Proof of Some Properties of the Cross Product of Three Vectors in $\mathbb{R}^4$ with Mathematica

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Abstract. In this paper, the definition of the cross product of three tangent vectors to $\mathbb{R}^4$ at the same point is stated, based on this definition a lemma is stated in which eight properties of said product are proposed and demonstrated. In addition, four theorems and two corollaries are stated and proved. One of the theorems constitutes an extension of the Jacobi identity. In some of the proofs, programs based on the paradigms: functional, rule-based and list-based from the Wolfram language, incorporated in Mathematica, are used.

Keywords: Cross product in $\mathbb{R}^4$ · Jacobi identity in $\mathbb{R}^4$ · Hyperparalelepiped · Hypervolume · Demonstrations using mathematica

1 Introduction

The cross product of three tangent vectors to $\mathbb{R}^4$ in the same point is formally defined in [6]. From that year to the present it has been used in different research works [1–3].

In this paper, the definition of the cross product of three tangent vectors to $\mathbb{R}^4$ given by Williams and Stein [6] is stated, based on this definition a lemma is stated in which eight properties of said product are proposed and demonstrated. Given five vectors tangent to $\mathbb{R}^4$ at the same point, an extension of the Jacobi identity is established (which is valid for three vectors tangent to $\mathbb{R}^3$ at the same point). Given two vectors tangent to $\mathbb{R}^4$ at the same point, we establish a formula to calculate the area of the parallelogram whose sides are these vectors. Given three vectors tangent to $\mathbb{R}^4$ at the same point, we proof that the volume of the parallelepiped whose edges are these three vectors is equal to the length

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of the cross product of them. A relation is established between the length of the cross product of three vectors tangent to $\mathbb{R}^4$ in the same point and the areas of the faces of the parallelepiped whose edges are these vectors. Finally, we proof that the absolute value of the quadruple scalar product of four vectors tangent to $\mathbb{R}^4$ at the same point is equal to the hyper-volume of the hyper-parallelepiped whose edges are these four vectors.

The Wolfram language [7], built into Mathematica [8], stands out from traditional computer languages by simultaneously supporting many programming paradigms, such as procedural, functional, rule-based, list-based, pattern-based, and more [4,5]. In this paper, certain demonstrations are carried out with the help of programs implemented based on these first three paradigms; in such a way that these programs are shorter and more efficient.

The structure of this paper is as follows: Sect. 2 introduce the mathematical definition of the Euclidean 4-space, a tangent vector, a vector field, the dot product of two points, the norm of a point, the dot product of two tangent vectors and a frame at a point. Then, Sect. 3 introduces the cross product of three tangent vectors. In this section properties, lemmas, theorems, and corollaries associated with this product are stated and demonstrated. Finally, Sect. 4 closes with the main conclusions of this paper.

2 Mathematical Preliminaries

Definition 1. Euclidean 4-space $\mathbb{R}^4$ *is the set of all ordered quadruples of real numbers. Such a quadruple* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *is called a* point *of* $\mathbb{R}^4$.

Definition 2. *A* tangent vector $\mathbf{v}_p$ *to* $\mathbb{R}^4$ *consist of two points of* $\mathbb{R}^4$*: its* vector part $\mathbf{v}$ *and its* point of application $\mathbf{p}$.

Definition 3. *Let* $\mathbf{p}$ *be a point of* $\mathbb{R}^4$. *The set* $T_p\left(\mathbb{R}^4\right)$ *consisting of all tangent vectors that have* $\mathbf{p}$ *as point of application is called the* tangent space *of* $\mathbb{R}^4$ *at* $\mathbf{p}$.

Definition 4. *A* vector field V *on* $\mathbb{R}^4$ *is a function that assigns to each point* $\mathbf{p}$ *of* $\mathbb{R}^4$ *a tangent vector* $V(\mathbf{p})$ *to* $\mathbb{R}^4$ *at* $\mathbf{p}$.

