=1,2,\dots,2n+2}{\operatorname{argmax}} \Big(A_i\underline{x} - B_i\Big).$$

Additionally, we assume that all rows of the matrix A are normalized by $\|A_i\| = 1$. We define a convex functional

$$\hat{J}(x) = \frac{1}{2}\Big|\hat{I}(A_\xi\underline{x} - B_\xi)\Big|^2,$$

where

$$\hat{I}x = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

We apply the Motzkin iterative method [9]

$$\underline{x}_{k+1} = \underline{x}_k - D\hat{J}(x_k), \quad k \geq 0. \tag{2}$$

for minimization the functional $\hat{J}(\hat{x})$. Each stationary point of $\hat{J}(\hat{x})$ is a solution of the problem (1). The choice of the initial guess will be determined further.

3 Overlapping of Two Simplices

Definition 2. *We define the n-dimensional simplex* $T = [t_1, t_2, \ldots, t_{n+1}]$ *as convex hull of affinely independent* $n+1$ *vertices* $t_i \in \mathbf{R}^n$, $i = 1, 2, \ldots, n+1$.

We suppose that the points t_i are affinely independent. We denote the boundary, the interior, the length of the longest edge, and the center of gravity of the n-dimensional nondegenerated simplex T by ∂T, $\mathring{T}$, $h(T)$ and $C_T = \frac{1}{n+1}\sum_{i=1}^{n+1} t_i$. We choose the cube corner $\hat{T} = [\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{n+1}]$, $\hat{t}_i = \{0, 0, \ldots, 1, \ldots, 0\}$, $i = 1, 2, \ldots, n$, $\hat{t}_{n+1} = \{0, 0, \ldots, 0\}$ for the reference simplex. The i-th coordinate of $\hat{t}_i$ $i = 1, 2, \ldots, n$ is equal to one, and all other coordinates are equal to zero. An arbitrary simplex T can be obtained from the reference simplex by the generic affine transformation $F_T : \underline{x} = A_T\underline{\hat{x}} + B_T$, $\hat{x} \in \hat{T}$, $x \in T$, where $A_T = \left(\underline{t}_1 - \underline{t}_{n+1}\ \underline{t}_2 - \underline{t}_{n+1} \cdots \underline{t}_n - \underline{t}_{n+1}\right)$ is the transition matrix and $B_T = \underline{t}_{n+1}$ is the translation vector.

The denotation μ_n stands for the Lebesgue measure in $\mathbf{R}^n$.

Definition 3. *The value*

$$\delta(T) = \frac{h(T)\mu_{n-1}(\partial T)}{2n\mu_n(T)},$$

is called measure of degeneracy [4, 20] of the simplex T.

Definition 4. *The measure of degeneracy related to an arbitrary triangulation* τ_h *is defined by*

$$\delta(\tau_h) = \max_{T \in \tau_h} \delta(T).$$

Our goal is to establish whether two nondegenerated simplices T and K have an intersection $T \cap K$ with $\mu_n(T \cap K) > 0$. In this case we say that both simplices intersect each other. We do not consider the case when $\mu_n(T \cap K) = 0$, $\mu_k(T \cap K) > 0$ for some k so that $1 \le k < n$.

Theorem 1. *Let* $M = A_K^{-1}A_T$ *and* $N = A_K^{-1}(B_T - B_K)$. *The n-dimensional nondegenerated simplices* T *and* K *have an intersection* $T \cap K$ *with* $\mu_n(T \cap K) > 0$, *if there are two points* $\hat{x}$, $\hat{y} \in \mathring{\hat{T}}$, *not necessarily different, so that*

$$\underline{\hat{y}} = M\underline{\hat{x}} + N. \tag{3}$$

Proof. From (3) it follows that there are two points $\hat{x}$, $\hat{y} \in \mathring{\hat{T}}$ so that $F_T(\hat{x}) = F_K(\hat{y})$. Since F_T and F_K are linear transformations, F_T maps $\hat{x}$ in $\mathring{T}$ and F_K maps $\hat{y}$ in $\mathring{K}$. The latter means that $\mathring{T} \cap \mathring{K} \neq \emptyset$, whence $\mu_n(T \cap K) > 0$. The two- and three-dimensional cases are illustrated in Fig. 1.□

By writing (3) in a scalar form

$$\hat{y}_i = \hat{\varphi}_i(\hat{x}), \quad i = 1, 2, \ldots, n \tag{4}$$

we obtain the following corollary.

