The Domination and Independent Domination Problems in Supergrid Graphs

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Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A subset D of $V(G)$ is a dominating set of G if every vertex not in D is adjacent to one vertex of D. A dominating set is called independent if it is an independent set. The domination number (resp., independent domination number) of G, denoted by $\gamma(G)$ (resp., $\gamma_{\text{ind}}(G)$), is the minimum cardinality of a dominating set (resp., independent dominating set) of G. The domination (resp., independent domination) problem is to compute a dominating set (resp., an independent dominating set) of G with size $\gamma(G)$ (resp., $\gamma_{\text{ind}}(G)$). Supergrid graphs are a natural extension for grid graphs. The domination and independent domination problems for grid graphs were known to be NP-complete. However, their complexities on supergrid graphs are still unknown. In this paper, we will prove these two problems for supergrid graphs to be NP-complete. Then, we compute $\gamma(R_{m\times n})$ and $\gamma_{\text{ind}}(R_{m\times n})$ for rectangular supergrid graphs $R_{m\times n}$ in linear time.

Keywords: Domination · Independent domination · Supergrid graph · Rectangular supergrid graph · Grid graph

1 Introduction

Let G be a graph. We will denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. Let $v \in V(G)$, and let $S \subseteq V(G)$. We use $G[S]$ to represent the subgraph induced by S. Denote by $N_G(v) = \{u \in V(G) | (u, v) \in E(G)\}$ the *open neighborhood* of vertex v, while its *closed neighborhood* is represented as $N_G[v] = N_G(v) \cup \{v\}$. Generally, let $N_G(S) = \cup_{v\in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. The *degree* of vertex v in G is the number of edges incident to v, and is denoted by $deg_G(v)$. Let D be a subset of $V(G)$. If $v \in D$ or $N_G(v) \cap D \neq \emptyset$, then we say that D *dominates* v. If D dominates every vertex of $S \subseteq V(G)$, then we

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call that D *dominates* S. When D dominates $V(G)$, D is called a *dominating set* of G. The *domination number* of a graph G is the minimum cardinality of a dominating set of G, and it is denoted by $\gamma(G)$. A *minimum dominating set* of G is a dominating set with size $\gamma(G)$. The *domination problem* is to compute a minimum dominating set of G, and it is a well-known NP-complete problem for general graphs [16]. This problem is still NP-complete for some special graph classes, including 4-regular planar graphs [16], grid graphs [10], cubic bipartite graphs [29], etc.

Many variants of the domination problem do exist and they are to compute a minimum dominating set with some additional properties, e.g., to be independent or to induce a connected graph. These problems arise in a lots of distributed network applications in which they are requested to locate the smallest number of centers in networks so that every vertex is nearby at least one center, where the set of centers forms a minimum dominating set, and some restricted conditions are satisfied if it is the domination variant. The concept of these domination related problems have many applications and have been widely studied in literature (see [21,22]); a rough estimate says that it occurs in more than 6000 papers to date [20]. In this paper, we will study the domination problem and its one variant, namely *independent domination problem*.

A set of vertices is said to be *independent* if its any two vertices are not adjacent. An *independent dominating set* of a graph G is a dominating set I such that I is independent. The independent domination number of a graph G is the minimum size of an independent dominating set in G, and it is denoted by $\gamma_{\text{ind}}(G)$. Since an independent dominating set of G is also a dominating set of G, $\gamma(G) \leqslant \gamma_{\text{ind}}(G)$ for any graph G. The *independent domination problem* is to compute an independent dominating set of G with size $\gamma_{\text{ind}}(G)$. This problem is NP-complete for general graphs [16]. It is still NP-complete for bipartite graphs, comparability graphs [11], planar graphs [5], chordal bipartite graphs [14], cubic bipartite graphs [29], grid graphs [10], etc. However, it is polynomial solvable when the input is restricted to some special graphs, including interval graphs [6], circular-arc graphs [7], bounded clique-width graphs [12], etc. For more related works and applications on independent domination, we refer the reader to the survey on independent domination in graphs given in [17], and more results regarding this problem in [1,2,4,13].

The *two-dimensional integer grid* G^∞ is an infinite graph in which its vertex set consists of all points of the Euclidean plane with integer coordinates, and two vertices are adjacent if and only if their (Euclidean) distance is 1. A *grid graph* is a finite and vertex-induced subgraph of G^∞. The *two-dimensional supergrid graph* S^∞ is the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if the difference of their x or y coordinates is not larger than 1. A supergrid graph G_s is a finite and undirected graph such that $V(G_s) \subset V(S^\infty)$ and $E(G_s) \subset E(S^\infty)$. For a vertex $v \in V(G_s)$, it is represented as (v_x, v_y), where v_x and v_y are the x and y coordinates of v respectively. Then, $1 \leqslant deg_{G_s}(v) \leqslant 8$. Grid graphs are not subclasses of supergrid graphs, and the converse is also true: these two

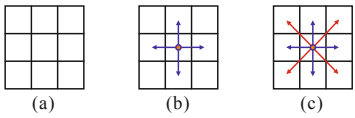


Fig. 1. (a) A set of lattices, (b) the neighbors of one lattice in a grid graph, and (c) the neighbors of one lattice in a supergrid graph, where each lattice is denoted by a vertex in a graph and arrow lines indicate the adjacent neighbors of one lattice.

graph classes contain common elements (vertices), but they are distinct since their edge sets are distinct. Clearly, grid graphs are bipartite [26], but supergrid graphs are not bipartite. A rectangular grid (or called complete grid) graph $G_{m\times n}$ has mn nodes with node $u = (u_x, u_y)$ adjacent to $v = (v_x, v_y)$ if and only if $|u_x - v_x| + |u_y - v_y| = 1$. A rectangular supergrid graph $R_{m\times n}$ is a supergrid graph with vertex set $\{(v_x, v_y)|1 \leqslant v_x \leqslant n \text{ and } 1 \leqslant v_y \leqslant m\}$ and edge set $\{(u, v)|0 \leqslant |u_x - v_x| \leqslant 1 \text{ and } 0 \leqslant |u_y - v_y| \leqslant 1\}$. Then, for $u \in V(G_{m\times n})$ and $v \in V(R_{m\times n})$, $2 \leqslant deg_{G_{m\times n}}(u) \leqslant 4$ and $3 \leqslant deg_{R_{m\times n}}(v) \leqslant 8$ if $m, n \geqslant 2$. Throughout this paper, we will denote by $(1, 1)$ the coordinates of the top-left vertex of a grid or supergrid graph.