Definition 5. *Let* U_1, U_2, U_3 *and* U_4 *be the vector fields on* $\mathbb{R}^4$ *such that*

$$\begin{aligned} U_1(\mathbf{p}) &= (1,0,0,0)_p & U_2(\mathbf{p}) &= (0,1,0,0)_p \\ U_3(\mathbf{p}) &= (0,0,1,0)_p & U_4(\mathbf{p}) &= (0,0,0,1)_p \end{aligned}$$

for each point $\mathbf{p}$ *of* $\mathbb{R}^4$. *We call* U_1, U_2, U_3, U_4 *—collectively—the* natural frame field *on* $\mathbb{R}^4$.

Definition 6. *The dot product of points* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *and* $\mathbf{q} = (q_1, q_2, q_3, q_4)$ *in* $\mathbb{R}^4$ *is the number*

$$\mathbf{p} \bullet \mathbf{q} = p_1 q_1 + p_2 q_2 + p_3 q_3 + p_4 q_4 \,.$$

Definition 7. *The* norm *of a point* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *is the number*

$$\|\mathbf{p}\| = (\mathbf{p} \bullet \mathbf{p})^{1/2} .$$

The norm is thus a real-valued function on $\mathbb{R}^4$; it has fundamental properties $\|\mathbf{p} + \mathbf{q}\| \leqq \|\mathbf{p}\| + \|\mathbf{q}\|$ and $\|a\mathbf{p}\| = |a| \, \|\mathbf{p}\|$, where $|a|$ is the absolute value of the number a.

Definition 8. *The* dot product *of* $\mathbf{v}, \mathbf{w} \in T_p\left(\mathbb{R}^4\right)$ *is the number*

$$\mathbf{v}_p \bullet \mathbf{w}_p = \mathbf{v} \bullet \mathbf{w} .$$

A fundamental result of linear algebra is the Schwartz inequality $|\mathbf{v} \bullet \mathbf{w}| \leqq \|\mathbf{v}\|\|\mathbf{w}\|$. This permits us to define the cosine of the angle ϑ between $\mathbf{v}$ and $\mathbf{w}$ by the equation

$$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos \vartheta ,$$

where $0 \leqq \vartheta \leqq \pi$.

In particular, if $\vartheta = \pi/2$, then $\mathbf{v} \bullet \mathbf{w} = 0$. Thus we shall define two vectors to be *orthogonal* provided their dot product is zero. A vector of length (norm) 1 is called a *unit vector.*

Definition 9. *A set* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ *of four orthogonal unit vectors tangent to* $\mathbb{R}^4$ *at* $\mathbf{p}$ *es called a* frame *at the point* $\mathbf{p}$.

For example, at each point $\mathbf{p}$ of $\mathbb{R}^4$, the vectors $U_1(\mathbf{p}), U_2(\mathbf{p}), U_3(\mathbf{p}), U_4(\mathbf{p})$ of Definition 5 constitute a frame at $\mathbf{p}$.

3 The Cross Product

Definition 10. *If* $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{x}$ *are tangent vectors to* $\mathbb{R}^4$ *at the same point* $\mathbf{p}$, *then the* cross product *of* $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{x}$ *is the tangent vector*

$$\mathbf{v} \times \mathbf{w} \times \mathbf{x} = - \begin{vmatrix} U_1(\mathbf{p}) & U_2(\mathbf{p}) & U_3(\mathbf{p}) & U_4(\mathbf{p}) \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix} .$$

Lemma 1. *If* $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{z}$ *are tangent vectors to* $\mathbb{R}^4$ *at the same point* $\mathbf{p}$, *then*

1. *The cross product* $\mathbf{v} \times \mathbf{w} \times \mathbf{x}$ *is* linear *in* $\mathbf{v}$, *in* $\mathbf{w}$ *and in* $\mathbf{x}$, *and satisfies the* alternation rules

$$\mathbf{v} \times \mathbf{w} \times \mathbf{x} = \mathbf{w} \times \mathbf{x} \times \mathbf{v} = \mathbf{x} \times \mathbf{v} \times \mathbf{w} = -\mathbf{x} \times \mathbf{w} \times \mathbf{v} = -\mathbf{w} \times \mathbf{v} \times \mathbf{x} = -\mathbf{v} \times \mathbf{x} \times \mathbf{w} .$$