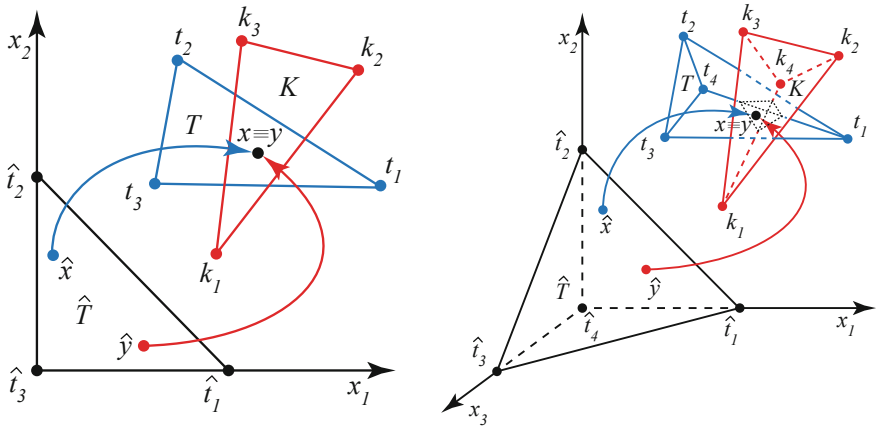

Fig. 1. Overlapping 2D and 3D simplices.

Corollary 1. *The n-dimensional simplices T and K have an intersection $T \cap K$ with $\mu_n(T \cap K) > 0$, if*

$$\hat{\varphi}_i(\hat{x}) > 0,\ i = 1, 2, \ldots, n,\ \textstyle\sum_{i=1}^{n} \hat{\varphi}_i(\hat{x}) < 1 \tag{5}$$

and

$$\hat{x}_i > 0,\ i = 1, 2, \ldots, n,\ \textstyle\sum_{i=1}^{n} \hat{x}_i < 1 \tag{6}$$

Proof. The relation $\hat{y} \in \mathring{\hat{T}}$ follows from (5). Additionally, $\hat{x} \in \mathring{\hat{T}}$ because of (6). Then the validity of the corollary is guaranteed by (4).□

In a matrix form the system (5)–(6) is equivalent to (1).

In Theorem 1, we consider the case when T and K are n-dimensional. Further, we solve more general problem when both simplices have different dimensions. Let $\hat{T}^l$ be a reference simplex with $\dim(\hat{T}^l) = l$.

Theorem 2. *The nondegenerated simplices T and K satisfy* $\dim(T) = k$, $\dim(K) = l$, $1 \le k \le l \le n$, $k + l < 2n$,

$$\mathring{T} \cap \mathring{K} \neq \emptyset \tag{7}$$

if there are two points $\hat{x} \in \mathring{\hat{T}}^k$ and $\hat{y} \in \mathring{\hat{T}}^l$, so that

$$\underline{\hat{y}} = M\underline{\hat{x}} + N, \tag{8}$$

where $M = Q_K^{-1} A_K^T A_T$ *and* $N = Q_K^{-1} A_K^T (B_T - B_K)$, $Q_K = A_K^T A_K$.

Proof. In this case, it is impossible to express $\hat{y}$ directly from the relation $F_T(\hat{x}) = F_K(\hat{y})$. That is why, we multiply the latter equality by A_K^T. Since the simplex K is nondegenerated we have $\operatorname{rank}(A_K^T A_K) = l$, i.e. Q_K is an invertible matrix. On the other hand, $\hat{x}$ and $\hat{y}$ belong to the reference simplices, which guarantees the validity of inequality (7).□

Analogously to Corollary 1, we create a system of linear inequalities to determine overlapping of two simplices.

Corollary 2. *Assume that*

$$\hat{y}_i = \hat{\varphi}_i(\hat{x}), \quad i = 1, 2, \ldots, l, \tag{9}$$

is a scalar form of (8). Then the nondegenerated simplices T and K satisfying $\dim(T) = k$, $\dim(K) = l$, $1 \le k \le l \le n$, $k + l < 2n$, *have an intersection* $T \cap K$ *with* $\mu_k(T \cap K) > 0$, *if*

$$\hat{\varphi}_i(\hat{x}) > 0, \, i = 1, 2, \ldots, l, \, \sum_{i=1}^{l} \hat{\varphi}_i(\hat{x}) < 1 \tag{10}$$

and

$$\hat{x}_i > 0, \, i = 1, 2, \ldots, k, \, \sum_{i=1}^{k} \hat{x}_i < 1. \tag{11}$$

Proof. The relations

$$\hat{x} \in \mathring{\hat{T}}^k \quad \text{and} \quad \hat{y} \in \mathring{\hat{T}}^l$$

follows from (10) and (11). Then the validity of the corollary is guaranteed by (9).□