An intuitive motivation of proposing supergrid graphs is as follows. Consider a set of lattices, depicted in Fig. 1(a), where each lattice is denoted as a vertex in a graph. For a grid graph, the neighbors of a lattice include its upper, down, left, and right lattices, see Fig. 1(b). However, in the real word and other applications, the neighbors of a lattice may also contain its upper-right, upper-left, down-right, and down-left adjacent lattices, see Fig. 1(c). For example, the sewing trace of a computerized sewing machine is such an application [23]. Thus, supergrid graphs can be used in these applications.

A brief summary of related works is given below. The domination problem on grid graphs was known to be NP-complete [10]. Many researchers have studied the domination numbers of rectangular grid graphs [8,9,19]. In [18], Gonçalves *et al.* computed $\gamma(G_{m\times n}) = \lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$ for $n \geqslant m \geqslant 16$. This result verified the conjecture in [8]. In [13], Crevals and Östergård computed $\gamma_{\text{ind}}(G_{m\times n}) = \lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$ for $n \geqslant m \geqslant 16$. In fact, $\gamma(G_{m\times n}) = \gamma_{\text{ind}}(G_{m\times n})$ for rectangular grid graph $G_{m\times n}$ by the result in [30]. In [28], the author computed $\gamma_{\text{ind}}(R_{m\times n}) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$. In this paper, we provide another way to compute $\gamma_{\text{ind}}(R_{m\times n})$. In [15], Gagnon *et al.* stated that $\gamma(R_{m\times n}) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$ is trivially known. Unfortunately, they do not provide any proof. In this paper, we will give a tight proof to verify $\gamma(R_{m\times n}) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$. In [23], we first introduced supergrid graphs and proved the Hamiltonian problems on supergrid graphs to be NP-complete. In [24,25,27], the Hamiltonian related properties on some special supergrid graphs have been studied. In this paper, we will prove that the domination and independent domination problems on (general) supergrid graphs are NP-complete. Then, we compute $\gamma(R_{m\times n}) = \gamma_{\text{ind}}(R_{m\times n}) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$ by a tight proof.

The rest of this paper is structured as follows. Section 2 introduces some notations and one related result in the literature. In Sect. 3, we prove that the domination and independent domination problems for (general) supergrid graphs are NP-complete. Section 4 computes $\gamma(R_{m\times n}) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$ by a formally proof. In Sect. 5, we use a known result to verify $\gamma_{\text{ind}}(R_{m\times n}) = \gamma(R_{m\times n})$. Finally, some conclusions are given in Sect. 6.

2 Preliminaries

In this section, we will introduce some notations and one result in the literature. For two sets X and Y, let $X - Y$ denote the set of elements in X that are not in Y. A path P in a graph G is a sequence of adjacent vertices starting from v_1 and ending at v_k, represented as $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{k-1} \rightarrow v_k$, where all the vertices $v_1, v_2, \cdots, v_k$ are distinct except that possibly the path is a cycle when $v_1 = v_k$. A path starting from vertex v_1 and ending at vertex v_k is denoted by (v_1, v_k)-path, and a path with n vertices is denoted by P_n if no ambiguous appears.

Let G_s be a supergrid graph, and let $v \in V(G_s)$. Then, $v = (v_x, v_y)$, where v_x and v_y are the x and y coordinates of v respectively, and $1 \leqslant deg_{G_s}(v) \leqslant 8$. Rectangular supergrid graphs form a special class of supergrid graph and they first appeared in [23], in which the Hamiltonian cycle problem is linear solvable. A rectangular supergrid graph $R_{m\times n}$ is a supergrid graph with vertex set $V(R_{m\times n}) = \{v = (v_x, v_y) | 1 \leqslant v_x \leqslant n \text{ and } 1 \leqslant v_y \leqslant m\}$ and edge set $E(R_{m\times n}) = \{(u, v) | 0 \leqslant |u_x - v_x| \leqslant 1 \text{ and } 0 \leqslant |u_y - v_y| \leqslant 1\}$. Then, $R_{m\times n}$ contains m rows and n columns of vertices. In this paper, w.l.o.g. we will assume that $m \leqslant n$. Let v be a vertex in $R_{m\times n}$ with $m \geqslant 2$. The vertex v is called a *corner* of $R_{m\times n}$ if $deg_{R_{m\times n}}(v) = 3$. There are four corners of $R(m, n)$ including *upper-left*, *upper-right*, *down-left*, and *down-right* corners coordinated as $(1, 1)$, $(n, 1)$, $(1, m)$, and (n, m), respectively (see Fig. 2). The edge (u, v) is called *horizontal* (resp., *vertical*) if $u_y = v_y$ (resp., $u_x = v_x$), and is said to be *crossed* if it is neither a horizontal nor a vertical edge. For example, Fig. 2 shows a rectangular supergrid graph $R_{8\times 10}$ and it also depicts the types of corners and edges. Note that a grid graph contains horizontal and vertical edges, but it contains no crossed edge. A sequence of consecutive horizontal (resp., vertical) edges, $e_1, e_2, \cdots, e_k$, is called a *horizontal* (resp., *vertical*) *edge-line*, denoted by L_k, where e_i and e_{i+1} contains a common vertex for $1 \leqslant i \leqslant k-1$. Let μ, ν be two vertices in L_k such that they are only incident to one edge. Then, a horizontal or vertical edge-line is denoted by (μ, ν)-edge-line. For instance, the bold consecutive horizontal edges in Fig. 2 indicate a horizontal edge-line L_6.