2. $\mathbf{v} \times \mathbf{w} \times \mathbf{x} \neq \mathbf{0}$ *if and only if* $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{x}$ *are linearly independent,*

3. $\mathbf{v} \times \mathbf{w} \times (\mathbf{x} \times \mathbf{y} \times \mathbf{z}) = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{x} \bullet \mathbf{w} & \mathbf{y} \bullet \mathbf{w} & \mathbf{z} \bullet \mathbf{w} \\ \mathbf{x} \bullet \mathbf{v} & \mathbf{y} \bullet \mathbf{v} & \mathbf{z} \bullet \mathbf{v} \end{vmatrix}$,
4. $\mathbf{v} \times \mathbf{w} \times (\mathbf{x} \times \mathbf{y} \times \mathbf{z}) \neq \mathbf{v} \times (\mathbf{w} \times \mathbf{x} \times \mathbf{y}) \times \mathbf{z} \neq (\mathbf{v} \times \mathbf{w} \times \mathbf{x}) \times \mathbf{y} \times \mathbf{z}$,
5. $\mathbf{v} \bullet \mathbf{w} \times \mathbf{x} \times \mathbf{y} = \begin{vmatrix} v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}$,
6. $\mathbf{v} \bullet \mathbf{w} \times \mathbf{x} \times \mathbf{y} \neq 0$ *if and only if* $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ *and* $\mathbf{y}$ *are linearly independent,*
7. *If any two vectors in* $\mathbf{v} \bullet \mathbf{w} \times \mathbf{x} \times \mathbf{y}$ *are reversed, the product changes sign. Explicitly*

$$\begin{gathered}
\mathbf{v} \bullet \mathbf{w} \times \mathbf{x} \times \mathbf{y} = \mathbf{v} \bullet \mathbf{x} \times \mathbf{y} \times \mathbf{w} = \mathbf{v} \bullet \mathbf{y} \times \mathbf{w} \times \mathbf{x} = \mathbf{w} \bullet \mathbf{v} \times \mathbf{y} \times \mathbf{x} = \mathbf{w} \bullet \mathbf{x} \times \mathbf{v} \times \mathbf{y} = \\
\mathbf{w} \bullet \mathbf{y} \times \mathbf{x} \times \mathbf{v} = \mathbf{x} \bullet \mathbf{v} \times \mathbf{w} \times \mathbf{y} = \mathbf{x} \bullet \mathbf{w} \times \mathbf{y} \times \mathbf{v} = \mathbf{x} \bullet \mathbf{y} \times \mathbf{v} \times \mathbf{w} = \mathbf{y} \bullet \mathbf{v} \times \mathbf{x} \times \mathbf{w} = \\
\mathbf{y} \bullet \mathbf{w} \times \mathbf{v} \times \mathbf{x} = \mathbf{y} \bullet \mathbf{x} \times \mathbf{w} \times \mathbf{v} = \\
-\mathbf{v} \bullet \mathbf{w} \times \mathbf{y} \times \mathbf{x} = -\mathbf{v} \bullet \mathbf{x} \times \mathbf{w} \times \mathbf{y} = -\mathbf{v} \bullet \mathbf{y} \times \mathbf{x} \times \mathbf{w} = -\mathbf{w} \bullet \mathbf{v} \times \mathbf{x} \times \mathbf{y} = -\mathbf{w} \bullet \mathbf{x} \times \mathbf{y} \times \mathbf{v} = \\
-\mathbf{w} \bullet \mathbf{y} \times \mathbf{v} \times \mathbf{x} = -\mathbf{x} \bullet \mathbf{v} \times \mathbf{y} \times \mathbf{w} = -\mathbf{x} \bullet \mathbf{w} \times \mathbf{v} \times \mathbf{y} = -\mathbf{x} \bullet \mathbf{y} \times \mathbf{w} \times \mathbf{v} = -\mathbf{y} \bullet \mathbf{v} \times \mathbf{w} \times \mathbf{x} = \\
-\mathbf{y} \bullet \mathbf{w} \times \mathbf{x} \times \mathbf{v} = -\mathbf{y} \bullet \mathbf{x} \times \mathbf{v} \times \mathbf{w},
\end{gathered}$$

8. $\mathbf{v} \bullet \mathbf{w} \times \mathbf{x} \times \mathbf{y} = -\mathbf{v} \times \mathbf{w} \times \mathbf{x} \bullet \mathbf{y}$.