4 The Overlap Detection Algorithm

In this section we analyze an overlapping of two convex polytopes. The ball $\mathcal{B}(T) = \text{Ball}(C_T, r_T)$ has a radius $r_T = \max_{i=1,2,\ldots,n+1} ||\underline{C}_T - \underline{t}_i||$ and a center C_T. The latter means that $\mathcal{B}(T)$ is the smallest ball with a center C_T, which covers the simplex T. Let $\mathcal{F}_n$ be the set of all nondegenerated linear maps from $\mathbf{R}^n$ to $\mathbf{R}^n$, and Ω_1 and Ω_2 are nondegenerated polytopes in $\mathbf{R}^n$. We suppose that the triangulations

$$\tau_h(\Omega_1) = \left\{ T_i = F_i(\hat{T}) \;\middle|\; F_i \in \mathcal{F}_n, \;\; \Omega_1 = \bigcup_{i=1}^{m} T_i, \right.$$
$$\left. \mu_n\left(T_i \cap T_j\right) = 0, \;\; 1 \le i < j \le m \right\}$$

and

$$\tau_H(\Omega_2) = \left\{ K_i = G_i(\hat{T}) \;\middle|\; G_i \in \mathcal{F}_n, \;\; \Omega_2 = \bigcup_{i=1}^{p} K_i, \right.$$
$$\left. \mu_n\left(K_i \cap K_j\right) = 0, \;\; 1 \le i < j \le p \right\}$$

have as small as possible $\text{Card}(\tau_h)$, $\delta(\tau_h)$, and $\text{Card}(\tau_H)$, $\delta(\tau_H)$. In the general case, the mesh parameters h and H are independent.

Petrov and Todorov [20] have obtained a dissipative algorithm for triangulating convex polytopes. For the sake of computational simplicity, they demonstrated their algorithm in the four-dimensional space. But the dissipation algorithm [20] can be applied successfully to all higher-dimensional spaces. In this

paper, we obtain the triangulations τ_h and τ_H of the domains Ω_1 and Ω_2 in the following way. Each polytope Ω_i is provided with the set of nodes

$$\mathcal{N}(\Omega_i) = \{\text{all vertices of } \Omega_i \text{ and the gravity center } G(\Omega_i)\}.$$

We apply the dissipation algorithm to triangulate the polytopes Ω_i, $i = 1, 2$. We could obtain optimal results by creating Delaunay triangulations (see Cignonit et al. [6]) of Ω_1 and Ω_2 but such divisions needs much more computational work.

The proposed algorithm has four steps:

(*i*) Creation of simplicial triangulations τ_h and τ_H for both polytopes;
(*ii*) Generation of the cover balls $\mathcal{B}(T)$ and $\mathcal{B}(K)$ for all simplices T and K in both triangulations;
(*iii*) Determination of all couples (T_i, K_j), $T_i \in \tau_h$ and $K_j \in \tau_H$ satisfying $\mu_n\left(\mathcal{B}(T_i) \bigcap \mathcal{B}(K_j)\right) > 0$;
(*iv*) Overlap detection test for all couples described in step (*iii*).

We denote the system of linear inequalities (5) with the constrains (6) by (W). The procedure for solving the system (W) by means of an iterative numerical method is denoted by Solve. The procedure Solve indicates the consistency or inconsistency of the system (1). This procedure has two nondegenerated polytopes as input and the existence or the lack of solutions of the linear system (W) as output. The initial guess x_0 for starting the Motzkin method is chosen to be the gravity center of the reference polytope. The cardinality of both triangulations affects the number of necessary operations to make a decision about the overlapping of both polytopes. The overlap detection algorithm is presented in pseudocode below.

Algorithm Polytopes Overlapping Detection;
  function Min(J : functional);
    var k : local;
    x_0 =GravityCenter($\hat{T}$);
    Min:=$\hat{J}(x_0)$;
    k=0;
    repeat
      $\gamma_k = \left\| D\hat{J}(x_k) \right\|$;
      $\xi = \underset{i=1,2,\ldots,2n+2}{\operatorname{argmax}} \left(A_i \underline{x} - B_i \right)$;
      $\underline{x}_{k+1} = \underline{x}_k - D\hat{J}(x_k)$;
      if $\hat{J}(x_{k+1})$ <Min **then** Min:=$\hat{J}(x_{k+1})$
      end if;
      k=k+1
    until $\gamma_k \geq \epsilon$
    Min:=$\hat{J}(x_k)$;
  end function;
procedure SOLVE(T, K : **simplices**);
  var i, j : local;