In our method, we need to partition a rectangular supergrid graph into two disjoint parts. The partition is defined as follows:

Definition 1. *Let S be a rectangular supergrid graph $R_{m\times n}$. A separation operation on S is a partition of S into two vertex-disjoint supergrid subgraphs S_1 and S_2, and is called vertical (resp., horizontal) if it consists of a set of horizontal*

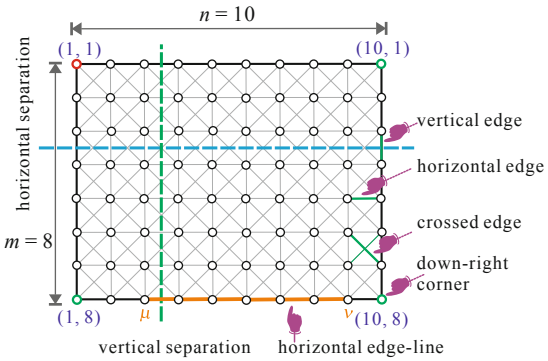


Fig. 2. A rectangular supergrid graph $R_{8\times 10}$, where bold dashed lines indicate vertical and horizontal separations, and a set of 6 bold edges indicates a horizontal edge-line L_6.

(resp., vertical) edges. For instance, the bold dashed vertical (resp., horizontal) line in Fig. 2 depicts a vertical (resp., horizontal) separation of $R_{8\times 10}$ which is partitioned into $R_{8\times 3}$ and $R_{8\times 7}$ (resp., $R_{3\times 10}$ and $R_{5\times 10}$).

In [9], Chang *et al.* computed the domination number $\gamma(P_n)$ of path P_n as follows:

Lemma 1. *(see [9].)* $\gamma(P_n) = \gamma(R_{1\times n}) = \lfloor \frac{n+2}{3} \rfloor = \lceil \frac{n}{3} \rceil$.

3 NP-completeness Results

In this section, the domination and independent domination problems for (general) supergrid graphs will be proved to be NP-complete. In 1990, Clark *et al.* [10] showed that the domination problem for (general) grid graphs is NP-complete. We will reduce it to the domination and independent domination problems on supergrid graphs.

Theorem 1. *(See [10].) The domination problem on (general) grid graphs is NP-complete.*

We will reduce the domination problem on grid graphs to the domination problem on supergrid graphs. Given a grid graph G_g, we will construct a supergrid graph G_s to satisfy that G_g contains a dominating set D with size $|D| \leqslant k$ if and only if G_s contains a dominating set $\hat{D}$ with size $|\hat{D}| \leqslant k + 2|E(G_g)|$. The construction steps are sketched as follows. First, we enlarge the input grid graph G_g such that each edge of G_g is transformed into a horizontal or vertical edge-line L_7 with 7 edges; i.e., enlarge each edge of G_g by 7 times. Let the enlarged grid graph be G'_g. For an example, Fig. 3(b) depicts grid graph G'_g enlarged from grid graph G_g in Fig. 3(a). In the second step, each horizontal or vertical (u, v)-edge-line of graph G'_g is replaced by a (u, v)-path which is a small supergrid graph

Algorithm 1: The supergrid graph construction algorithm

Input: A grid graph G_g. (see Fig. 3(a))
Output: A supergrid graph G_s. (see Fig. 3(d))
Method: // an algorithm constructing a supergrid graph from a grid graph
1. enlarge G_g to a grid graph G'_g such that each edge of G_g is transformed into a horizontal or vertical edge-line L_7 with 7 edges; (see Fig. 3(b))
2. **for** each horizontal or vertical (u, v)-path in G'_g, where $u, v \in V(G_g)$, replace it by a snake path to connect u and v;
   // this replaced path is a small supergrid graph and is called snake (u, v)-path, denoted by $S(u, v)$ (see Fig. 3(c)), where $u, v \in V(G_g)$ are called connectors of $S(u, v)$
3. the constructed graph is a supergrid graph G_s (see Fig. 3(d)), and **output** G_s.

and is an undirected path P_8, where $u, v \in V(G_g)$. The path connecting u and v is called a *snake* (u, v)*-path*, denoted by $S(u, v)$. Figure 3(c) depicts a snake (u, v)-path. Note that there exists no directionality for edges on $S(u, v)$. Finally, the constructed graph is a supergrid graph G_s. For instance, Fig. 3(d) shows the supergrid graph G_s constructed from grid graph G_g in Fig. 3(a). Note that each snake path is an undirected path P_8 with 8 vertices. Algorithm 1 presents the details of our construction.

Next, we will show that there exists an arrangement of snake paths in G_s such that they are disjoint except their connectors. The arrangement rule is sketched as follows: Consider a grid graph G_g. Since every grid graph is a subgraph of a rectangular grid graph (or called complete grid graph) with maximum number of edges, we can only consider the arrangement of snake paths for a rectangular grid graph $G_{m\times n}$, where $G_g = G_{m\times n}$. Let u and v be two adjacent vertices of $G_{m\times n}$ such that $u_x \leqslant v_x$ and $u_y \leqslant v_y$. Let P_v^u be the enlarged horizontal or vertical edge-line of edge (u, v) in G_g. We arrange snake paths of G_s by the following rule:

Case 1: $v_x = u_x + 1$. In this case, P_v^u is a horizontal edge-line. There are two subcases:

Case 1.1: u_y is odd. If u_x is odd, then the snake path $S(u, v)$ is placed above the edge-line P_v^u; otherwise, it is placed below P_v^u.

Case 1.2: u_y is even. If u_x is odd, then the snake path $S(u, v)$ is placed below the edge-line P_v^u; otherwise, it is placed above P_v^u.

Case 2: $v_y = u_y + 1$. In this case, P_v^u is a vertical edge-line. There are two subcases:

Case 2.1: u_x is odd. If u_y is odd, then the snake path $S(u, v)$ is placed to be the right of edge-line P_v^u; otherwise, it is placed to the left of P_v^u.