Proof. The first and second properties follow immediately from Definition 10. The third property is deduced with the assistance of Mathematica. To do this, both terms are calculated and subtracted from the equality. Then the rule-based programming paradigm is used to substitute $\bar{v}$ for $\text{Array}[v, 4] = \{v[1], v[2], v[3], v[4]\}$, ..., $\bar{z}$ for $\text{Array}[z, 4] = \{z[1], z[2], z[3], z[4]\}$ and the result is the null vector, this validates equality.

$$\bar{v} \times \bar{w} \times (\bar{x} \times \bar{y} \times \bar{z}) - \text{Det}\left[\begin{pmatrix} \bar{x} & \bar{y} & \bar{z} \\ \bar{x}.\bar{w} & \bar{y}.\bar{w} & \bar{z}.\bar{w} \\ \bar{x}.\bar{v} & \bar{y}.\bar{v} & \bar{z}.\bar{v} \end{pmatrix}\right] /. \text{OverBar}[a_] \to \text{Array}[a, 4]//\text{Simplify}$$

$$\{0, 0, 0, 0\}$$

The fourth property is deduced from the third property. The sixth and seventh properties are deduced from the fifth property. The last property is deduced from the seventh property and the commutativity of the dot product.

Theorem 1. *The cross product* $\mathbf{v} \times \mathbf{w} \times \mathbf{x}$ *is orthogonal to* $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{x}$, *and has length such that*

$$\|\mathbf{v} \times \mathbf{w} \times \mathbf{x}\|^2 = \begin{vmatrix} \mathbf{v} \bullet \mathbf{v} & \mathbf{v} \bullet \mathbf{w} & \mathbf{v} \bullet \mathbf{x} \\ \mathbf{w} \bullet \mathbf{v} & \mathbf{w} \bullet \mathbf{w} & \mathbf{w} \bullet \mathbf{x} \\ \mathbf{x} \bullet \mathbf{v} & \mathbf{x} \bullet \mathbf{w} & \mathbf{x} \bullet \mathbf{x} \end{vmatrix}.$$

Proof. The sixth property of the Lemma 1 shows the perpendicularity of $\mathbf{v}\times\mathbf{w}\times\mathbf{x}$ with $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$. On the other hand, by the definition of norm and the eighth and third properties of Lemma 1, we have to

$$\begin{aligned}\|\mathbf{v}\times\mathbf{w}\times\mathbf{x}\|^2 &= (\mathbf{v}\times\mathbf{w}\times\mathbf{x})\bullet(\mathbf{v}\times\mathbf{w}\times\mathbf{x}) = -(\mathbf{v}\times\mathbf{w}\times\mathbf{x})\times\mathbf{v}\times\mathbf{w}\bullet\mathbf{x}\\ &= -\mathbf{v}\times\mathbf{w}\times(\mathbf{v}\times\mathbf{w}\times\mathbf{x})\bullet\mathbf{x} = -\begin{vmatrix}\mathbf{v} & \mathbf{w} & \mathbf{x}\\ \mathbf{v}\bullet\mathbf{w} & \mathbf{w}\bullet\mathbf{w} & \mathbf{x}\bullet\mathbf{w}\\ \mathbf{v}\bullet\mathbf{v} & \mathbf{w}\bullet\mathbf{v} & \mathbf{x}\bullet\mathbf{v}\end{vmatrix}\bullet\mathbf{x}\\ &= \begin{vmatrix}\mathbf{v}\bullet\mathbf{v} & \mathbf{v}\bullet\mathbf{w} & \mathbf{v}\bullet\mathbf{x}\\ \mathbf{w}\bullet\mathbf{v} & \mathbf{w}\bullet\mathbf{w} & \mathbf{w}\bullet\mathbf{x}\\ \mathbf{x}\bullet\mathbf{v} & \mathbf{x}\bullet\mathbf{w} & \mathbf{x}\bullet\mathbf{x}\end{vmatrix}\end{aligned}$$