determine $A_T,\ B_T$ **from** $T = A_T\hat{T} + B_T$;
determine $A_K,\ B_k$ **from** $K = A_K\hat{T} + B_K$;
$M := A_K^{-1}A_T$; $N := A_K^{-1}\Big(B_T - B_K\Big)$;
$\hat{\underline{\Phi}}(\hat{x}) := M\underline{\hat{x}} + N$;
% The functions $\hat{\varphi}_i(\hat{x})$ are coordinates of the vector $\hat{\underline{\Phi}}(\hat{x})$.
for $i = 1$ **to** $n + 1$ **do**
    for $j = 1$ **to** n **do**
        $A_{ij} = \hat{\varphi}_i(\hat{t}_{n+1}) - \hat{\varphi}_i(\hat{t}_j)$
    end for
end for;
$\Big(A_{ij}\Big)_{i=n+2,n+3,\ldots,2n+1,j=1,2,\ldots,n} := -I_{n+1}$;
$A_{2n+2} := (1, 1, \ldots 1)$;
for $i = 1$ **to** $n + 1$ **do**
    $B_i = \hat{\varphi}_i(\hat{t}_{n+1})$
end for;
$\Big(B_i\Big)_{i=n+2,n+3,\ldots,2n+1} := \underline{0}_n$; $B_{2n+2} := 1$;
$\hat{J}(x) := \frac{1}{2}\Big|\hat{I}(A_\xi\underline{x} - B_\xi)\Big|^2$;
if $\Big|\mathrm{Min}(\hat{J})\Big| < \varepsilon$ **then** Solution := **exists**
else Solution := **nonexists**
end if;
end procedure;
% The Main Algorithm.
begin
set Overlap = **false**;
set ε;
define Ω_1, Ω_2;
compile $\tau_h(\Omega_1)$, $\tau_H(\Omega_2)$;
for $i = 1$ **to** m **do**
    for $j = 1$ **to** p **do**
        if $\mu_n\left(\mathcal{B}(T_i)\bigcap\mathcal{B}(K_j)\right) > 0$ **then**
            Solve(T_i, K_j);
            if Solution = **exists then**
                % An intersection is detected.
                Overlap = **true**;
                print('The simplices', T_i, 'and', K_j, 'are overlapping each other.')
            end if
        end if
    end for
end for
if Overlap = **true then**
    print('The polytopes have a common subset')

60: **end if**
61: **end.**

Algorithm 1 is explained in the specific case when $\dim(\Omega_1) = \dim(\Omega_2) = n$ for the sake of simplicity. The same algorithm can be extended to the case $\dim(\Omega_1) = k$, $\dim(\Omega_2) = l$, $2 \leq k + l < 2n$ by applying Corollary 2.

5 Overlap Detection Examples

In this section, we compare some low-cost numerical methods to establish overlap between polytopes.

We begin the illustration of the theory from the previous paragraph with a three-dimensional example. By the first two examples, we illustrate that the intersection between the tested polytopes is found. An axonometric drawing of the considered polyhedra in Example 1 is presented. The three-dimensional shadow of the tested polytopes in Example 2 is shown. The real advantages of the proposed algorithm are demonstrated by the overlapping of two twelve-dimensional polytopes in Example 3.

Further, we use the denotation $\mathbf{1}_k = (1, i = 1, 2, \ldots, k)$.

Example 1. *We define a transformation* $L_3 : \underline{x} = D_3\underline{\hat{x}} + E_3$, $L_3 \in \mathcal{F}_3$ *by* $D_3 = \begin{pmatrix} 1 & 0.3 & 0 \\ 0 & 1 & 0.2 \\ 0.4 & 0 & 1 \end{pmatrix}$ and $E_3 = 0.275598 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
Let us consider the polyhedra:

$$\begin{aligned} &\Omega_1 = [t_1(0,0,0), t_2(1,0,0), t_3(0,1,0), \\ &t_4(0,0,1), t_5(-1,0,0), t_6(0,-1,0), t_7(0,0,-1)] \text{ and} \\ &\hat{\Omega}_2 = [\hat{k}_8(0,0,0), \hat{k}_9(1,0,0), \hat{k}_{10}(0,1,0), \hat{k}_{11}(0,0,1), \\ &\hat{k}_{12}(1,1,0), \hat{k}_{13}(0,1,1), \hat{k}_{14}(1,0,1), \hat{k}_{15}(1,1,1)]. \end{aligned}$$

The parallelohedron Ω_2 *is obtained by*

$$\Omega_2 = L_3(\hat{\Omega}_2). \tag{12}$$

The main purpose in this example is to determine the presence or absence of an intersection between the polyhedra Ω_1 *and* Ω_2*.*