Case 2.2: u_x is even. If u_y is odd, then the snake path $S(u, v)$ is placed to be the left of edge-line P_v^u; otherwise, it is placed to the right of P_v^u.

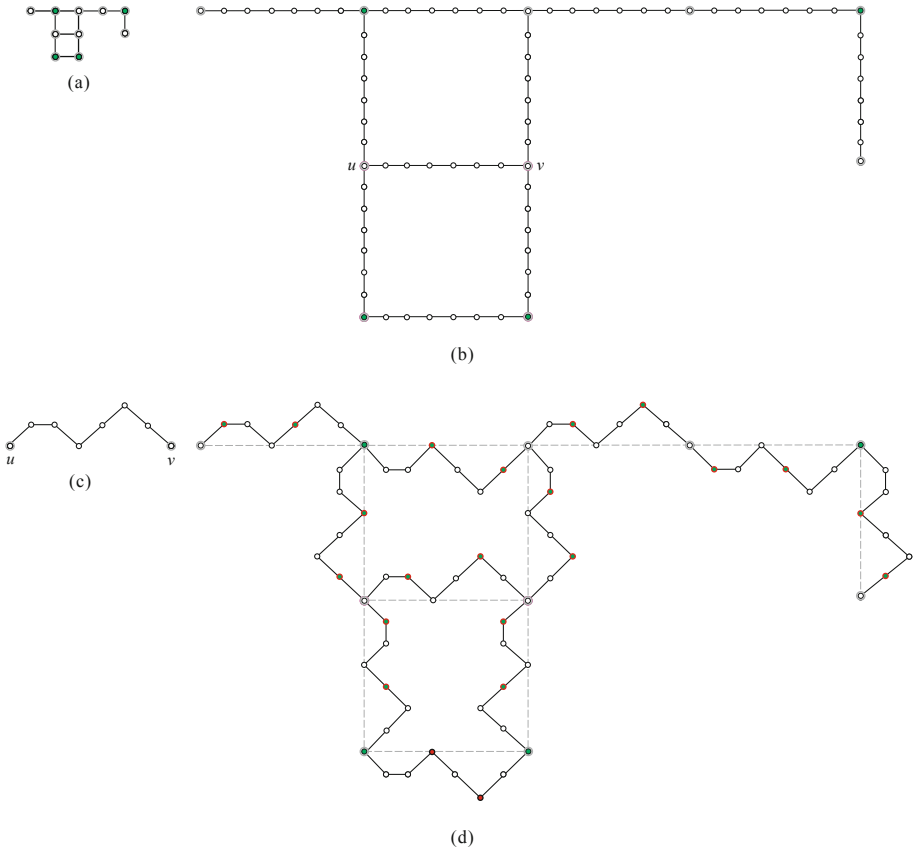


Fig. 3. (a) A grid graph G_g, (b) a grid graph G'_g by enlarging each edge of G_g 7 times, (c) a snake (u, v)-path $S(u, v)$ to replace the enlarged edge-line of G'_g, and (d) a constructed supergrid graph G_s obtained from G'_g by replacing each enlarged edge-line with a snake path in (c), where solid lines indicate the edges of G_g and G_s, double circles represent the vertices of G_g, and solid circles indicate the vertices in a dominating set of G_g or G_s.

For example, Fig. 4 depicts the arrangement of snake paths in G_s for $G_g = G_{16\times16}$. The above arrangement rule is called *Rule AS* (Arrange Snake paths) and it holds the following property:

Lemma 2. *Rule AS arranges the snake paths of G_s such that these paths are disjoint except their connectors.*

Proof. Consider a square with vertices (i, j), $(i+1, j)$, $(i, j+1)$, $(i+1, j+1)$ in a grid graph G_g. There are four cases:

Case 1: i, j are even or odd. By Rule AS, the snake paths with four connectors (i, j), $(i+1, j)$, $(i, j+1)$, $(i+1, j+1)$ are arranged disjoint.

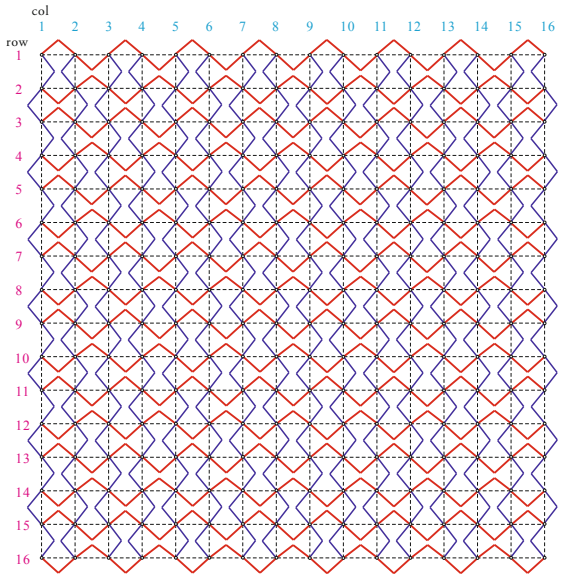


Fig. 4. An arrangement of snake paths in G_s from $G_g = G_{16\times16}$, where bold triangle lines indicate the snake paths of G_s.

Case 2: i is even and j is odd, or i is odd and j is even. By Rule AC, the snake paths with the four connectors (i, j), $(i+1, j)$, $(i, j+1)$, $(i+1, j+1)$ are arranged disjoint.

We have considered any case to locate snake paths. In a square of a grid graph G_g, there are at most two snake paths to be placed inside the square of paths in G_s and they are located inside the paths square across from each other. By the construction of snake paths, any two vertices of two adjacent snake paths are not adjacent except their connectors which are in G_g, and the height of each snake path equals to two. Thus, the lemma holds true.

Clearly, Algorithm 1, together with Rule AS, can be done in polynomial time. Thus, the following lemma holds true.