Theorem 2. *Let* $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{z}$ *be tangent vectors to* $\mathbb{R}^4$ *at the same point* $\mathbf{p}$, *then*

$$\begin{aligned}&\mathbf{v}\times\mathbf{w}\times(\mathbf{x}\times\mathbf{y}\times\mathbf{z})+\mathbf{v}\times\mathbf{x}\times(\mathbf{w}\times\mathbf{y}\times\mathbf{z})+\mathbf{v}\times\mathbf{y}\times(\mathbf{z}\times\mathbf{x}\times\mathbf{w})+\\ &\quad\mathbf{v}\times\mathbf{z}\times(\mathbf{w}\times\mathbf{x}\times\mathbf{y})+\mathbf{w}\times\mathbf{x}\times(\mathbf{v}\times\mathbf{y}\times\mathbf{z})+\mathbf{w}\times\mathbf{y}\times(\mathbf{z}\times\mathbf{x}\times\mathbf{v})+\\ &\quad\mathbf{w}\times\mathbf{z}\times(\mathbf{v}\times\mathbf{x}\times\mathbf{y})+\mathbf{x}\times\mathbf{y}\times(\mathbf{z}\times\mathbf{w}\times\mathbf{v})+\mathbf{x}\times\mathbf{z}\times(\mathbf{v}\times\mathbf{w}\times\mathbf{y})+\\ &\qquad\mathbf{y}\times\mathbf{z}\times(\mathbf{x}\times\mathbf{w}\times\mathbf{v})=0\,.\end{aligned}$$

Proof. Taking into account that the Jacobi identity exists for three vectors tangent to $\mathbb{R}^3$, we proceeded to develop a Mathematica program in which the possible combinations of the cross product of five vectors tangent to $\mathbb{R}^4$ are considered; in such a way that, an extension of the Jacobi identity was obtained for five vectors tangent to $\mathbb{R}^4$. The program returned twelve possible combinations, of which the third was chosen for representativeness reasons. Obviously, from this choice the other possibilities can be deduced.

```
vecs = {x̄, ȳ, z̄, v̄, w̄};
perms = Select[Permutations[vecs, {3}], OrderedQ];
Tu = Tuples[{Identity, Reverse}, {Length[perms]}];
permsTu = MapThread[#1@@{#2}&, {#, perms}]&@Tu;
Table[
  permsk = permsTu[[k]];
  g = Complement[vecs, #]&@permsk;
  f = MapThread[Cross[Sequence@@#1, Cross@@#2]&, {g, permsk}];
  s = Plus@@(f /. OverBar[a_] → Array[a, 4]);
  If[s===Table[0, 4], Plus@@f == 0, Nothing],
  {k, Length[Tu]}][[3]]
```

$$\bar{v}\times\bar{w}\times(\bar{x}\times\bar{y}\times\bar{z})+\bar{v}\times\bar{x}\times(\bar{w}\times\bar{y}\times\bar{z})+\bar{v}\times\bar{y}\times(\bar{z}\times\bar{x}\times\bar{w})+\bar{v}\times\bar{z}\times(\bar{w}\times\bar{x}\times\bar{y})+$$
$$\bar{w}\times\bar{x}\times(\bar{v}\times\bar{y}\times\bar{z})+\bar{w}\times\bar{y}\times(\bar{z}\times\bar{x}\times\bar{v})+\bar{w}\times\bar{z}\times(\bar{v}\times\bar{x}\times\bar{y})+\bar{x}\times\bar{y}\times(\bar{z}\times\bar{w}\times\bar{v})+$$
$$\bar{x}\times\bar{z}\times(\bar{v}\times\bar{w}\times\bar{y})+\bar{y}\times\bar{z}\times(\bar{x}\times\bar{w}\times\bar{v})==0$$