Solution. We triangulate both polyhedra as follows:
$\Omega_1 = \Big\{T_1 = [t_1, t_2, t_3, t_4],\quad T_2 = [t_1, t_3, t_4, t_5],\quad T_3 = [t_1, t_2, t_4, t_6],\quad T_4 = [t_1, t_2, t_3, t_7],\quad T_5 = [t_1, t_4, t_5, t_6],\ T_6 = [t_1, t_3, t_5, t_7],\quad T_7 = [t_1, t_2, t_6, t_7],\quad T_8 = [t_1, t_5, t_6, t_7]\Big\}$ and $\hat{\Omega}_2 = \Big\{\hat{K}_1 = [\hat{k}_9, \hat{k}_{10}, \hat{k}_{11}, \hat{k}_{15}],\ \hat{K}_2 = [\hat{k}_8, \hat{k}_9, \hat{k}_{10}, \hat{k}_{11}],\ \hat{K}_3 = [\hat{k}_{10}, \hat{k}_{11}, \hat{k}_{13}, \hat{k}_{15}],\ \hat{K}_4 = [\hat{k}_9, \hat{k}_{11}, \hat{k}_{14}, \hat{k}_{15}],\ \hat{K}_5 = [\hat{k}_9, \hat{k}_{10}, \hat{k}_{12}, \hat{k}_{15}]\Big\}$. The polyhedron Ω_1 is an octahedron, which is optimally divided into eight cube corners all of them with measure $\delta(T_i) = 3.34607$ $i = 1, 2, \ldots, 8$. The polyhedron $\hat{\Omega}_2$ is a cube, which is optimally partitioned [19] into four cube corners $\hat{K}_i$ $i = 2, 3, 4, 5$ and one

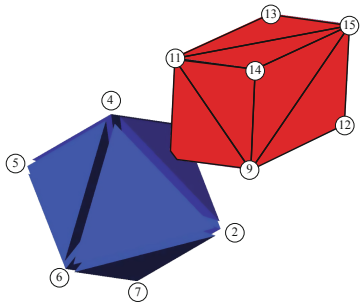

Fig. 2. An overlapping of Ω_1 and Ω_2 in the three-dimensional case.

Table 1. A comparison between the Motzkin method and the other iterative methods.

Method	CGM	BBM	MM	RM
$\epsilon \setminus$ initial guesses	$\underline{x}_0 = \underline{C}_{\hat{T}}$	$\underline{x}_0 = \underline{C}_{\hat{T}}$, $\underline{x}_1 = 5 \cdot \mathbf{1}_3$	$\underline{x}_0 = \underline{C}_{\hat{T}}$	$\underline{x}_0 = \underline{C}_{\hat{T}}$, $\lambda = 1.9$
10^{-6}	66	87	169	11

regular tetrahedron K_1. The cube $\hat{\Omega}_2$ is deformed by means of (12). The measures of degeneracy before and after the deformation of the cube $\hat{\Omega}_2$ are: $\delta(\hat{K}_1) = 2.44949$, $\delta(\hat{K}_i) = 3.34607$, $i = 2, 3, 4, 5$ and $\delta(K_1) = 3.04601$, $\delta(K_2) = 2.90384$, $\delta(K_3) = 4.42111$, $\delta(K_4) = 4.0847$, $\delta(K_5) = 4.32041$. The octahedron Ω_1 is not deformed. By applying Algorithm 1, we establish an intersection between T_1 and K_1, i.e. an overlapping between Ω_1 and Ω_2 is detected, see Fig. 2. The point $x_0(0.317466, 0.333466, 0.341466)$ belongs to the intersection $\mathring{\Omega}_1 \cap \mathring{\Omega}_2$. The corresponding points in the reference simplex are: $\hat{x}_0(0.00760254, 0.317466, 0.333466)$ and $\hat{y}_0(0.0277923, 0.0469174, 0.0547506)$. The vertex k_8 of Ω_2 belongs to the interior $\mathring{\Omega}_1$ of the octahedron.

We compare the Motzkin method (MM) [9] with the BB-method [23], the conjugate gradient method for quadratics (CGM) [28], and the relaxation method (RM) with various steplengths [26]. Let γ_k be the Euclidean norm of the Fréchet derivative of the objective functional. The stop criterion for all these methods is $\gamma_k < \epsilon$. An initial guess for all methods is chosen to be the gravity center of the reference finite element. We present the number of iterations for satisfying the stop criteria in Table 1. Additional initial guesses are necessary for the two-point methods.

Table 1 indicates that the Motzkin method has the best performance among other considered methods in this example. The row relaxation method strongly depends on the relaxation parameter λ. The results in Table 2 indicate that the overrelaxation is more suitable in this example.□

Table 2. The performance of the row relaxation method. The initial guess is chosen to be $\underline{x}_0 = \underline{C}_{\hat{T}}$ and $\epsilon = 10^{-6}$. The number ν_λ stands for the necessary iterations for satisfying the stop criterion.