Lemma 3. *Given a grid graph G_g, Algorithm 1, together with Rule AS, constructs a supergrid graph G_s in polynomial time.*

To prove our NP-completeness result, we need to observe some domination properties of snake paths. Note that a snake path is a simple undirected path P_8. By Lemma 1, the following properties of snake (u, v)-path $S(u, v)$ can be easily verified:

Proposition 1. *Let $\hat{D}$ be a dominating set of G_s constructed by Algorithm 1, and let $S(u, v)$ be a snake path with connectors u and v. By Lemma 1, the following statements hold true:*

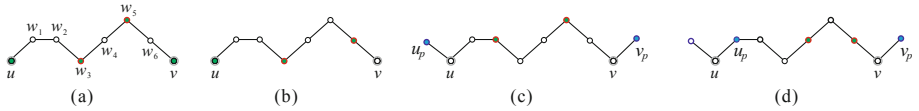


Fig. 5. The minimum dominating set of snake path $S(u,v)$ in G_s for (a) $u,v \in \hat{D}$, (b) $u \in \hat{D}$ and $v \notin \hat{D}$, (c) $u,v \notin \hat{D}$ and, $(N_{G_s}(u) - S(u,v)) \cap \hat{D} \neq \emptyset$ and $(N_{G_s}(v) - S(u,v)) \cap \hat{D} \neq \emptyset$, and (d) $u,v \notin \hat{D}$ and, $(N_{G_s}(u) - S(u,v)) \cap \hat{D} = \emptyset$ or $(N_{G_s}(v) - S(u,v)) \cap \hat{D} = \emptyset$, where $\hat{D}$ is a dominating set of G_s and solid circles indicate the vertices in $\hat{D}$.

(1) If $u,v \in \hat{D}$, then $\gamma(S(u,v) - (N_{G_s}[u] \cup N_{G_s}[v])) = 2$ (See Fig. 5(a)).
(2) If $u \in \hat{D}$ and $v \notin \hat{D}$, then $\gamma(S(u,v) - N_{G_s}[u]) = \gamma(S(u,v) - (N_{G_s}[u] \cup \{v\})) = 2$, and there exists a vertex v_d in $\hat{D} \cap S(u,v)$ such that v_d dominates v if $\hat{D} - S(u,v)$ does not dominate v (See Fig. 5(b)).
(3) If $u,v \notin \hat{D}$, then
(3-1) *if $(N_{G_s}(u) - S(u,v)) \cap \hat{D} \neq \emptyset$ and $(N_{G_s}(v) - S(u,v)) \cap \hat{D} \neq \emptyset$, then $\gamma(S(u,v) - \{u,v\}) = 2$ (See Fig. 5(c));*
(3-2) *if $(N_{G_s}(u) - S(u,v)) \cap \hat{D} = \emptyset$ or $(N_{G_s}(v) - S(u,v)) \cap \hat{D} = \emptyset$, then $\gamma(S(u,v) - \{u,v\}) = 3$ (See Fig. 5(d)).*

Let $\hat{D}$ be a dominating set of G_s, and let H be a subgraph of G_s. We will denote the restriction of $\hat{D}$ to H by $\hat{D}_{|H}$. In the above proposition, we can see that for any dominating set $\hat{D}$ of supergrid graph G_s and snake (u,v)-path $S(u,v)$, $|\hat{D}_{|S(u,v)-\{u,v\}}| \geqslant 2$. In the following, we will prove that grid graph G_g has a dominating set D with size $|D| \leqslant k$ if and only if supergrid graph G_s contains a dominating set $\hat{D}$ with size $|\hat{D}| \leqslant k + 2|E(G_g)|$. We first prove the only if part as follows.

Lemma 4. *Assume that grid graph G_g contains a dominating set D with size $|D| \leqslant k$. Then, supergrid graph G_s contains a dominating set $\hat{D}$ with size $|\hat{D}| \leqslant k + 2|E(G_g)|$*

Proof. Consider an edge (u,v) of G_g. Let $S(u,v) = u \to w_1 \to w_2 \to w_3 \to w_4 \to w_5 \to w_6 \to v$ be the snake (u,v)-path in G_s constructed from edge (u,v) of G_g, as shown in Fig. 5(a). Initially, let $\hat{D} = D$. We then consider three cases depending on whether $u,v \in D$. These cases include (1) $u,v \in D$, (2) $u \in D$ and $v \notin D$, and (3) $u,v \notin D$. For each snake (u,v)-path $S(u,v)$ in any case, we compute two vertices $w_\imath, w_\jmath$, $6 \geqslant \imath, \jmath \geqslant 1$, of $S(u,v) - \{u,v\}$, together with u and v, to dominate $S(u,v)$, and then let $\hat{D} = \hat{D} \cup \{w_\imath, w_\jmath\}$. These two vertices are depicted in Figs. 5(a)–(c). We can prove the constructed set $\hat{D}$ to be a dominating set of G_s after computing all snake paths of G_s. Because of the space limitation, we omit the details of proof.

During our construction of $\hat{D}$, $|\hat{D}| = |\hat{D}| + 2$ after computing one snake path of G_s. Since there exist $|E(G_g)|$ snake paths of G_s, we construct a dominating set $\hat{D}$ of G_s with size $|\hat{D}| = |D| + 2|E(G_g)| \leqslant k + 2|E(G_g)|$. For example, Fig. 3(a) shows a dominating set D of G_g with $|D| = 4$, and the dominating set $\hat{D}$ of

G_s constructed from D is shown in Fig. 3(d), where $|\hat{D}| = |D| + 2|E(G_g)| = 4 + 2 \times 11 = 26$.

Next, we will prove the if part as follows.

Lemma 5. *Assume that supergrid graph G_s contains a dominating set $\hat{D}$ with size $|\hat{D}| \leqslant k + 2|E(G_g)|$. Then, grid graph G_g has a dominating set D with size $|D| \leqslant k$.*

Proof. We will show there exists a dominating set D of G_g with size k or less. The construction steps are described as follows. First, we compute a dominating set D' of G_s from $\hat{D}$ such that $|D'| \leqslant |\hat{D}|$. Second, we remove all vertices not in G_g from D' and obtain a set D. Then, we can prove D to be a dominating set of grid graph G_g.