Theorem 3. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are tangent vectors to* $\mathbb{R}^4$ *at the same point* $\mathbf{p}$*, then the area,* S*, of the parallelogram with sides* $\mathbf{v}$ *and* $\mathbf{w}$ *is*

$$S=\sqrt{\begin{vmatrix}\mathbf{v}\bullet\mathbf{v} & \mathbf{v}\bullet\mathbf{w}\\ \mathbf{w}\bullet\mathbf{v} & \mathbf{w}\bullet\mathbf{w}\end{vmatrix}}\,.$$

Proof. Figure 1 shows the parallelogram. If h is the distance from $\mathbf{w}$ to the segment from $\mathbf{p}$ to $\mathbf{v}$ and L is the length of this segment, then the area will be $A=Lh$. The L length can be expressed as

$$L=\sqrt{\mathbf{v}\bullet\mathbf{v}}\,.$$

The point Q (Fig. 2) is on the segment generated by $\mathbf{v}$, i.e.

$$Q=t\mathbf{v};\quad t\in\mathbb{R}\,.$$

According to Fig. 2 it is true that

$$(\mathbf{w}-Q)\bullet\mathbf{v}=0\,.$$

After solving this equation we obtain

$$t=\frac{\mathbf{v}\bullet\mathbf{w}}{\mathbf{v}\bullet\mathbf{v}}\,.$$

So that

$$\begin{aligned}h&=\sqrt{(\mathbf{w}-Q)\bullet(\mathbf{w}-Q)}\\&=\sqrt{\mathbf{w}\bullet\mathbf{w}+t^2\,\mathbf{v}\bullet\mathbf{v}-2t\,\mathbf{v}\bullet\mathbf{w}}\,.\end{aligned}$$

Finally, after making the respective substitutions and simplifications with the Mathematica assistance, we obtain

$$\begin{aligned}S&=Lh\\&=\sqrt{\begin{vmatrix}\mathbf{v}\bullet\mathbf{v} & \mathbf{v}\bullet\mathbf{w}\\ \mathbf{w}\bullet\mathbf{v} & \mathbf{w}\bullet\mathbf{w}\end{vmatrix}}\,.\end{aligned}$$

A more intuitive description of S is

$$\begin{aligned}S&=\sqrt{\begin{vmatrix}\|\mathbf{v}\|^2 & \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\vartheta\\ \|\mathbf{w}\|\,\|\mathbf{v}\|\cos\vartheta & \|\mathbf{w}\|^2\end{vmatrix}}\\&=\|\mathbf{v}\|\,\|\mathbf{w}\|\sqrt{\begin{vmatrix}1 & \cos\vartheta\\ \cos\vartheta & 1\end{vmatrix}}\\&=\|\mathbf{w}\|\,\|\mathbf{v}\|\sin\vartheta\,,\end{aligned}$$

where $0\leqq\vartheta\leqq\pi$ is the smaller of the two angles from $\mathbf{v}$ to $\mathbf{w}$ (Fig. 1).

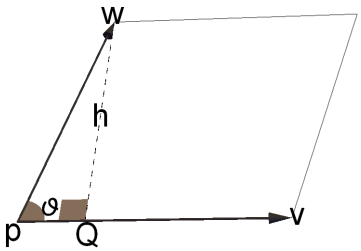

Fig. 1. Parallelogram with edges $\mathbf{v}$ and $\mathbf{w}$.

Theorem 4. *If* $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{x}$ *are tangent vectors to* $\mathbb{R}^4$ *at the same point* $\mathbf{p}$, *then* $\|\mathbf{v} \times \mathbf{w} \times \mathbf{x}\|$ *is the volume,* V, *of the parallelepiped with edges* $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{x}$.

Proof. Figure 2 shows the parallelepiped. If h is the distance from $\mathbf{x}$ to the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$ and S is the area of this parallelogram, then the volume will be $V = Sh$. The S area can be expressed as

$$S = \sqrt{\begin{vmatrix} \mathbf{v} \bullet \mathbf{v} & \mathbf{v} \bullet \mathbf{w} \\ \mathbf{w} \bullet \mathbf{v} & \mathbf{w} \bullet \mathbf{w} \end{vmatrix}}.$$

The point Q (Fig. 2) is on the plane generated by $\mathbf{v}$ and $\mathbf{w}$, i.e.