λ	1.2	1.25	1.5	1.6	1.7	1.9
ν_λ	110	99	17	13	11	11

Example 2. *We continue with a four-dimensional example. Here, we use a deformation map* $L_4 : \underline{x} = D_4\underline{\hat{x}} + E_4$, $L_4 \in \mathcal{F}_4$ *by*

$$D_4 = \begin{pmatrix} 1 & 0.3 & 0.2 & 0.1 \\ 0.1 & 1 & 0.2 & 0.2 \\ 0.1 & 0.1 & 1 & 0.1 \\ 0.2 & 0.1 & 0.2 & 1 \end{pmatrix} \quad \text{and} \quad E_4 = 0.23 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

We define the polytopes:

$$\Omega_1 = \Big[t_1(0,0,0,0), t_2(1,0,0,0), t_3(0,1,0,0), t_4(0,0,1,0), t_5(0,0,0,1),$$

$$t_6(-1,0,0,0), t_7(0,-1,0,0), t_8(0,0,-1,0), t_9(0,0,0,-1)\Big]$$

and $\hat{\Omega}_2 = \Big[\hat{k}_1(0,0,0,0), \hat{k}_2(1,0,0,0), \hat{k}_3(0,1,0,0), \hat{k}_4(0,0,1,0),$
$\hat{k}_5(1,1,0,0), \hat{k}_6(0,1,1,0), \hat{k}_7(1,0,1,0), \hat{k}_8(1,1,1,0), \hat{k}_9(0,0,0,1),$
$\hat{k}_{10}(1,0,0,1), \hat{k}_{11}(0,1,0,1), \hat{k}_{12}(0,0,1,1), \hat{k}_{13}(1,1,0,1),\ \hat{k}_{14}(0,1,1,1),$
$\hat{k}_{15}(1,0,1,1),\ \hat{k}_{16}(1,1,1,1),\ \hat{k}_{17}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)]$.
The polytope Ω_2 *is obtained by*

$$\Omega_2 = L_4(\hat{\Omega}_2). \tag{13}$$

The objects of intersection in this example are: Ω_1 *and* Ω_2.

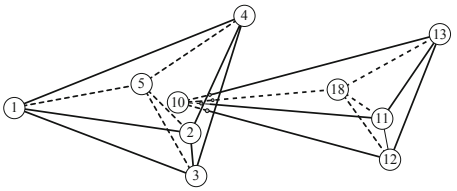

Fig. 3. The intersecting pentatopes T_1 and K_1.

Solution. The polytope Ω_1 is an orthoplex. Therefore, we divide it into sixteen tesseract corners:

$$\Omega_1 = \Big\{T_1 = [t_1, t_2, t_3, t_4, t_5],\ \ T_2 = [t_1, t_3, t_4, t_5, t_6],\ \ T_3 = [t_1, t_2, t_4, t_5, t_7],$$
$$T_4 = [t_1, t_2, t_3, t_5, t_8],\ \ T_5 = [t_1, t_2, t_3, t_4, t_9],\ \ T_6 = [t_1, t_4, t_5, t_6, t_7],$$
$$T_7 = [t_1, t_3, t_5, t_6, t_8],\ \ T_8 = [t_1, t_3, t_4, t_6, t_9],\ \ T_9 = [t_1, t_2, t_5, t_7, t_8],$$
$$T_{10} = [t_1, t_2, t_4, t_7, t_9],\ \ T_{11} = [t_1, t_2, t_3, t_8, t_9],\ \ T_{12} = [t_1, t_5, t_6, t_7, t_8],$$
$$T_{13} = [t_1, t_4, t_6, t_7, t_9],\ \ T_{14} = [t_1, t_3, t_6, t_8, t_9],\ \ T_{15} = [t_1, t_2, t_7, t_8, t_9],$$
$$T_{16} = [t_1, t_6, t_7, t_8, t_9]\Big\}$$

all of them with a measure of degeneracy $\delta(T_i) = 4.24264$ $i = 1, 2, \ldots 16$. The polytope $\hat{\Omega}_2$ is a tesseract, which is optimally refined into 24 tesseract corners as it is done in [20]. All simplices K_i in

$$\hat{\tau}_H(\hat{\Omega}_2) = \left\{ \hat{K}_i \in [\hat{T}] \ \mid \ \Omega_2 = \bigcup_{i=1}^{24} \hat{K}_i,\ \mu_n\left(\hat{K}_i \cap \hat{K}_j\right) = 0,\ 1 \le i < j \le 24 \right\}$$

have the same measure of degeneracy $\delta(\hat{K}_i) = 4.24264$. After the deformation (13) the measure $\delta(\tau_H)$ of