First, we compute D' from $\hat{D}$ as follows. Initially, let $D' = \hat{D}$. Consider a snake path $S(u, v)$ of G_s constructed from edge (u, v) of G_g, as depicted in Fig. 5(a). By Proposition 1, $|\hat{D}_{|S(u,v)-\{u,v\}}| \geqslant 2$. We will construct a set D' of G_s obtained from $\hat{D}$ such that $|D'| \leqslant |\hat{D}|$, $|D'_{|S(u,v)-\{u,v\}}| \geqslant 2$, and removing all vertices not in G_g from D' results in a dominating set D of G. We consider three cases of (1) $u, v \notin D'$, (2) $u \in D'$ and $v \notin D'$, and (3) $u, v \in D'$. First consider the case of $u, v \notin D'$. In case of Fig. 5(d), $|D'_{|S(u,v)-\{u,v\}}| \geqslant 3$ by Statement (3-2) of Proposition 1, and we then set u to be in D', i.e., $D' = D' \cup \{u\}$. Then, snake (u, v)-path $S(u, v)$ satisfies that $u \in D'$ and $v \notin D'$. It is then computed in case of $u \in D'$ and $v \notin D'$. For the other cases, we compute D' on $S(u, v)$ to satisfy $|D'_{|S(u,v)-\{u,v\}}| = 2$. We can verify that D' is a dominating set of G_s and satisfies the following properties: (due to space limitation, we omit the details of proof)

(p1) $|D'| \leqslant |\hat{D}|$,
(p2) every snake path contains exactly two vertices not in G_g, and
(p3) for each snake (u, v)-path $S(u, v)$ with $u, v \notin D'$, there exist $z_1 \in N_{G_g}(u)$ and $z_2 \in N_{G_g}(v)$ such that $z_1, z_2 \in D'$.

We finally compute a dominating set D of G_g from D' by the following steps:

(1) initially, let $D = D'$;
(2) remove all vertices of D not in D_g from D;
(3) the resultant set D will be a dominating set of G_g.

Assume by contradiction that $|D| > k$. Let $|D| = k + x$, $x > 0$. Since $D' \cap S(u, v)$ contains exactly two vertices not in G_g for each snake path $S(u, v)$, we get that $|D| = |D'| - 2|E(G_g)|$. Then,
$|D'| \leqslant |\hat{D}| = k + 2|E(G_g)| = (k + x) + 2|E(G_g)| - x = |D| + 2|E(G_g)| - x$, and hence
$|D| = |D'| - 2|E(G_g)| \leqslant (|D| + 2|E(G_g)| - x) - 2|E(G_g)| = |D| - x = (k + x) - x = k$, a contradiction. Thus, $|D| \leqslant k$.

By Lemmas 4 and 5, we summarize the following lemma:

Lemma 6. *Let G_g be a grid graph and let G_s be the supergrid graph constructed from G_g by Algorithm 1 and Rule AS. Then, G_g contains a dominating set D with size $|D| \leqslant k$ if and only if G_s contains a dominating set $\hat{D}$ with $|\hat{D}| \leqslant k + 2|E(G_g)|$.*

Obviously, the domination problem for supergrid graphs is in NP. By Theorem 1, Lemmas 2–3, and Lemma 6, we conclude the following theorem:

Theorem 2. *The domination problem on supergrid graphs is NP-complete.*

A dominating set of the supergrid graphs constructed in Lemmas 4 and 5 can be easily modified as an independent dominating set (see Figs. 5(a)–(c)). Thus, the independent domination problem on supergrid graphs is also NP-complete, and, hence, the following theorem holds true.

Theorem 3. *The independent domination problem on supergrid graphs is NP-complete.*

4 The Domination Number of Rectangular Supergrid Graphs

In this section, we will compute $\gamma(R_{m\times n})$ for rectangular supergrid graph $R_{m\times n}$ with $n \geqslant m$. To simplify notation, we will use % to denote the modulo operation. In addition, we use $\mathcal{R}_\imath$ to denote the set of vertices of row $\imath$ in $R_{m\times n}$. By Lemma 1, $\gamma(R_{1\times n}) = \lceil \frac{n}{3} \rceil$. Next, we consider $R_{2\times n}$ as follows:

Lemma 7. $\gamma(R_{2\times n}) = \lceil \frac{n}{3} \rceil$.

Proof. By the structure of $R_{2\times n}$, a vertex of $R_{2\times n}$ dominates at most 6 vertices including its 5 neighbors and itself. Let D be any dominating set of $R_{2\times n}$. Then, $6 \cdot |D| \geqslant |V(R_{2\times n})| = 2n$, and, hence, $|D| \geqslant \lceil \frac{n}{3} \rceil$. That is, $\gamma(R_{2\times n}) \geqslant \lceil \frac{n}{3} \rceil$. Let D_1 be the minimum dominating set of $\mathcal{R}_1$. By Lemma 1, $|D_1| = \lceil \frac{n}{3} \rceil$. For each $v \in D_1$, v dominates $(v_x - 1, 2)$, $(v_x, 2)$, and $(v_x + 1, 2)$ in $\mathcal{R}_2$. In addition, v dominates at most three vertices $(v_x - 1, 1)$, $(v_x, 1)$, and $(v_x + 1, 1)$ in $\mathcal{R}_1$. Since D_1 dominates all vertices of $\mathcal{R}_1$ in $R_{2\times n}$, it also dominates all vertices of $\mathcal{R}_2$. Thus, D_1 is a dominating set of $R_{2\times n}$. Then, $\gamma(R_{2\times n}) \leqslant |D_1| = \lceil \frac{n}{3} \rceil$. Therefore, $\gamma(R_{2\times n}) = \lceil \frac{n}{3} \rceil$. For example, Fig. 6 shows the minimum dominating set of $R_{2\times n}$ for $2 \leqslant n \leqslant 10$.