$$Q = t\mathbf{v} + s\mathbf{w}; \quad s, t \in \mathbb{R}.$$

According to Fig. 2 it is true that

$$\begin{cases} (\mathbf{x} - Q) \bullet \mathbf{v} = 0, \\ (\mathbf{x} - Q) \bullet \mathbf{w} = 0. \end{cases}$$

After solving this system we obtain

$$t = \frac{\begin{vmatrix} \mathbf{v} \bullet \mathbf{x} & \mathbf{v} \bullet \mathbf{w} \\ \mathbf{w} \bullet \mathbf{x} & \mathbf{w} \bullet \mathbf{w} \end{vmatrix}}{\begin{vmatrix} \mathbf{v} \bullet \mathbf{v} & \mathbf{v} \bullet \mathbf{w} \\ \mathbf{w} \bullet \mathbf{v} & \mathbf{w} \bullet \mathbf{w} \end{vmatrix}}, \quad s = \frac{\begin{vmatrix} \mathbf{v} \bullet \mathbf{v} & \mathbf{v} \bullet \mathbf{x} \\ \mathbf{w} \bullet \mathbf{v} & \mathbf{w} \bullet \mathbf{x} \end{vmatrix}}{\begin{vmatrix} \mathbf{v} \bullet \mathbf{v} & \mathbf{v} \bullet \mathbf{w} \\ \mathbf{w} \bullet \mathbf{v} & \mathbf{w} \bullet \mathbf{w} \end{vmatrix}}.$$

So that

$$\begin{aligned} h &= \sqrt{(\mathbf{x} - Q) \bullet (\mathbf{x} - Q)} \\ &= \sqrt{\mathbf{x} \bullet \mathbf{x} + t^2\, \mathbf{v} \bullet \mathbf{v} + s^2\, \mathbf{w} \bullet \mathbf{w} - 2t\, \mathbf{v} \bullet \mathbf{x} - 2s\, \mathbf{w} \bullet \mathbf{x} + 2st\, \mathbf{v} \bullet \mathbf{w}}. \end{aligned}$$

Finally, after making the respective substitutions and simplifications with the Mathematica assistance, we obtain

$$\begin{aligned} V &= Sh \\ &= \sqrt{\begin{vmatrix} \mathbf{v} \bullet \mathbf{v} & \mathbf{v} \bullet \mathbf{w} & \mathbf{v} \bullet \mathbf{x} \\ \mathbf{w} \bullet \mathbf{v} & \mathbf{w} \bullet \mathbf{w} & \mathbf{w} \bullet \mathbf{x} \\ \mathbf{x} \bullet \mathbf{v} & \mathbf{x} \bullet \mathbf{w} & \mathbf{x} \bullet \mathbf{x} \end{vmatrix}} \\ &= \|\mathbf{v} \times \mathbf{w} \times \mathbf{x}\|. \end{aligned}$$

A more intuitive description of the length of a cross product is

$$\|\mathbf{v}\times\mathbf{w}\times\mathbf{x}\| = \sqrt{\begin{vmatrix} \|\mathbf{v}\|^2 & \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\vartheta & \|\mathbf{v}\|\,\|\mathbf{x}\|\cos\varphi \\ \|\mathbf{w}\|\,\|\mathbf{v}\|\cos\vartheta & \|\mathbf{w}\|^2 & \|\mathbf{w}\|\,\|\mathbf{x}\|\cos\gamma \\ \|\mathbf{x}\|\,\|\mathbf{v}\|\cos\varphi & \|\mathbf{x}\|\,\|\mathbf{w}\|\cos\gamma & \|\mathbf{x}\|^2 \end{vmatrix}}$$

$$= \|\mathbf{v}\|\,\|\mathbf{w}\|\,\|\mathbf{x}\| \sqrt{\begin{vmatrix} 1 & \cos\vartheta\cos\varphi \\ \cos\vartheta & 1 & \cos\gamma \\ \cos\varphi\cos\gamma & 1 \end{vmatrix}}$$

where $0 \leqq \vartheta, \varphi, \gamma \leqq \pi$ is the smaller of the two angles from $\mathbf{v}$ to $\mathbf{w}$,$\mathbf{w}$ to $\mathbf{x}$ and $\mathbf{v}$ to $\mathbf{x}$, respectively (Fig. 2).