$$\tau_H(\Omega_2) = \Big\{ K_i = L_4(\hat{K}_i) \ \mid \ \ 1 \le i \le 24 \Big\}$$

grows up to 5.85361. The latter indicates a slight deformation of Ω_2. Obviously, $\Omega_2 = \bigcup_{i=1}^{24} K_i$, $\mu_n(K_i \cap K_j) = 0$, $1 \le i < j \le 24$. The execution of Algorithm 1 indicates an intersection of the simplices $T_1[t_1, t_2, t_3, t_4, t_5]$ and $K_1[k_{10}, k_{11}, k_{12}, k_{13}, k_{18}]$, see Fig. 3. The vertex k_{10} of Ω_2 belongs to the interior of the orthoplex. The point $x_0(0.237771, 0.246095, 0.238852, 0.250095)$ belongs to the intersection between the cross polytope and the parallelotope, see Fig. 4. The preimages of x_0 in the reference simplex are: $\hat{y}_0(0.96391, 0.00147, 0.01128, 0.00583)$ and $\hat{x}_0(0.02719, 0.23777, 0.24609, 0.23885)$.□

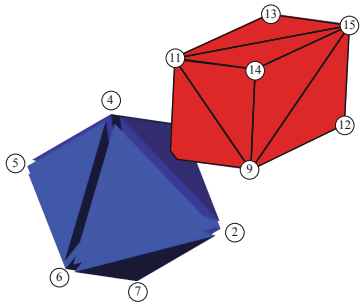

Fig. 4. An overlapping of Ω_1 and Ω_2 in the four-dimensional case.

The advantages of the Motzkin method will be demonstrated in the third example.

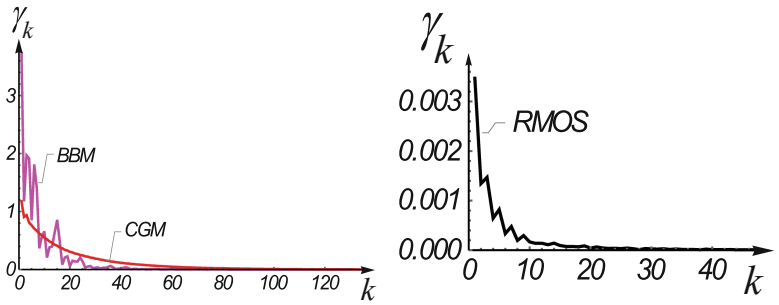

Fig. 5. The gradient descent obtained by the Barzilai-Borwein method, the conjugate gradient method, and the relaxation method with the optimal steplength.

Example 3. *Here, we consider overlapping of two twelve-dimensional simplices. Each vertex* $\hat{k}_i$, $i = 1, 2, \dots 12$ *of the Freudenthal element*

$$\hat{K}[\hat{k}_1(1,0,\dots,0), \hat{k}_2(1,1,\dots,0,0), \dots, \hat{k}_{12}(1,1,\dots,1,1), \hat{k}_{13}(0,0,\dots,0)]$$

has the first i coordinates equal to one and the other coordinates are equal to zero. The thirteenth node is located at the origin. The simplex $\hat{K}$ *is deformed by the linear transformation* $K = D_{12}\hat{K} + E_{12}$, *where*

$$D_{12} = \begin{pmatrix}
1 & 0.3 & 0.2 & 0.1 & 0.3 & 0.2 & 0.1 & 0.3 & 0.2 & 0.1 & 0.3 & 0.2 \\
0.1 & 1 & 0.2 & 0.2 & 0.3 & 0.2 & 0.1 & 0.3 & 0.2 & 0.1 & 0.3 & 0.3 \\
0.1 & 0.1 & 1 & 0.1 & 0.3 & 0.2 & 0.1 & 0.1 & 0.1 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.1 & 0.2 & 1 & 0.3 & 0.2 & 0.1 & 0.3 & 0.2 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.2 & 0.1 & 1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.2 & 0.2 & 0.2 \\
0.3 & 0.1 & 0.3 & 0.1 & 0.3 & 1 & 0.1 & 0.3 & 0.2 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.2 & 0.1 & 0.3 & 0.2 & 1 & 0.3 & 0.1 & 0.2 & 0.1 & 0.2 \\
0.2 & 0.1 & 0.3 & 0.1 & 0.3 & 0.2 & 0.1 & 1 & 0.2 & 0.1 & 0.1 & 0.3 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.3 & 0.2 & 0.1 & 0.3 & 1 & 0.1 & 0.3 & 0.2 \\
0.2 & 0.1 & 0.2 & 0.1 & 0.3 & 0.2 & 0.1 & 0.3 & 0.2 & 1 & 0.3 & 0.2 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.3 & 0.2 & 0.1 & 0.3 & 0.2 & 0.1 & 1 & 0.2 \\
0.2 & 0.1 & 0.2 & 1 & 0.3 & 0.2 & 0.1 & 0.3 & 0.2 & 0.1 & 0.3 & 1
\end{pmatrix}$$

and

$$E_{12} = (0.0744333, 0.0745333, 0.0746333, 0.0747333, 0.0748333, 0.0749333, 0.0750333, 0.0751333, 0.0752333, 0.0753333, 0.0754333, 0.0755333)^T.$$

We study the intersection between the reference cube corner $\hat{T}^{12}$ *and the deformed Freudenthal element* K.