By similar arguments in proving Lemma 7, we compute $\gamma(R_{3\times n})$ as follows:

Lemma 8. $\gamma(R_{3\times n}) = \lceil \frac{n}{3} \rceil$.

Proof. By the structure of $R_{3\times n}$, a vertex of $R_{3\times n}$ dominates at most 9 vertices including its 8 neighbors and itself. Let D be any dominating set of $R_{3\times n}$. Then, $9 \cdot |D| \geqslant |V(R_{3\times n})| = 3n$, and, hence, $|D| \geqslant \lceil \frac{n}{3} \rceil$. That is, $\gamma(R_{3\times n}) \geqslant \lceil \frac{n}{3} \rceil$. Let D_2 be the minimum dominating set of $\mathcal{R}_2$ in $R_{3\times n}$. By Lemma 1, $|D_2| = \lceil \frac{n}{3} \rceil$. By the same arguments in proving Lemma 7, D_2 is a dominating set of $R_{3\times n}$. Thus, $\gamma(R_{3\times n}) \leqslant \lceil \frac{n}{3} \rceil$. It follows from the above upper bound and lower bound of $\gamma(R_{3\times n})$ that $\gamma(R_{3\times n}) = \lceil \frac{n}{3} \rceil$. For example, Fig. 7 shows the minimum dominating set of $R_{3\times n}$ for $3 \leqslant n \leqslant 9$.

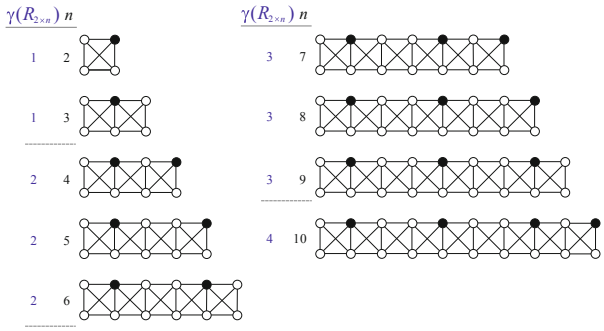


Fig. 6. $\gamma(R_{2\times n}) = \lceil \frac{n}{3} \rceil$ for $2 \leqslant n \leqslant 10$, where solid circles indicate the vertices in the minimum dominating set of $R_{2\times n}$.

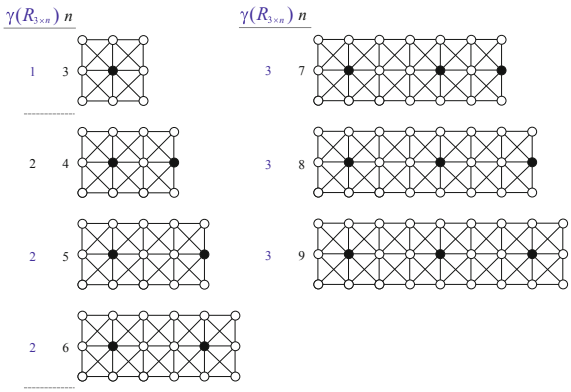


Fig. 7. $\gamma(R_{3\times n}) = \lceil \frac{n}{3} \rceil$ for $3 \leqslant n \leqslant 9$, where solid circles indicate the vertices in the minimum dominating set of $R_{3\times n}$.

Now, we consider $R_{m\times n}$ with $n \geqslant m \geqslant 3$. We will prove the following lemma by induction.

Lemma 9. *Let $R_{m\times n}$ be a rectangular supergrid graph with $n \geqslant m \geqslant 3$. Then, $\gamma(R_{m\times n}) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$.*

Proof. Let $D_\imath$ be the minimum dominating set of $\mathcal{R}_\imath$, the set of vertices of row $\imath$ in $R_{m\times n}$. By the construction in [9], $D_\imath = \{(2+3j, \imath) | 0 \leqslant j \leqslant \lfloor \frac{n}{3} \rfloor - 1\}$, and $D_\imath = D_\imath \cup \{(n, \imath)\}$ if $n\%3 \neq 0$. We claim that $R_{m\times n}$ contains a minimum dominating set D such that $D = \cup_{0\leqslant \imath \leqslant \lfloor \frac{m}{3} \rfloor - 1} D_{2+3\imath}$, and $D = D \cup D_m$ if $m\%3 \neq 0$.

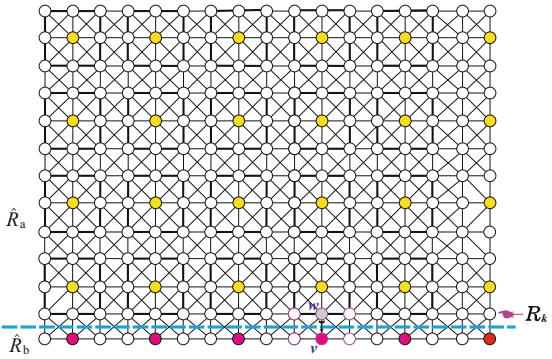


Fig. 8. The minimum dominating set of $R_{(k+1)\times n}$ when $k\%3 = 0$, where solid circles indicate the vertices in the minimum dominating set.