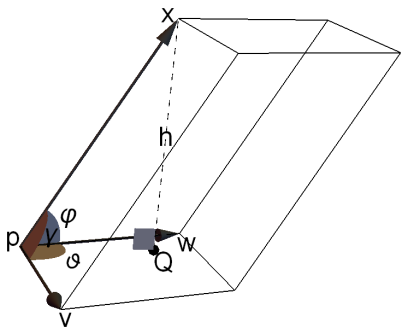

Fig. 2. Parallelepiped with edges $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$.

Corollary 1. *If* $\mathbf{v}_1$, $\mathbf{v}_2$ *and* $\mathbf{v}_3$ *are tangent vectors to* $\mathbb{R}^4$ *at the same point* $\mathbf{p}$ *and* $S_{i,j}$ *is the area of the parallelogram with sides* $\mathbf{v}_i$ *and* $\mathbf{v}_j$, *then*

$$\|\mathbf{v}_1\times\mathbf{v}_2\times\mathbf{v}_3\|^2 = S_{23}^2\|\mathbf{v}_1\|^2 + S_{13}^2\|\mathbf{v}_2\|^2 + S_{12}^2\|\mathbf{v}_3\|^2 + \\ + 2\,\mathbf{v}_1\bullet\mathbf{v}_2\,\mathbf{v}_1\bullet\mathbf{v}_3\,\mathbf{v}_2\bullet\mathbf{v}_3 - 2\|\mathbf{v}_1\|^2\,\|\mathbf{v}_2\|^2\,\|\mathbf{v}_3\|^2\,.$$

Proof. The proof is immediate from the Theorem 1 and the Theorem 3.

Corollary 2. *If* $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ *and* $\mathbf{y}$ *are tangent vectors to* $\mathbb{R}^4$ *at the same point* $\mathbf{p}$, *then* $|\mathbf{v}\bullet\mathbf{w}\times\mathbf{x}\times\mathbf{y}|$ *is the hypervolume of the hyperparallelepiped with edges* $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ *and* $\mathbf{y}$.

Proof. Figure 3 shows the hyperparallelepiped. By Theorem 4 the volume of the parallelepiped with edges $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{y}$ is $V = \|\mathbf{w}\times\mathbf{x}\times\mathbf{y}\|$. If h is the height, then $h = \|\mathbf{v}\|\,|\cos\theta|$, $0 \leqq \theta \leqq \pi$ is the smaller of the two angles from $\mathbf{v}$ to $\mathbf{w}\times\mathbf{x}\times\mathbf{y}$. Therefore, the hypervolume of the hyperparalelepiped is

$$W = h\,V = \|\mathbf{v}\|\,\|\mathbf{w}\times\mathbf{x}\times\mathbf{y}\|\,|\cos\theta| = |\mathbf{v}\bullet\mathbf{w}\times\mathbf{x}\times\mathbf{y}|\,.$$

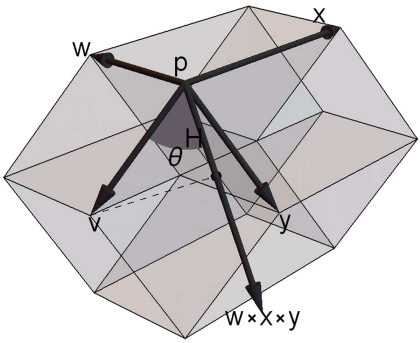

Fig. 3. Hyperparallelepiped with edges $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{y}$.

4 Conclusions

In this paper, the definition of the cross product of three vectors tangent to $\mathbb{R}^4$ at the same point has been stated, based on this definition a lemma has been enunciated in which eight properties of said product have been proposed and demonstrated. Furthermore, four theorems and two corollaries have been stated and proved. One of the theorems constitutes an extension of the Jacobi identity. In some of the proofs, programs based on the paradigms: functional, rule-based and list-based have been used; all of these supported by the Wolfram language, incorporated in Mathematica.

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