Solution. Since $k_{13} \in \mathring{\hat{T}}^{12}$, the measure $\mu_{12}(\hat{T} \cap K)$ is positive. We calculate the transitional matrix A_K of the element K and write $A_T = I_{12}$, $B_T = \underline{0}_{12}$, $B_K = E_{12}$. The cube corners have the lowest rate of divergence [18]. Despite this $\delta(\hat{T}^{12}) = 10.93477$. To obtain a real comparison between the considered

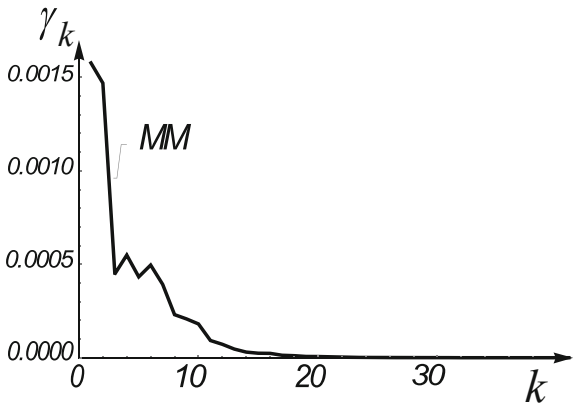

Fig. 6. The gradient descent obtained by the Motzkin method.

methods, we choose the strongly deformed element K with $\delta(K) = 132.726$. This affect the condition number of the matrix $\kappa(M_{12}) = 67.1266$. Thus we obtain ill-conditioned system (1).

We compare the Motzkin method with the BB-method, the conjugate gradient method, and the relaxation method with the optimal steplength (RMOS) [16]. An initial guess for all methods is chosen to be the gravity center of the cube corner. The relaxation method is divergent with a zero initial guess. We present the number of iterations for satisfying the stop criteria in Table 3. Additional initial guesses are necessary for the two-point methods. Table 3 indicates that the Motzkin method is superior to other considered methods in this example. Intermediate results on gradient descent are presented in Figs. 5 and 6.

Table 3. A comparison between the Motzkin method and the other iterative methods. The table present the number of necessary iterations for satisfying the stop criterion.

Method	CGM	BBM	MM	RMOS
$\epsilon \setminus$ initial guesses	$\underline{x}_0 = \underline{C}_{\hat{T}}$	$\underline{x}_0 = \hat{\underline{t}}_{13}$, $\underline{x}_1 = \underline{C}_{\hat{T}}$	$\underline{x}_0 = \underline{C}_{\hat{T}}$	$\underline{x}_0 = \underline{C}_{\hat{T}}$
10^{-6}	284	122	26	168
10^{-8}	370	164	41	287
10^{-12}	545	255	72	539

We establish good performance for the row relaxation method in the low-dimensional spaces and bad behavior of the same method in the higher-dimensional cases.

6 Conclusion

A feasible iterative algorithm for polytopes overlapping detection is developed in this paper. The algorithm is described in pseudocode. The only restriction on the polytopes tested for an intersection is both of them be convex. Concave polytopes can be studied for intersection by this algorithm if each of them is previously divided into convex subdomains. Then each part can be triangulated by simplicial finite elements. The algorithm is designed to be used in any Euclidean space. The investigation for overlapping requires triangulating of the considered polytopes. For this purpose, we use the dissipation algorithm. Thus we obtain triangulations of the tested polytopes with a small number of elements in order to reduce the number of couples of simplices. The overlappings of simplices with the same and with different dimensions are analyzed. Due to the high complexity of the Fourier-Motzkin elimination for complicated polytopes in $\mathbf{R}^n$ $n \geq 10$, the system (W) is solved iteratively. The proposed algorithm is implemented and tested in the three- and four-dimensional cases.

The numerical tests indicate that the row iterative methods have better performance than the other iterative methods. Additionally, the Motzkin method is superior to all other tested methods. The real advantages of the method are demonstrated by intersecting twelve-dimensional polytopes. Graphical presentations are shown in the three and four-dimensional spaces.

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