We will prove this claim by induction on m. Initially, let $m = 3$. By Lemma 8, the claim holds true. Assume that the claim is true when $m = k \geqslant 3$. Then, $R_{k\times n}$ has a minimum dominating set $\hat{D}$ such that $\hat{D} = \cup_{0\leqslant \imath\leqslant \lfloor \frac{k}{3}\rfloor -1} D_{2+3\imath}$, and $\hat{D} = \hat{D}\cup D_k$ if $k\%3 \neq 0$. Let $m = k+1$. Consider that $k\%3 = 0$. We first make a horizontal separation on $R_{(k+1)\times n}$ to obtain two disjoint subgraphs $\hat{R}_a = R_{k\times n}$ and $\hat{R}_b = R_{1\times n}$, where $\hat{R}_b = \mathcal{R}_{k+1}$ is the set of vertices of row $k+1$ in $R_{(k+1)\times n}$, as depicted in Fig. 8. Let D be a minimum dominating set of $R_{(k+1)\times n}$. Suppose that $D\cap\mathcal{R}_k \neq \emptyset$, where $\mathcal{R}_k$ is the set of vertices of row k as shown in Fig. 8. Then, $|D_{|\hat{R}_a}| > |\hat{D}|$. Let $w \in D\cap\mathcal{R}_k$ and $v \in \hat{R}_b$ with $w_x = v_x$, see Fig. 8. Then, $N_{R_{(k+1)\times n}}[w]\cap(\mathcal{R}_k\cup\hat{R}_b) = N_{R_{(k+1)\times n}}[v]\cap(\mathcal{R}_k\cup\hat{R}_b)$. Let $D = D-\{w\}\cup\{v\}$. Then, D is still a minimum dominating set of $R_{(k+1)\times n}$. Thus, there exists a minimum dominating set D of $R_{(k+1)\times n}$ such that $D\cap\mathcal{R}_k = \emptyset$, and hence $\hat{D} = D_{|\hat{R}_a}$. Then, $N_{R_{(k+1)\times n}}[\hat{D}]\cap\hat{R}_b = \emptyset$. So, no vertex of $D_{|\hat{R}_a} = \hat{D}$ dominates vertex of $\hat{R}_b$. To dominate $\hat{R}_b$, it needs at least $\lceil\frac{n}{3}\rceil$ vertices by Lemma 1. Let $\hat{D}_b$ be a such minimum dominating set of $\hat{R}_b$. Then, $D = \hat{D}\cup\hat{D}_b$ is a minimum dominating set of $R_{(k+1)\times n}$. By the same arguments above and Lemmas 1 and 7, the claim holds when $k\%3 = 1$ or $k\%3 = 2$. Thus, the claim holds true when $n = k+1$. By induction, the claim holds true for $m \geqslant 3$. By the claim, $R_{m\times n}$ contains a minimum dominating set D with $|D| = \lceil\frac{m}{3}\rceil\lceil\frac{n}{3}\rceil$. Thus, $\gamma(R_{m\times n}) = \lceil\frac{m}{3}\rceil\lceil\frac{n}{3}\rceil$ for $n \geqslant m \geqslant 3$.

It immediately follows from Lemmas 1 and 7–9 that the following theorem holds.

Theorem 4. $\gamma(R_{m\times n}) = \lceil\frac{m}{3}\rceil\lceil\frac{n}{3}\rceil$ *for* $m, n \geqslant 1$.

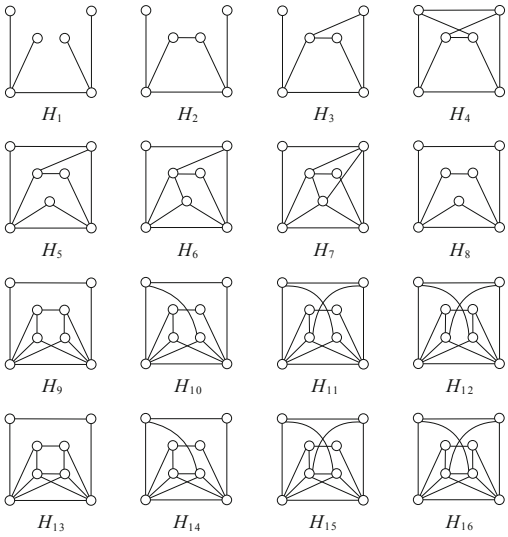


Fig. 9. A set of forbidden induced subgraphs in a graph G with $\gamma(G) = \gamma_{\text{ind}}(G)$ [30].

5 The Independent Domination Number of Rectangular Supergrid Graphs

In [28], Klobučar computed the independent domination number $\gamma_{\text{ind}}(R_{m\times n})$ of $R_{m\times n}$. In this section, we will use a different method to compute it. Since an independent dominating set of a graph G is also a dominating set, $\gamma(G) \leqslant \gamma_{\text{ind}}(G)$ for any graph G. In [3], Allan and Laskar proved that $K_{1,3}$-free graphs are graphs with equal domination and independent domination numbers. By the structure of $R_{m\times n}$, we can see that rectangular supergrid graphs are not $K_{1,3}$-free graphs, i.e., they contain induced subgraph $K_{1,3}$. However, Topp and Volkmann [30] showed that if a graph G contains no induced subgraph isomorphic to one of the graphs $H_1, H_2, \cdots, H_{16}$ in Fig. 9, then $\gamma(G) = \gamma_{\text{ind}}(G)$. The following theorem shows their result:

Theorem 5. *(see [30]). If a graph G contains no induced subgraph isomorphic to one of graphs $H_1, H_2, \cdots, H_{16}$ as shown in Fig. 9, then $\gamma(G) = \gamma_{\text{ind}}(G)$.*

By inspecting the structure of rectangular supergrid graphs, they contain H_1 and H_2 as induced subgraphs. However, rectangular supergrid graphs $R_{m\times n}$ are H_3-free graphs, i.e., they contain no induced subgraph isomorphic to H_3. Thus, $\gamma_{\text{ind}}(R_{m\times n}) = \gamma(R_{m\times n})$. In fact, we construct a minimum dominating set of $R_{m\times n}$ that is also an independent dominating set. It follows from Theorems 4 and 5 that the following theorem holds true.

Theorem 6. $\gamma_{\text{ind}}(R_{m\times n}) = \gamma(R_{m\times n}) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$ *for* $m, n \geqslant 1$.

6 Concluding Remarks

In this paper, we prove that the domination and independent domination problems on general supergrid graphs are NP-complete. Then, we provide a formally proof to compute the minimum dominating set of a rectangular supergrid graph. In addition, we use a simple method to verify the minimum dominating set of a rectangular supergrid graph is also a minimum independent dominating set.

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